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Extreme operators on AL-spaces

by

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Abstract. For two AL-spaces E and F, we find a necessary and sufficient condition that the unit ball V in the space of all bounded linear operators from E into F coincide with $(\operatorname{conv}\operatorname{ex} V)^{-}$ where the closure is taken with respect to the strong operator topology. We also present some related results for positive operators and consider the norm approximation by $\operatorname{conv}\operatorname{ex} V$.

1. Introduction. A Banach lattice E is called an *abstract Lebesgue space* (briefly, AL-*space*) if its norm is additive on the positive cone. By the Kakutani representation theorem, the most general example of an AL-space is the space $L^1(\mu)$ for some positive measure μ. Besides, any AL-space E can be identified as a Banach lattice with the space $N(Y)$ of all order continuous (= normal) bounded Radon measures on a (unique to within homeomorphism) compact hyperstonian space Y. The space $C(Y)$ of all real valued continuous functions on Y is now a Banach lattice isomorphic with the Banach dual E', and the space $M(Y)$ of all bounded Radon measures on Y is a Banach lattice isomorphic with E'' (see H. H. Schaefer [6], II, 8.5, 9.2, and 9.3).

Let E, F be any AL-spaces and Y, X their corresponding hyperstonian spaces. Throughout, we will identify E with $N(Y)$, E' with $C(Y)$, E'' with $M(Y)$ and, respectively, F with $N(X)$, F' with $C(X)$, F'' with $M(X)$. We denote by V and U the unit balls in the spaces of bounded operators $L(E,F)$ and $L(F',E')$, respectively. By a standard argument, U is a compact subset of $L(F',E')$ with respect to the weak* operator topology $\sigma\big(L(F',E'), F'\otimes E\big)$. Thus, by the Krein–Milman theorem, the convex hull of the extreme points of U is always dense in U for this topology. It is not so with V: it may even happen that in V there are no extreme points whatsoever. Nevertheless, in Section 3 we obtain certain Krein–Milman type theorems for V by using the technique of the adjoint embedding $S\to S'$ of V into U.

The set $\operatorname{ex} U$ of all extreme points of U was characterized (in a more general setting) by M. Sharir in [8]. The extreme points of V have been identified by the present author in [1]. In the sequel we will make constant use of the below-specified results on extreme operators.

The set V' of all adjoints of operators from V coincides with the set U_0 of all order continuous contractions in U. Also, we have $\operatorname{ex} V' = V' \cap \operatorname{ex} U$ ([1], Proposition). The set $\operatorname{ex} U$ consists exactly of the so-called *nice operators*, i.e. operators $T \in L(C(X), C(Y))$ of the form $Tf(y) = r(y)f(\varphi(y))$ where $\varphi\colon Y \to X$ is a continuous map and $r \in C(Y)$ with $|r| = 1$ ([8], Theorem 2; see also [1], Theorem 1). Moreover, $\operatorname{ex} V'$ consists of all nice operators that are induced by continuous open maps φ ([1], Corollary). Analogous results for positive operators are stated in Section 4 of the present paper.

Our main aim is to characterize those pairs E, F of AL-spaces for which all operators from V can be approximated, with respect to the strong operator topology, by convex combinations of $\operatorname{ex} V$ (Section 3). If E and F are finite dimensional, then V is compact, so that such an approximation can be carried out by the direct use of the Krein–Milman theorem. In [2] C. W. Kim has shown that the approximation is still possible in case $E = F = l^1$. Our Theorem 1 and Corollary 2 of Section 3 extend this last result. Similar approximation problem for positive operators is considered in Section 4. In Section 5 we present an explicit representation of the convex hull of $\operatorname{ex} U$. Section 6 shows that, in general, the norm closure of the convex hull of $\operatorname{ex} V$ does not coincide with V, even if the strong operator closure does. Finally, in Section 7 we observe that convex combinations of extreme contractions attain their norm on the unit ball of E.

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2. Density of adjoint contractions. For any two Banach spaces G, H the algebraic tensor product $G'' \otimes H$ can be viewed as the set of all finite rank operators in $L(G', H)$. The ε-unit ball U_ε in $G'' \otimes H$ can thus be identified with the set of all finite rank contractions ([6], IV, 2). Clearly, the ε-unit ball of $G \otimes H$ consists of those finite rank contractions in $L(G', H)$ whose adjoints take H' into G. Let us recall that the Banach space H is said to have the *metric approximation property* if the identity operator in $L(H, H)$ can be approximated in the topology of compact convergence by elements of the ε-unit ball in $H' \otimes H$ ([6], IV, 2.2).

LEMMA 1. *Let G, H be Banach spaces. If H has the metric approximation property (e.g., if H is an AM-space), then every contraction in $L(G', H)$ can be approximated in the strong operator topology by elements of the ε-unit ball W_ε in $G \otimes H$. If G is a Banach lattice and H is an AM-space, then, in addition, every positive contraction can be approximated by positive members of W_ε.*

Proof. By [6], IV 2.3 (c), the ε-unit ball U_ε in $G'' \otimes H$ is dense in the unit ball of $L(G', H)$ with respect to the strong operator topology. Further, by [6], IV 5.4 Cor. 3, W_ε is dense in U_ε for the weak, and hence for strong, operator topology. This completes the proof of the first part of the lemma.

