with this property, is at least $r/2$. Thus we assume the opposite situation prevails.

To recapitulate, I_2 has more than $3r/2$ elements. To each element i in I_2 there are indices j, k such that $|s_i - s_j| < h$ and $|t_i - t_k| < h$. At least one of j, k belongs to I_1, by the construction of I_1. For at least r elements i of I_2 (we call these I_2') at least one of j, k belongs to I_2. We write I_2'' if $j \in I_1$ and $k \in I_2$, and I_2''' otherwise. Then one of the sets I_2'', I_2''' has at least $r/2$ members.

Assuming that I_2'' has at least $r/2$ members, we finally attain a contradiction. For I_1 has fewer than $r/2$ members, so that I_2'' contains two elements i_0 and i_{00} such that for some j, we have $|s_{i_0} - s_j| < h$ and $|s_{i_{00}} - s_j| < h$, whence $|s_{i_0} - s_{i_{00}}| < 2h$. Also, $|t_{i_0} - t_k| < h$, with some k in I_2. Thus (s_{i_0}, t_{i_0}) is included in the first method of selection, a contradiction.

5. To complete our estimation of $\|I(R, u)\|_{2r}^{2r}$, we recall that this was expressed as an integral over $F^{(2r)}$, and that the integral over $F^{(2r)} \setminus H_R$ was found to be negligible. The measure of H_R was just found to have order $(R^{2\eta-2})^{ar/2} = R^{(\eta-1)ar}$. The integrand, moreover, is in $L^p(\mu^{2r})$, for $1 < p < 2a$, and its norm in L^p has order $R^{\eta 2r}$. The integral over H_R, therefore has order $R^{\eta 2r} \cdot R^{(\eta-1)arq}$, wherein $q = (p-1)p^{-1} > 0$. As η decreases to 0, the exponent approaches $-arq$, so that $\|I(r, u)\|_{2r}^{2r}$ has magnitude $B_r R^{-cr}$, for any $c < aq$. The number was subject to the inequality $q < 1-(2a)^{-1}$, so that c is subject to the inequality $c < a - 1/2$. This allows us to conclude that the density of the measure λ belongs to a certain Hölder class, depending on a.

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Singular integrals on generalized Lipschitz and Hardy spaces

by

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Abstract. Let $d(x, y)$ be a quasi-distance and μ a measure, both defined on X, such that (X, d, μ) is a normalized space of homogeneous type. Singular integral kernels are defined on (X, d, μ). Norm inequalities are given for the singular integral operators, associated with these kernels, acting on atomic Hardy spaces and their duals.

Introduction. Let X be a topological space and $d(x, y)$ a non-negative function defined on $X \times X$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) There exists a constant k such that

$$d(x, y) \leqslant k\big(d(x, z) + d(z, y)\big).$$

(iv) The balls with center at x and radius $r > 0$,

$$B(x, r) = \{y\colon d(x, y) < r\},$$

are a basis of neighbourhoods of x.

Moreover, we shall assume that there is a regular Borel measure μ such that for every ball $B(x, r)$, $x \in X$, $r > 0$, there exist two positive and finite constants c_1, c_2 such that

$$c_1 r \leqslant \mu\big(B(x, r)\big) \leqslant c_2 r. \tag{1}$$

This property of the measure μ implies that if $b > 0$ and $\varepsilon > 0$, then,

$$\int\limits_{d(x,z) > b > 0} d(x, z)^{-1-\varepsilon} d\mu(x) \leqslant cb^{-\varepsilon}. \tag{2}$$

The triple (X, d, μ), satisfying the requirements above, shall be called a *normalized homogeneous space* (see [3]).