For the proof of the second part, note first that the identity map in the AM-space H can be approximated by positive contractions in U_ε (see [6], a remark preceding IV, Theorem 2.4). Since the evaluation map taking a Banach lattice into its second dual is positive, Cor. 3 in [6], IV 5.4 holds true also for positive parts of the respective unit balls. Since H has the metric approximation property ([6] IV 2.4), the proof is concluded by an argument analogous to that used in the first part.

The following consequence of Lemma 1 will be used in the sequel.

COROLLARY 1. *Let E, F be any AL-spaces. Then V' is dense in U for the strong operator topology in $L(F', E')$. The same holds for the positive parts of V' and U, respectively.*

Proof. Put $G = F$ and $H = E'$ in Lemma 1.

3. Approximation by the convex hull of $\operatorname{ex} V$. By the Krein–Milman theorem, the convex hull of $\operatorname{ex} U$ is dense in U for the weak* operator topology (in fact, by Theorem 2.1 of Morris and Phelps [4], it is dense in U even with respect to the strong operator topology).

PROPOSITION 1. *The convex hull of $\operatorname{ex} V$ is dense in V for the strong operator topology in $L(E, F)$ if and only if $\operatorname{ex} V'$ is dense in $\operatorname{ex} U$ for the weak* operator topology in $L(F', E')$.*

Proof. If $\operatorname{conv} \operatorname{ex} V$ is dense in V for the weak (strong) operator topology, then clearly $\operatorname{conv} \operatorname{ex} V'$ is dense in V' for the weak* operator topology in $L(F', E')$. Thus, by Milman's "converse" of the Krein–Milman theorem, $\operatorname{ex} V'$ is dense in $\operatorname{ex} U$ for the weak* operator topology (V' is dense in U by Corollary 1).

Conversely, if $\operatorname{ex} V'$ is dense in $\operatorname{ex} U$, then evidently $\operatorname{conv} \operatorname{ex} V'$ is dense in $\operatorname{conv} \operatorname{ex} U$, so that also in U and, moreover, in V'. Therefore, $\operatorname{conv} \operatorname{ex} V$ is dense in V for the weak (and thus for the strong) operator topology in $L(E, F)$.

Our next task is to identify those AL-spaces for which the equivalent conditions of Proposition 1 are satisfied.

By the representation theorem [7], 26.4.7 for AL-spaces, there exists for E a unique well-ordered family $\mathfrak{m}_\sigma, -1 \leqslant \sigma < \tau$, of cardinal numbers such that set $\{\sigma\colon \mathfrak{m}_\sigma \neq 0\}$ is cofinal in τ, each $\mathfrak{m}_\sigma$ for $\sigma \geqslant 0$ is either equal to zero or to one, or else is uncountable, and E is Banach lattice isomorphic with the l^1-join

$$l^1(\mathfrak{m}_{-1}) + \sum_{0 \leqslant \sigma < \tau} \mathfrak{m}_\sigma L^1(\mu_{\omega_\sigma}),$$

where, for any infinite cardinal number α, μ_α denotes the product measure on the Cantor cube $\{0,1\}^\alpha$. We denote by M_α the corresponding hyperstonian space for $L^1(\mu_\alpha)$. Now the hyperstonian space Y of E can be viewed as a compactification of the direct topological sum of the space $\beta\mathfrak{m}_{-1}$ (= the corresponding hyperstonian space for $l^1(\mathfrak{m}_{-1})$) and the spaces M_{ω_σ}, $\mathfrak{m}_\sigma$ copies of each. The measure algebra of μ_α is homogeneous (see [7], 28.4), so, by an easy application of Maharam's theorem, each non-empty closed-and-open subset of M_α is homeomorphic with M_α.

Lemma 2. *There exists a continuous open map* $\varphi\colon M_\alpha\to M_\beta$ *if and only if* $\alpha\geqslant\beta$.

Proof. Suppose $\alpha\geqslant\beta$. The natural projection of $\{0,1\}^\alpha$ onto $\{0,1\}^\beta$ induces a lattice homomorphism $T\colon L^\infty(\mu_\beta)\to L^\infty(\mu_\alpha)$. The corresponding lattice homomorphism $C(M_\beta)\to C(M_\alpha)$ takes 1 into 1, so that it is an extreme contraction ([6], III 9.1) and is induced by a continuous map $\varphi\colon M_\alpha\to M_\beta$. Since the projection is measure preserving, T' takes $L^1(\mu_\alpha)$ into $L^1(\mu_\beta)$, whence T has a pre-adjoint, so that it is order continuous and φ has to be an open map ([6], II 9.3, Proposition).

Conversely, if there is an open continuous map $\varphi\colon M_\alpha\to M_\beta$, then, as $\varphi(M_\alpha)$ is a closed-and-open subset of M_β, we may assume that φ is onto. Thus, the associated lattice homomorphism $T\colon C(M_\beta)\to C(M_\alpha)$ is one-to-one. Therefore, the range of the pre-adjoint $S\colon L^1(\mu_\alpha)\to L^1(\mu_\beta)$ of T is dense in $L^1(\mu_\beta)$. This implies that the density character β of $L^1(\mu_\beta)$ (see [7], 24.4.9 (A)) is less than or equal to the density character α of $L^1(\mu_\alpha)$.

Let now

$$l^1(\mathfrak{n}_{-1})+\sum_{0\leqslant\varrho<\nu}\mathfrak{n}_\varrho L^1(\mu_{\omega_\varrho})$$

be the like representation for the AL-space F.

Lemma 3. *If* $\sup\{\varrho\colon \mathfrak{n}_\varrho\neq 0\}\leqslant\inf\{\sigma\colon \mathfrak{m}_\sigma\neq 0\}$, *then* $\operatorname{ex}V'$ *is dense in* $\operatorname{ex}U$ *for the strong operator topology.*

Proof. By the Stone–Weierstrass theorem the continuous characteristic functions form a linearly dense subset of the Banach space $C(X)$. Thus, by [5], III 4.5, it suffices to show that for any finite partition $X_1,\ldots,X_n$ of X into non-empty closed-and-open sets and for any $T\in\operatorname{ex}U$ there exists $T_0\in\operatorname{ex}V'$ such that $T\chi_i=T_0\chi_i$ $(i=1,\ldots,n)$, where by χ_i we denote the characteristic function of X_i. Since T is extreme, $|T\chi_i|$ are characteristic functions of certain closed-and-open subsets $Y_1,\ldots,Y_n$ forming a partition of Y (see [1], Theorem 1). By Corollary in [1], it suffices to find a continuous open map φ such that $\varphi^{-1}(X_i)=Y_i$. Clearly, without any loss of generality we may assume that $Y_i=Y$ and $X_i=M_{\omega_\varrho}=X$. Thus it remains to show that there exists a continuous open map $\varphi\colon Y\to X$.

By the l^1-join representation, Y is the closure of the topological direct sum Y_0 of closed-and-open subsets Y_t, each homeomorphic either with certain M_{ω_σ} or with $\beta\mathfrak{m}_{-1}$. Evidently, the set $Y\setminus Y_0$ is closed and nowhere dense in Y.

By Lemma 2 and by assumption, for each Y_t there exists a continuous open map $\varphi_t\colon Y_t\to X$. Putting all these maps together we obtain the continuous open map $\varphi=\bigcup\varphi_t$ from Y_0 into X. The formula $\nu\to\nu\circ\varphi^{-1}$ defines a positive contraction $S\colon N(Y)\to M(X)$. For any closed nowhere dense subset A of X the inverse image $\varphi^{-1}(A)$ is closed in Y_0 and has empty interior, so the closure of $\varphi^{-1}(A)$ is nowhere dense in Y. Thus $S\nu$ along with ν vanishes on closed nowhere dense subsets, implying $S\nu\in N(X)$ ([6], II 9). It is now not hard to see that the adjoint operator $S'\in L\big(C(X),C(Y)\big)$ satisfies

$$S'f(y)=f\big(\varphi(y)\big)$$

for all $f\in C(X)$ and $y\in Y_0$. Thus, by continuity, S' is a lattice homomorphism and takes 1 into 1, whence $S'\in\operatorname{ex}V'$. By virtue of Corollary in [1], there exists a continuous open map $\varphi\colon Y\to X$ (inducing S' and extending φ).

Lemma 4. *If* $\operatorname{ex}V'$ *is dense in* $\operatorname{ex}U$ *for the weak* operator topology, then* $\sup\{\varrho\colon \mathfrak{n}_\varrho\neq 0\}\leqslant\inf\{\sigma\colon \mathfrak{m}_\sigma\neq 0\}$.

Proof. Suppose that, on the contrary, $\mathfrak{n}_\varrho\neq 0$ and $\mathfrak{m}_\sigma\neq 0$ for some $\varrho>\sigma$. Then, in particular, F has a non-atomic part, so X contains a closed-and-open subset X_0 homeomorphic with some M_{ω_ϱ}.

If $\mathfrak{m}_{-1}\neq 0$, then Y has an isolated point y_0. The extreme contraction $T\colon C(X)\to C(Y)$ induced by a continuous map φ satisfying $\varphi(y_0)\in X_0$ and $\varphi(y)\notin X_0$ for $y\neq y_0$ cannot be approximated by $\operatorname{ex}V'$ for the weak* operator topology. Indeed, for any continuous open map φ the image of y_0 is not in X_0 (X_0 has no isolated points), so that denoting by χ the characteristic function of X_0 and by δ the Dirac measure concentrated in y_0 we have $\langle T\chi,\delta\rangle=1$ and $\langle S\chi,\delta\rangle=0$ for any $S\in\operatorname{ex}V'$.