Let $\varphi(x)$ be a real or complex valued function on X, square integrable on bounded subsets of X. The mean value of $\varphi(x)$, on a ball B, $\mu(B)^{-1}\int_B \varphi(x)\, d\mu(x)$, shall be denoted by $m_B(\varphi)$. We shall say that this function φ belongs to Lip(γ), $0 \leqslant \gamma \leqslant 1$, if there exists a finite constant c, such that for every ball B

$$(3) \qquad \Big(\mu(B)^{-1} \int_B |\varphi(x) - m_B(\varphi)|^2 \, d\mu(x)\Big)^{1/2} \leqslant c\mu(B)^{\gamma}$$

holds. We observe that our definition of Lip(0) coincides with the usual definition of BMO (Bounded mean oscillation, see [6]). The least constant c such that (3) holds shall be called the γ-Lipschitz norm of φ and shall be denoted by $\|\varphi\|_{\mathrm{Lip}(\gamma)}$. The norm of a function φ is equal to zero if and only if φ is equal to a constant almost everywhere. Therefore, if we declare two functions equivalent when they differ in a constant, the γ-Lipschitz norms define Banach spaces that we shall denote by $\mathit{Lip}(\gamma)$. We denote by $\bar{\varphi}$ the equivalence class of a function φ (see [7]).

Let $0 < p \leqslant 1$. A *p-atom* on (X, d, μ) is a function $a(x)$ whose support is contained in a ball B satisfying:

$$(4) \qquad \begin{aligned} &\text{(i)} \quad \Big(\mu(B)^{-1} \int_B |a(x)|^2 \, d\mu(x)\Big)^{1/2} \leqslant \mu(B)^{-1/p}, \\ &\text{(ii)} \quad \int a(x)\, d\mu(x) = 0. \end{aligned}$$

(See [2] and [7].) A p-atom can be identified with a linear functional on $\mathit{Lip}(1/p-1)$ by

$$\langle L_a, \bar{\varphi}\rangle = \int a(x)\varphi(x)\, d\mu(x).$$

This can be shown as follows. By part (ii) of (4), we have

$$\int a(x)\varphi(x)\, d\mu(x) = \int a(x)\big(\varphi(x) - m_B(\varphi)\big)\, d\mu(x),$$

then, by Schwarz inequality, (3) and part (i) of (4), we get

$$(5) \qquad \Big|\int a(x)\varphi(x)\, d\mu(x)\Big| \leqslant \Big(\int_B |a(x)|^2 \, d\mu(x)\Big)^{1/2} \Big(\int_B |\varphi(x) - m_B(\varphi)|^2 \, d\mu(x)\Big)^{1/2}$$
$$\leqslant \|\varphi\|_{\mathrm{Lip}(1/p-1)}.$$

This also shows that $\|L_a\| \leqslant 1$. For the sake of simplicity, we shall write a instead of L_a, therefore, $\langle a, \bar{\varphi}\rangle$ means $\langle L_a, \bar{\varphi}\rangle$.

For any sequence of p-atoms $\{a_i\}_{i=1}^{\infty}$ and any sequence of numbers $\{\lambda_i\}_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$, the series of functionals on $\mathit{Lip}(1/p-1)$,

$$\sum_{i=1}^{\infty} \lambda_i a_i$$

is absolutely convergent. We shall say that a linear functional f belongs to $\mathscr{H}^p(X, d, \mu)$ or simply to $\mathscr{H}^p$, if there exist a sequence $\{a_i\}_{i=1}^{\infty}$ of p-atoms and a sequence of numbers $\{\lambda_i\}_{i=1}^{\infty}$ with $\sum |\lambda_i|^p < \infty$, such that

$$f = \sum \lambda_i a_i.$$

The norm of f is defined as

$$\|f\|_{\mathscr{H}^p}^p = \inf\Big\{\sum |\lambda_i|^p, f = \sum \lambda_i a_i\Big\}.$$