If $\mathfrak{m}_{-1}=0$, then, by assumption, for some $\alpha<\beta$ the spaces Y and X contain respectively M_α and M_β as closed-and-open subsets. We may also assume that M_β is a proper subset of X. Now taking $\chi_{M_\beta}\oplus\mu_\alpha$ as the appropriate functional for $\big(L(F',E'),F'\oplus E\big)$, we can see by Lemma 2 that no operator induced by a continuous map φ satisfying $\varphi(M_\alpha)\subset M_\beta$ and $\varphi(Y\setminus M_\alpha)\subset X\setminus M_\beta$ can be approximated by elements of $\operatorname{ex}V'$.

Putting Proposition 1 and the last two lemmas together we obtain the following result.

Theorem 1. *Let* E, F *be AL-spaces. Then the following conditions are equivalent*:

(1) $\sup\{\varrho\colon \mathfrak{n}_\varrho\neq 0\}\leqslant\inf\{\sigma\colon \mathfrak{m}_\sigma\neq 0\}$,

(2) $\operatorname{ex}V'$ *is dense in* $\operatorname{ex}U$ *for the strong operator topology in* $L(F',E')$,

(3) $\operatorname{ex} V'$ *is dense in* $\operatorname{ex} U$ *for the weak* operator topology in* $L(F', E')$,

(4) $\operatorname{conv}\operatorname{ex} V$ *is dense in* V *for the strong (weak) operator topology in* $L(E, F)$.

In the special case $E = F$ we have the following

Corollary 2. *Let* $E = F$. *Then* $\operatorname{conv}\operatorname{ex} V$ *is dense in* V *for the strong operator topology if and only if* E *is homogeneous, i.e. if there is only one non-zero* $\mathfrak{m}_\sigma$. *If, in addition,* E *is separable, then* $\operatorname{conv}\operatorname{ex} V$ *is dense in* V *if and only if* E *is either non-atomic or purely atomic.*

In the light of Lemma 2 the following corollary is also evident.

Corollary 3. *If* $\sigma < \varrho$ *whenever* $\mathfrak{m}_\sigma \neq 0$ *and* $\mathfrak{n}_\varrho \neq 0$ *(in particular, if* E *is purely atomic and* F *non-atomic), then* $\operatorname{ex} V = \emptyset$.

4. Positive operators. Let V_+ and U_+ denote the positive parts of V and U. If $E = F$, then the elements of V_+ are often called *sub-Markov operators*.

Proposition 2. *Let* E, F *be AL-spaces. Then* V'_+ *is an extreme subset of* U_+, *dense in* U_+ *with respect to the strong operator topology.*

In particular, $\operatorname{ex} V'_+ = V'_+ \cap \operatorname{ex} U_+$.

Proof. By Corollary 1, V'_+ is dense in U_+. We will prove that V'_+ is extreme in U_+. The set $L_0 = \{S' \colon S \in L(E, F)\}$ is an ideal in $L(F', E')$ (more precisely, the ideal of all order continuous operators in $L(F', E')$, see Lemma 3 in [1]). Thus, it suffices to show that for any vector lattice L, for any ideal L_0 in L, and for any convex subset Q of L_+, the set $Q \cap L_0$ is extreme in Q. Let $T \in Q \cap L_0$ be a non-trivial convex combination $T = \alpha T_1 + \beta T_2$ with $T_1, T_2 \in Q$. Then $0 \leqslant T_1 \leqslant \alpha^{-1} T$, so that $T_1 \in L_0$ and, analogously, $T_2 \in L_0$, whence $Q \cap L_0$ is an extreme subset of Q.

By [6], III 9.1, we can almost immediately characterize the extreme points of U_+ in the following way:

Proposition 3. *Let* X, Y *be any compact Hausdorff spaces. Then for any* $T \in L(C(X), C(Y))$ *the following conditions are equivalent:*

(a) T *is an extreme point in the set of positive contractions,*

(b) T *is an algebra homomorphism,*

(c) T *is a lattice homomorphism and* $T1$ *is a characteristic function,*

(d) *there exists a (unique) closed-and-open subset* Y_0 *of* Y *and a (unique) continuous map* $\varphi \colon Y_0 \to X$ *such that*

$$Tf(y) = \begin{cases} f(\varphi(y)) & \text{if } \; y \in Y_0, \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in C(X)$.

By the already mentioned Proposition 9.3, II in [6] and by our Propositions 2 and 3, we can identify the extreme points of V_+ as those

operators $S \in L(E, F)$, whose adjoints $T = S'$ satisfy condition (d) of Proposition 3 with φ continuous and open. Now all results of Section 3 remain valid if we replace V and U by their positive parts V_+ and U_+ (except Corollary 3, where the assertion $\operatorname{ex} V = \emptyset$ should now be substituted by $\operatorname{ex} V_+ = \{0\}$). Indeed, the obvious slight change is needed only in the first part of the proof of Lemma 3: now $Y_1, \ldots, Y_n$ need not be a partition of Y. The other proofs remain unchanged. In particular, we have the following result.