This is not a norm in the ordinary sense, unless $p = 1$. However, $\mathscr{H}^p$ with $\|f\|_{\mathscr{H}^p}$ becomes a complete metrizable topological vector space. The dual space of $\mathscr{H}^p$ can be identified with $\mathit{Lip}(1/p-1)$ through the bilinear functional (see [4], [5] and [7]),

$$\langle f, \bar{\varphi}\rangle = \sum \lambda_i \int a_i(x)\varphi(x)\, d\mu(x)$$

where $f = \sum \lambda_i a_i$, $\sum |\lambda_i|^p < \infty$ and $\bar{\varphi} \in \mathit{Lip}(1/p-1)$. The norm of $\bar{\varphi}$ as an element of the dual space of $\mathscr{H}^p$ is equivalent to the norm of $\bar{\varphi}$ in $\mathit{Lip}(1/p-1)$.

Statement of the results

DEFINITION 1. Let $K(x, y)$ be a measurable function defined on $X \times X$. We shall say that $K(x, y)$ is a *singular integral kernel* if the following assumptions hold:

(i) For any ball B and $\varepsilon > 0$, if $A = \{(x, y): d(x, y) > \varepsilon\} \cap (B \times B)$, then

$$\iint_A |K(x, y)|^2 \, d\mu(x)\, d\mu(y) < \infty.$$

(ii) The operator K_η, $\eta > 0$, defined as

$$K_\eta(f)(x) = \int_{X \sim B(x,\eta)} K(x, y) f(y)\, d\mu(y),$$

which by (i) is well defined for any $f(y)$ with bounded support and in $L^2(X, \mu)$, satisfies

$$\|K_\eta f\|_2 \leqslant c\|f\|_2.$$

The constant c is finite and independent of $f(y)$ and η.

(iii) For any f in $L^2(X, \mu)$ with bounded support,

$$\lim_{\eta \to 0} K_\eta f = Kf$$

exists in $L^2(X, \mu)$.

(iv) If $d(x, z) > 2d(y, z)$, then $K(x, y)$ satisfies

$$|K(x, y) - K(x, z)| \leqslant c\, d(y, z)^{\varepsilon} / d(x, z)^{1+\varepsilon},$$

where $0 < \varepsilon \leqslant 1$ and c is a finite constant.

By (ii) and (iii), the operator K can be extended as a continuous linear operator on $L^2(X, \mu)$. Therefore, the adjoint K^* is well defined. Let $R > 0$ and let $\chi_R(x)$ be the characteristic function of the ball $B(z, R)$, where z is an arbitrary point of X, then

(v) The limit of $K^*(\chi_R)$ for R tending to infinity exists weakly in L^2 on bounded sets and it is equal to a finite constant.

As a reference to Definition 1, see [1].

Let $a(x)$ be a p-atom in (X, d, μ) and $K(x, y)$ be a singular integral kernel. Since $a(x)$ belongs to $L^2(X, \mu)$, the function $K(a)(x)$ is well defined and belongs to $L^2(X, \mu)$.

THEOREM 1. *Let $2/(2+\varepsilon) < p \leqslant 1$ and let $a(x)$ be a p-atom with support contained in $B = B(z, \sigma)$. Then, the function $M(x) = K(a)(x)$ satisfies:*

$$\int |M(x)|^2 d\mu(x) \leqslant c\sigma^{-(2/p)+1}, \tag{6}$$

$$\int |M(x)|^2 d(x, z)^{(2/p)-1+\varepsilon} d\mu(x) \leqslant c\sigma^{\varepsilon}, \tag{7}$$

$$\int M(x)\, d\mu(x) = 0. \tag{8}$$

Remark. A function $M(x)$ satisfying (6) and (7) is absolutely integrable on X.

DEFINITION 2. We shall say that a measurable function $M(x)$, defined on X, is a (p, ε)-*molecule* if there exist a point $z \in X$, $\sigma > 0$ and $0 < c < \infty$ such that (6), (7) and (8) hold.

The definition of molecule and Theorem 3 for the Hardy space $\mathscr{H}^1$ is due to R. R. Coifman.

THEOREM 2. *Let $M(x)$ be a (p, ε)-molecule. Then, for every φ belonging to* Lip$(1/p-1)$, *the function $M(x)\cdot\varphi(x)$ is absolutely integrable on X and the linear functional given by*

$$L_M(\bar{\varphi}) = \int M(x)\varphi(x)\, d\mu(x)$$

is well defined and bounded on Lip$(1/p-1)$.

THEOREM 3. (Decomposition of a molecule into atoms.) *Let $M(x)$ be a (p, ε)-molecule such that the constant c in (6) and (7) is smaller than a positive number A. Then, there exist: a constant B, a sequence $\{a_i\}_{i=1}^{\infty}$ of p-atoms and a numerical sequence $\{\lambda_i\}_{i=1}^{\infty}$ with $\sum |\lambda_i|^p \leqslant B$, such that the functional L_M, defined as in the statement of Theorem 2, satisfies*

$$L_M(\bar{\varphi}) = \sum_{i=1}^{\infty} \lambda_i \langle a_i, \bar{\varphi}\rangle,$$

for every $\bar{\varphi} \in$ Lip$(1/p-1)$.

Theorem 3 shows that the linear functional L_M, which by Theorem 2 belongs to the dual space of *Lip*$(1/p-1)$, also belongs to $\mathscr{H}^p$ and that $\|L_M\|_{\mathscr{H}^p} \leqslant B$.

DEFINITION 3. Let $K(x, y)$ be a singular integral kernel and $\varepsilon > (1/p)-1$. Let $B = B(z, \sigma)$ and $2kB = B(z, 2k\sigma)$. For $\varphi \in$ Lip$(1/p-1)$, we define the function $K_B^{\#}(\varphi)(y)$ on B as

$$K_B^{\#}(\varphi)(y) = \lim_{\eta\to 0} \int_{2kB\sim B(y,\eta)} K(x, y)\varphi(x)\, d\mu(x) + \int_{X\sim 2kB} \big(K(x, y) - K(x, z)\big)\varphi(x)\, d\mu(x),$$

where the limit is the weak-L^2 limit on B.

DEFINITION 4. Let $K(x, y)$ be a singular integral kernel and $\varepsilon > (1/p)-1$. For every $\bar{\varphi} \in$ *Lip*$(1/p-1)$, we define $K^{\#}(\bar{\varphi})$ as the class of all the functions $\psi(x)$ on X such that, for any ball B there exists a finite constant c_B satisfying

$$\psi(y) = K_B^{\#}(\varphi)(y) + c_B,$$

almost everywhere on B.

It will be shown later (see Lemma 3) that the class $K^{\#}(\bar{\varphi})$ is not empty and if $\psi(x) \in K^{\#}(\bar{\varphi})$, then $\psi_1(x) \in K^{\#}(\bar{\varphi})$ if and only if the difference $\psi(x) - \psi_1(x)$ is equal to a constant, almost everywhere on X.

THEOREM 4. *Let $K(x, y)$ be a singular integral kernel. For any $\bar{\varphi} \in$ Lip$(1/p-1)$, if $\psi \in K^{\#}(\bar{\varphi})$ then $\psi \in$* Lip$(1/p-1)$ *and there is a finite constant c such that,*

$$\|\psi\|_{\mathrm{Lip}(1/p-1)} \leqslant c\|\bar{\varphi}\|_{Lip(1/p-1)}.$$

Therefore, $K^{\#}$ defines a bounded linear operator from Lip$(1/p-1)$ into Lip$(1/p-1)$ and $\|K^{\#}\| \leqslant c$ (for previous results, see [8] and [5]).