Proposition 4. $(\operatorname{conv}\operatorname{ex} V)^- = V$ *if and only if* $(\operatorname{conv}\operatorname{ex} V_+)^- = V_+$, *the closures being taken with respect to the strong (weak) operator topology.*

5. The convex hull of $\operatorname{ex} U$. We will give a more explicit description of $\operatorname{conv}\operatorname{ex} U$. Throughout this section Y and X denote compact Hausdorff spaces and we assume that Y is Stonian (= extremally disconnected). Under these assumptions, $L(C(X), C(Y))$ is a Banach lattice ([6], II 7.7 and IV 1.5). By U we denote the unit ball in $L(C(X), C(Y))$.

Lemma 5. *If* $\varphi_1, \ldots, \varphi_n$ *are continuous maps from* Y *into* X, *then there exists a finite partition* $Z_1, \ldots, Z_m$ *of* Y *into closed-and-open subsets such that for any* Z_k *and any* $i, j \in \{1, \ldots, n\}$ *the open set*

$$\{y \in Z_k \colon \varphi_i(y) \neq \varphi_j(y)\}$$

is either empty or dense in Z_k.

Proof. It is well known (and easy to see) that the family A of all sets of the form $G \cup N$ with G open and N nowhere dense is an algebra of subsets of Y. For any i, j the set

$$F(i, j) = \{y \in Y \colon \varphi_i(y) = \varphi_j(y)\}$$

is closed, so $F(i, j) \in A$. Thus the partition generated by all sets $F(i, j)$ consists of the sets $A_k = G_k \cup N_k$ $(k = 1, \ldots, m)$ with G_k open and N_k nowhere dense. For any non-empty G_k we put $Z_k = \overline{G}_k$. Since Y is Stonian, the Z_k are closed-and-open and pairwise disjoint. Moreover, since the finite union $\bigcup N_k$ is a nowhere dense set, $Z_1, \ldots, Z_m$ form a partition of Y. Finally, for any $1 \leqslant k \leqslant m$ the open set

$$\{y \in G_k \colon \varphi_i(y) \neq \varphi_j(y)\}$$

is either empty or coincides with G_k, whence the assertion follows. ∎

Under the above assumptions on Y and X, the nice operators in $L(C(X), C(Y))$ (see Section 1) coincide with $\operatorname{ex} U$ ([8]; cf. [1]). For any continuous map $\varphi \colon Y \to X$ the positive nice operator $f \to f \circ \varphi$ will be denoted by T_φ.

THEOREM 2. *Let X, Y be compact Hausdorff spaces and let Y be Stonian. For any operator $T \in L(C(X), C(Y))$ the following conditions are equivalent:*

(a) *T is a convex combination of nice operators (i.e. $T \in \operatorname{convex} U$);*

(b) *there exist a partition $Z_1, \ldots, Z_m$ of Y into closed-and-open subsets, an array of real numbers β_{ij} $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i)$ with $\sum_j |\beta_{ij}| \leqslant 1$ for $1 \leqslant i \leqslant m$, and a corresponding array of continuous maps $\psi_{ij}\colon Z_i \to X$ with $\{z \in Z_i\colon \psi_{ij}(z) \neq \psi_{ik}(z)\}^- = Z_i$, for $j \neq k$, such that T is represented as*

$$T = \sum_{i=1}^{m} \chi_{Z_i} \sum_{j=1}^{n_i} \beta_{ij} T_{\psi_{ij}}.$$

Proof. (a)$\Rightarrow$(b). Let $T = \sum \alpha_k r_k T_{\varphi_k}$ with $\alpha_k \geqslant 0$, $\sum \alpha_k = 1$, $r_k \in C(Y)$, $|r_k| = 1$, φ_k continuous, and $k = 1, \ldots, n$. By Lemma 5 there exists a partition $Z_1, \ldots, Z_m$ of Y into closed-and-open sets such that for each i any two mappings φ_j, φ_k either coincide on Z_i or differ on a dense subset of Z_i. Since the sets $\{y\colon r_k(y) = 1\}$ are closed-and-open, we may (and will) assume that the r_k are constant on each Z_i. For each $1 \leqslant i \leqslant m$ we can partition the set $\{1, \ldots, n\}$ into subsets $J_{i1}, \ldots, J_{in_i}$ so that any φ_j, φ_k coincide on Z_i if j and k are in the same atom of the partition and differ on a dense subset of Z_i if j and k belong to different atoms. Assertion (b) is satisfied by letting ψ_{ij} be any representative of the set $\{\varphi_k\colon k \in J_{ij}\}$ and by putting $\beta_{ij} = \sum \alpha_k r_k(z)$, where $z \in Z_i$ and k runs over J_{ij}.