THEOREM 5. *Let $K(x, y)$ be a singular integral kernel and $2/(2+\varepsilon) < p \leqslant 1$. Let f belong to $\mathscr{H}^p$, that is, $f = \sum \lambda_i a_i$, where the a_i's are p-atoms and $\sum |\lambda_i|^p < \infty$. Then, the operator*

$$Kf = \sum \lambda_i K(a_i)$$

is well defined. Moreover, K is linear and there is a constant c independent of f such that

$$\|Kf\|_{\mathscr{H}^p} \leqslant c\|f\|_{\mathscr{H}^p}.$$

The operator $K^{\#}$ is the dual operator of K.

Proofs of the results. First, we shall prove two lemmas that will be needed in the sequel.

LEMMA 1. *Let φ belong to* Lip(γ), $0 \leqslant \gamma \leqslant 1$. *Let* $r_j = a^j\sigma$, $a > 1$, $\sigma > 0$ *and j a non-negative integer. If we denote by m_j the mean value*

$$m_j = m_{B(z,r_j)}(\varphi),$$

then, the following estimates hold:

(i) *If* $0 < \gamma \leqslant 1$, *then*

$$|m_j| \leqslant c\|\varphi\|_{\mathrm{Lip}(\gamma)}(a^j\sigma)^\gamma + |m_0|.$$

(ii) *If* $\gamma = 0$, *then*

$$|m_j| \leqslant c\|\varphi\|_{\mathrm{Lip}(0)}j + |m_0|,$$

where the constant c is finite and does not depend on j, σ or φ.

Proof. Let B_1 and B_2 be two balls satisfying $B_2 \supset B_1$. Then,

$$m_{B_1}(\varphi) - m_{B_2}(\varphi) = \mu(B_1)^{-1}\int_{B_1}\big(\varphi(x) - m_{B_2}(\varphi)\big)\,d\mu(x).$$

Taking absolute values and enlarging the domain of integration, we have

$$(9)\quad |m_{B_1}(\varphi) - m_{B_2}(\varphi)| \leqslant \big(\mu(B_2)\mu(B_1)^{-1}\big)\mu(B_2)^{-1}\int_{B_2}|\varphi(x) - m_{B_2}(\varphi)|\,d\mu(x)$$
$$\leqslant \mu(B_2)\mu(B_1)^{-1}\|\varphi\|_{\mathrm{Lip}(\gamma)}\mu(B_2)^\gamma.$$

Now, writing m_j as

$$m_j = \sum_{i=1}^{j}(m_i - m_{i-1}) + m_0,$$

we get

$$(10)\qquad |m_j| \leqslant \sum_{i=1}^{j}|m_i - m_{i-1}| + |m_0|$$

and applying (9) to $B_1 = B(z, r_{i-1})$ and $B_2 = B(z, r_i)$, we obtain

$$|m_i - m_{i-1}| \leqslant ca\|\varphi\|_{\mathrm{Lip}(\gamma)}(a^i\sigma)^\gamma,$$

where c is a finite constant depending on the homogeneous space only. Therefore, using this estimate of $|m_i - m_{i-1}|$ in (10), it follows that

$$|m_j| \leqslant ca\|\varphi\|_{\mathrm{Lip}(\gamma)}\sum_{i=1}^{j}(a^i\sigma)^\gamma + |m_0|.$$

Computing the sum on the right-hand side, separately for $0 < \gamma \leqslant 1$ and $\gamma = 0$, we obtain the estimates (i) and (ii) claimed in the statement of the lemma. ■

LEMMA 2. *Let φ belong to* Lip(γ), $0 \leqslant \gamma < 1$, *and let $K(x, y)$ be a singular integral kernel with $\varepsilon > \gamma$. Then, if $\sigma > 0$ and $Y \in B(z, \sigma/2)$, the estimate*

$$(11)\quad \int_{X\sim B(z,\sigma)}|K(x, y) - K(x, z)|\,|\varphi(x)|\,d\mu(x)$$
$$\leqslant cd(y, z)^\varepsilon\big(\sigma^{\gamma-\varepsilon}\|\varphi\|_{\mathrm{Lip}(\gamma)} + \sigma^{-\varepsilon}|m_{B(z,\sigma)}\varphi|\big)$$

holds, with c a finite constant not depending on φ, σ, z and $y \in B(z, \sigma/2)$.

Proof. Let $y \in B(z, \sigma/2)$. Then, $d(x, z) > \sigma = 2(\sigma/2) > 2d(y, z)$. Therefore, from (iv) in Definition 1, we have

$$|K(x, y) - K(x, z)| \leqslant cd(y, z)^\varepsilon d(x, z)^{-1-\varepsilon}.$$

Thus, the integral in (11) is smaller than or equal to

$$(12)\quad cd(y, z)^\varepsilon\int_{X\sim B(z,\sigma)}d(x, z)^{-1-\varepsilon}|\varphi(x)|\,d\mu(x)$$
$$= cd(y, z)^\varepsilon\sum_{i=0}^{\infty}\int_{B(z,a^{i+1}\sigma)\sim B(z,a^i\sigma)}d(x, z)^{-1-\varepsilon}|\varphi(x)|\,d\mu(x)$$
$$\leqslant cd(y, z)^\varepsilon\sum_{i=0}^{\infty}(a^i\sigma)^{-1-\varepsilon}\int_{B(z,a^{i+1}\sigma)}|\varphi(x)|\,d\mu(x).$$

Now, if as in Lemma 1, we denote $m_{B(z,a^i\sigma)}(\varphi)$ by m_i, we have

$$(13)\quad \int_{B(z,a^{i+1}\sigma)}|\varphi(x)|\,d\mu(x) \leqslant \int_{B(z,a^{i+1}\sigma)}|\varphi(x) - m_{i+1}|\,d\mu(x) + c|m_{i+1}|a^{i+1}\sigma.$$

By Lemma 1 and the definition of the Lip(γ)-norm of φ, the right-hand side of (13) turns out to be smaller than or equal to

$$c\cdot\big((a^i\sigma)^{1+\gamma}\|\varphi\|_{\mathrm{Lip}(\gamma)} + a^i\sigma|m_0|\big),$$

for $0 < \gamma < 1$, and

$$cia^i\sigma(\|\varphi\|_{\mathrm{Lip}(0)} + |m_0|),$$

for $\gamma = 0$.

Therefore, by the estimate we have just obtained, (12) is smaller than or equal to

$$cd(y, z)^\varepsilon\Big(\|\varphi\|_{\mathrm{Lip}(\gamma)}\sum_{i=0}^{\infty}(a^i\sigma)^{\gamma-\varepsilon} + |m_0|\sum_{i=0}^{\infty}(a^i\sigma)^{-\varepsilon}\Big),$$

for $0 < \gamma < 1$, or

$$cd(y, z)^\varepsilon(\|\varphi\|_{\mathrm{Lip}(0)} + |m_0|)\sum_{i=0}^{\infty}i\cdot(a^i\sigma)^{-\varepsilon},$$

for $\gamma = 0$.

In either case, these expressions are equal to

$$cd(y,z)^{\varepsilon}(\|\varphi\|_{\mathrm{Lip}(\gamma)}\sigma^{\gamma-\varepsilon}+|m_0|\sigma^{-\varepsilon}),$$

with c a suitable chosen constant which does not depend on φ, σ, z or y. ■

Proof of Theorem 1. We begin by proving (6). Since $M(x) = K(a)(x)$, from (ii) and (iii) in Definition 1, we have that

$$\int |M(x)|^2 d\mu(x) \leqslant c\int |a(x)|^2 d\mu(x),$$

and by part (i) of (4), we get

$$\int |a(x)|^2 d\mu(x) \leqslant \mu\big(B(z,\sigma)\big)^{-(2/p)+1}.$$

Then, taking into account that by (1), $\mu\big(B(z,\sigma)\big) \simeq \sigma$, we obtain

$$\int |M(x)|^2 d\mu(x) \leqslant c\sigma^{-(2/p)+1}.$$

Let us show (7). We have

$$(14)\quad \int |M(x)|^2 d(x,z)^{(2/p)-1+\varepsilon} d\mu(x) = \int_{B(z,2k\sigma)} |M(x)|^2 d(z,x)^{(2/p)-1+\varepsilon} d\mu(x) + \int_{X\sim B(z,2k\sigma)} |M(x)|^2 d(z,x)^{(2/p)-1+\varepsilon} d\mu(x).$$