(b)$\Rightarrow$(a). By an inductive argument with respect to m, it is not hard to show that there exists a finite sequence (α_k), $k = 1, \ldots, n$, of non-negative real numbers with $\sum \alpha_k = 1$ such that for each $1 \leqslant i \leqslant m$ there exists a partition $J_{i1}, \ldots, J_{in_i}$ of $\{1, \ldots, n\}$ satisfying $\beta_{ij} = \sum \varrho_{ik} \alpha_k$, where $|\varrho_{ik}| = 1$ and k runs over J_{ij}. By putting $\varphi_k(z) = \psi_{ij}(z)$ and $r_k(z) = \varrho_{ik}$ for $z \in Z_i$ and $k \in J_{ij}$, we obtain $T = \sum \alpha_k r_k T_{\varphi_k}$.

LEMMA 6. *Let φ_1, φ_2 be continuous maps from Y into X. Then T_{φ_1} and T_{φ_2} are disjoint in $L(C(X), C(Y))$ if and only if*

$$A = \{y\colon \varphi_1(y) \neq \varphi_2(y)\}$$

is a dense subset of Y.

Proof. Sufficiency. Suppose $y \in A$ and let $f_1 + f_2 = 1$ be a continuous partition of identity in $C(X)$ with $f_i(\varphi_j(y)) = \delta_{ij}$. We have $T_{\varphi_i} f_j(y) = 0$ for $i \neq j$, so that $T1(y) = T(f_1 + f_2)(y) = 0$, where $T = T_{\varphi_1} \wedge T_{\varphi_2}$. Therefore, $T1 = 0$ on A and, consequently, $T = 0$.

Necessity. If $\varphi_1(y) = \varphi_2(y)$ on a non-empty open subset of Y, then the equality holds on a non-empty closed-and-open subset Z. Thus $T_{\varphi_i} \geqslant \chi_Z T_{\varphi_1} = \chi_Z T_{\varphi_2} \neq 0$ for $i = 1, 2$, whence $T_{\varphi_1} \wedge T_{\varphi_2} \neq 0$.

As a consequence of Lemma 6 we obtain the following corollary.

COROLLARY 4. *If $T \in \operatorname{convex} U$ is represented as in* (b), *Theorem* 2, *then its modulus can be expressed by the formula*

$$|T| = \sum_{i=1}^{m} \chi_{Z_i} \sum_{j=1}^{n_i} |\beta_{ij}| T_{\psi_{ij}}.$$

In particular, $\chi_{Z_i} |T| \in \operatorname{convex} U_+$ for $1 \leqslant i \leqslant m$.

Proof. The first part is straightforward and the second follows from Proposition 4.

6. Failure of the approximation for norm topology. In this section we show that for non-atomic F no non-zero compact contraction can be approximated in norm by $\operatorname{convex} V$. Observe that F is non-atomic if and only if X has no isolated points.

LEMMA 7. *Let ν be a non-atomic positive measure in $N(X)$. Then for any finite family $\{\varphi_1, \ldots, \varphi_n\}$ of continuous open maps from Y into X and for any $\varepsilon > 0$ there exist non-empty closed-and-open subsets $Y_0 \subset Y$ and $X_0 \subset X$ such that $\nu(X_0) < \varepsilon$ and $\varphi_i(Y_0) \subset X_0$ for $1 \leqslant i \leqslant n$.*

Proof. Since ν is non-atomic, there exists a non-empty closed-and-open subset X_1 of $\varphi_1(Y)$ such that $\nu(X_1) < \varepsilon/n$. Clearly, $Y_1 = \varphi_1^{-1}(X_1)$ is a non-empty closed-and-open subset of Y and $\varphi_1(Y_1) \subset X_1$. Suppose, by induction, that for $m < n$ we have already found non-empty closed-and-open subsets $X_1, \ldots, X_m$ in X with $\nu(X_i) < \varepsilon/n$, and Y_m in Y with $\varphi_i(Y_m) \subset X_i$ $(i = 1, \ldots, m)$. Since ν is non-atomic and the set $\varphi_{m+1}(Y_m)$ is closed-and-open, $\varphi_{m+1}(Y_m)$ contains a non-empty closed-and-open subset X_{m+1} with $\nu(X_{m+1}) < \varepsilon/n$. The inductive assumption is now satisfied for $Y_{m+1} = Y_m \cap \varphi_{m+1}^{-1}(X_{m+1})$. Thus the induction works up to n and letting $X_0 = X_1 \cup \ldots \cup X_n$, $Y_0 = Y_n$, we get the desired result.

Let (χ_j), $1 \leqslant j \leqslant n$, be a family of disjoint characteristic functions in $E' = C(Y)$ and let (ν_j), $1 \leqslant j \leqslant n$, be a family of positive elements of norm $\leqslant 1$ in $F = N(X)$. These two families determine a finite rank positive contraction defined by

$$(+) \qquad T_0 = \sum \nu_j \otimes \chi_j \in V'_+.$$

Clearly, $\|\nu_j\| \leqslant \|T_0\|$ whenever $\chi_j \neq 0$, and for some j, say, $j = 1$, the equality holds.