Since $(2/p)-1+\varepsilon>0$, for the first integral on the right-hand side, we have

$$\int_{B(z,2k\sigma)} |M(x)|^2 d(x,z)^{(2/p)-1+\varepsilon} d\mu(x) \leqslant c\sigma^{(2/p)-1+\varepsilon}\int |M(x)|^2 d\mu(x).$$

Then, by (6) already proved, we obtain

$$\int_{B(z,2k\sigma)} |M(x)|^2 d(x,z)^{(2/p)-1+\varepsilon} d\mu(x) \leqslant c\sigma^{\varepsilon}.$$

As for the second integral on the right-hand side of (14), if $B_i = B(z, 2k\cdot 2^i\sigma)$, then

$$(15)\quad \int_{X\sim B(z,2k\sigma)} |M(x)|^2 d(x,z)^{(2/p)-1+\varepsilon} d\mu(x) = \sum_{i=0}^{\infty}\int_{B_{i+1}\sim B_i} |M(x)|^2 d(x,z)^{(2/p)-1+\varepsilon} d\mu(x) \leqslant c\sum_{i=0}^{\infty}(2^i\sigma)^{(2/p)-1+\varepsilon}\int_{B_{i+1}\sim B_i} |M(x)|^2 d\mu(x).$$

Let us estimate the integral

$$\int_{B_{i+1}\sim B_i} |M(x)|^2 d\mu(x).$$

For $x\notin B_i$, $y\in B(z,\sigma)$, we have,

$$2k\cdot 2^i\sigma \leqslant d(x,z) \leqslant k\big(d(x,y)+d(y,z)\big) \leqslant kd(x,y)+k\sigma.$$

Therefore, $d(x,y) > (2^{i+1}-1)\sigma > 0$. Thus, for $x\notin B_i$,

$$M(x) = \lim_{\eta\to 0}\int_{X\sim B(x,\eta)} K(x,y)a(y)d\mu(y) = \int K(x,y)a(y)d\mu(y).$$

Since, by part (ii) of (4), the integral of $a(y)$ is zero, we get

$$M(x) = \int \big(K(x,y)-K(x,z)\big)a(y)d\mu(y).$$

Now, since $x\notin B_i$ and $y\in B(z,\sigma)$, the inequalities

$$d(x,z) > 2k\cdot 2^i\sigma \geqslant 2\sigma > 2d(y,z)$$

hold. Therefore, by (iv) of Definition 1 and part (i) of (4),

$$|M(x)| \leqslant c\int d(y,z)^{\varepsilon} d(x,z)^{-1-\varepsilon}|a(y)|d\mu(y) \leqslant c\sigma^{\varepsilon+1-(1/p)} d(x,z)^{-1-\varepsilon}.$$

From this estimate for $M(x)$, it follows easily that

$$\int_{B_{i+1}\sim B_i} |M(x)|^2 d\mu(x) \leqslant c\sigma^{2\varepsilon+2-(2/p)}(2^i\sigma)^{-1-2\varepsilon}.$$

Therefore, since $(2/p)-2-\varepsilon<0$, the sum of the last member of (15) is smaller than or equal to

$$c\sigma^{\varepsilon}\sum_{i=0}^{\infty}2^{i((2/p)-2-\varepsilon)} = c\sigma^{\varepsilon}.$$

This ends the proof of (7).