Let now $T_i \in \operatorname{ex} V'_+$ for $1 \leqslant i \leqslant m$ and let T be a convex combination

$$T = \sum \alpha_i T_i \in \operatorname{convex} V'_+.$$

LEMMA 8. *If F is non-atomic, then $\|T_0 - T\| \geqslant \|T_0\|/2$.*

Proof. By the remarks following Proposition 3, each T_i is induced by a continuous open map $\varphi_i\colon Y_i \to X$ with Y_i closed-and-open. For χ_j

in $(+)$ we denote $\{y\colon \chi_j(y)=1\}=Z_j$. According to the previous remark, we have $\|\nu_1\|=\|T_0\|$. Let k be the maximal integer $\leqslant m$ such that

$$Z_1\cap\bigcap_{i\in K}Y_i\neq\emptyset$$

for a subset $K\subset\{1,\ldots,m\}$ of cardinality k. Without loss of generality we may assume that either $k=0$ or $K=\{1,\ldots,k\}$. Now we consider two cases.

(1) $a_1+\ldots+a_k>\|T_0\|/2$. Applying Lemma 7 with $\nu=\nu_1+\ldots+\nu_n$ and with

$$Z=Z_1\cap\bigcap_{i\in K}Y_i$$

instead of Y, we find closed-and-open subsets $Y_0\subset Z$ and $X_0\subset X$ such that $\nu(X_0)<\varepsilon$ and $\varphi_i(Y_0)\subset X_0$ for $1\leqslant i\leqslant m$. Letting χ be the characteristic function of X_0 we get

$$T\chi(y)\geqslant\sum_{i=1}^{k}a_i\chi\big(\varphi_i(y)\big)=\sum_{i=1}^{k}a_i>\|T_0\|/2.$$

On the other hand, we have

$$T_0\chi=\sum\langle\nu_j,\chi\rangle\chi_j\leqslant\langle\nu,\chi\rangle<\varepsilon,$$

whence $\|T_0-T\|+\varepsilon\geqslant\|T_0\|/2$ for any $\varepsilon>0$.

(2) $k=0$ or $a_1+\ldots+a_k\leqslant\|T_0\|/2$. By the maximality of k,

$$T1(y)=\sum_{i=1}^{m}a_iT_i1(y)=\sum_{i=1}^{k}a_i\quad\text{whenever}\quad y\in Z.$$

On the other hand, for any $y\in Z$ we have $T_01(y)=\sum\langle\nu_j,1\rangle\chi_j(y)\geqslant\langle\nu_1,1\rangle\chi_1(y)=\|T_0\|$. Thus, $\|T_0-T\|\geqslant\|T_0\|/2$, which concludes the proof of the lemma.

By an argument used in the proof of Lemma 1, each positive finite rank contraction in V'_+ can be approximated in norm by contractions of the form $(+)$. Since the norm closure of all finite rank positive contractions coincides with the compact operators in V_+ ([6], IV 4.6, Cor. 1), we obtain the following quick consequence of Lemma 8.

THEOREM 3. *Let E, F be AL-spaces and suppose that F is non-atomic. Then for any compact positive contraction $S_0\in V_+$ and any $S\in\operatorname{conv}\operatorname{ex}V_+$ we have $\|S_0-S\|\geqslant\|S_0\|/2$.*

In case $E=F$, the set of positive contractions V_+ forms a topological semigroup with respect to the norm topology. Both $\operatorname{ex}V_+$ and $\operatorname{convex}V_+$ are subsemigroups of V_+.

COROLLARY 5. *If $E=F$ is a non-atomic AL-space, then the norm closure of $\operatorname{convex}V_+$ does not contain any non-zero weakly compact operator.*

Proof. If $0\neq S\in V_+$ is weakly compact, then, by the Dunford–Pettis property of E, the operator S^2 is compact ([6], II 9.9, Cor. 1). By Theorem 3 and by the continuity of composition in V_+, S cannot be approximated in norm by convex combinations of $\operatorname{ex}V_+$.

Now we return to arbitrary contractions. First, let us recall that if $S_0\in L(E,F)$ is compact, then its modulus $|S_0|$ is also compact ([6], IV 1.5 (ii), III 11.4, II 8.5, and IV 8.1). For any $S\in L\big(N(Y),N(X)\big)$ and any closed-and-open subset Z in Y we denote by $S\iota_Z$ the operator defined by $\nu\to S(\nu|Z)$, where $\nu|Z$ is the restriction of ν to Z. Clearly, $(S\iota_Z)'=\chi_ZS'$ and $|S\iota_Z|=|S|\iota_Z$. Theorem 3 can now be extended in the following manner.

THEOREM 4. *Let E, F be any AL-spaces with F non-atomic. Then for any compact contraction $S_0\in V$ and any $S\in\operatorname{convex}V$ we have $\|S_0-S\|\geqslant\|S_0\|/2$.*

Proof. Let $T=S'$ be represented as in (b) of Theorem 2. We have then $\|S_0\|=\||S_0|\|=\||S_0|'\|=\max\|\chi_{Z_i}|S_0|'\|=\max\||S_0|\iota_{Z_i}\|$. Suppose, say, $\|S_0\|=\||S_0|\iota_{Z_1}\|$. Thus

$$\|S_0-S\|\geqslant|S_0\iota_{Z_1}-S\iota_{Z_1}|\geqslant\||S_0|\iota_{Z_1}-|S|\iota_{Z_1}\|\geqslant\||S_0|\iota_{Z_1}\|/2=\|S_0\|/2,$$

the last inequality by Theorem 3 and Corollary 4.

7. A remark on norm attaining operators. An operator T attains its norm if $\|Tx\|=\|T\|$ for some x of norm 1. Norm attaining operators on various Banach spaces were treated in [3] and [9]. To the author's best knowledge still little is known about norm attaining operators on AL-spaces (cf. [9], p. 299). We will show that all operators from $\operatorname{convex}V$ attain their norm. Thus, in view of previous results of Section 3, we have a fairly rich set of norm attaining operators in e.g. $L(L^1[0,1],L^1[0,1])$.

PROPOSITION 5. *Any operator $S\in\operatorname{convex}V$ attains its norm.*

Proof. It suffices to show that if $T\in\operatorname{convex}U$, then T' attains its norm on $N(Y)$. Let T be represented as in (b) of Theorem 2. Then $T'=\sum_i\sum_j\beta_{ij}T'_{\varphi_{ij}}\iota_{Z_i}$ and $\|T'\|=\max_i\|\sum_j\beta_{ij}T'_{\varphi_{ij}}\iota_{Z_i}\|\leqslant\max_i\sum_j|\beta_{ij}|$. Suppose, say,

$$\|T'\|=\sum_j|\beta_{1j}|.$$

By the properties of φ_{ij} (see (b) of Theorem 2), there exists $y\in Z_1$ such that all points $x_j=\varphi_{1j}(y)$ $(1\leqslant j\leqslant n_1)$ are distinct. By Urysohn's lemma there is a function $f\in C(X)$ with $|f|\leqslant1$, such that $f(x)=\operatorname{sign}\beta_{1j}$ in a neighborhood of x_j $(1\leqslant j\leqslant n_1)$. Moreover, by the continuity of φ_{ij}, there exists a neighborhood G of y such that $f\big(\varphi_{1j}(z)\big)=\operatorname{sign}\beta_{1j}$ in G.

If ν is a normal Radon probability measure concentrated on G, then we have

$$\langle S\nu, f\rangle = \langle T'\nu, f\rangle = \sum_j |\beta_{1j}| = \|T'\| = \|S\|,$$

which concludes the proof of the proposition.

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Smooth and $\boldsymbol{R}$-analytic negligibility of subsets and extension of homeomorphisms in Banach spaces

by

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Abstract. It is proved that, if A is a compact set in the space $E = E' \times l_p(A)$, where E' is an infinite-dimensional, separable Banach space, then $E \setminus A$ and E are $\boldsymbol{R}$-analytically isomorphic. It is also established that, if A is a closed subspace of infinite codimension in a separable Banach space or in an arbitrary Hilbert space E, then $E \setminus A$ and E are $\boldsymbol{R}$-analytically isomorphic. In the smooth category analogous facts hold true if E is any infinite-dimensional weakly compactly generated (WCG-) Banach space. It is shown that any embedding of a compact subset A of a Banach space E admits an extension to an autohomeomorphism of E which is $\boldsymbol{R}$-analytic off A provided that either A is finite-dimensional and $E = E' \times l_p(A)$ for a separable infinite-dimensional Banach space E' or $E = E' \times l_p(A)$, where E' has an unconditional Schauder basis. Other results of this type are proved.

Introduction. Let us say that a closed subset A of a manifold M is *smoothly* or ***R**-analytically negligible in* M if $M \setminus A$ and M are smoothly or $\boldsymbol{R}$-analytically isomorphic. Negligibility of subsets was investigated by Renz [17], Moulis [14], West [21], Burghelea and Kuiper [5], Szigeti [18]. The most general theorem known in this field so far was established by Renz; it stated (in its weaker form) that compact sets are smoothly negligible in smooth Banach spaces with unconditional Schauder bases. The main result of the first part of this paper is the theorem stating that compact subsets are $\boldsymbol{R}$-analytically negligible in any infinite-dimensional separable Banach space. This is a strengthening of the result of Renz concerning smooth negligibility. (The only fact concerning $\boldsymbol{R}$-analytic negligibility known earlier was obtained by Burghelea and Kuiper [5]; it stated that $l_2 \setminus \{0\}$ and l_2 are $\boldsymbol{R}$-analytically isomorphic.) We observe also that our theorem does not extend to all infinite-dimensional Banach spaces; e.g. one-point sets are not $\boldsymbol{R}$-analytically negligible in the space $c_0(A)$ with uncountable A.

In the second part of this paper we deal with questions concerning the extension of embeddings of compact subsets K of a Banach space E into the space E. Let us recall that Renz proved in [17] that if $E = l_\alpha$