Let us show (8). By the remark following the statement of Theorem 1, $M(x)$ is absolutely integrable. Therefore,

$$\int M(x)d\mu(x) = \lim_{R\to\infty}\int_{B(z,R)} M(x)d\mu(x) = \lim_{R\to\infty}\int K(a)(x)\chi_R(x)d\mu(x) = \lim_{R\to\infty}\int a(x)K^*(\chi_R)(x)d\mu(x).$$

Then, since $a(x)$ is supported on a bounded set, by (v) of Definition 1 and part (ii) of (4), we have

$$\int M(x)d\mu(x) = c\int a(x)d\mu(x) = 0. \ ■$$

Proof of Theorem 2. Since a molecule is an integrable function, then the integrability of $M(x)\varphi(x)$ is equivalent to the integrability of $M(x)\big(\varphi(x)-m_{B(z,\sigma)}(\varphi)\big)$. Moreover, since the integral of $M(x)$ is equal to zero,

$$L_M(\varphi) = \int M(x)\varphi(x)\,d\mu(x) = \int M(x)\big(\varphi(x)-m_{B(z,\sigma)}(\varphi)\big)\,d\mu(x).$$

Therefore, we can work with $\varphi(x)-m_{B(z,\sigma)}(\varphi)$ instead of $\varphi(x)$ or, equivalently, with functions $\varphi(x)$ such that $m_{B(z,\sigma)}(\varphi)=0$.

Let us define $B_i = B(z, 2^i\sigma)$ for i a non-negative integer and $B_{-1}=\emptyset$. Then,

$$(16)\quad \int |M(x)||\varphi(x)|\,d\mu(x) = \sum_{i=0}^{\infty}\int_{B_i\sim B_{i-1}} |M(x)||\varphi(x)|\,d\mu(x)$$
$$\leqslant \sum_{i=0}^{\infty}\int_{B_i\sim B_{i-1}} |M(x)|\big(|\varphi(x)-m_i|+|m_i|\big)\,d\mu(x)$$
$$\leqslant \sum_{i=0}^{\infty}\Big(\int_{B_i\sim B_{i-1}} |M(x)|^2\,d\mu(x)\Big)^{1/2}\times$$
$$\times\Big[\Big(\int_{B_i}|\varphi(x)-m_i|^2\,d\mu(x)\Big)^{1/2}+|m_i|\,\mu(B_i)^{1/2}\Big].$$

By (1), we have

$$(17)\qquad \mu(B_i)\leqslant c_2 2^i\sigma.$$

By Lemma 1 and recalling that $m_0 = m_{B(z,\sigma)}(\varphi)=0$, we obtain

$$(18)\qquad |m_i|\leqslant\begin{cases} c(2^i\sigma)^{((1/p)-1)}\|\varphi\|_{\mathrm{Lip}(1/p-1)}, & \text{for}\quad p<1,\\ ci\,\|\varphi\|_{\mathrm{Lip}(0)}, & \text{for}\quad p=1.\end{cases}$$

The definition of $\mathrm{Lip}(1/p-1)$ and (17) imply

$$(19)\qquad \Big(\int_{B_i}|\varphi(x)-m_i|^2\,d\mu(x)\Big)^{1/2}\leqslant \|\varphi\|_{\mathrm{Lip}(1/p-1)}\,\mu(B_i)^{(1/p)-1/2}$$
$$\leqslant c\,\|\varphi\|_{\mathrm{Lip}(1/p-1)}(2^i\sigma)^{(1/p)-1/2}.$$

As for the integral of $|M(x)|^2$ on $B_i\sim B_{i-1}$, we have that if $i\geqslant 1$, then

$$\int_{B_i\sim B_{i-1}}|M(x)|^2\,d\mu(x)\leqslant (2^{i-1}\sigma)^{(-(2/p)+1-\varepsilon)}\int |M(x)|^2\,d(z,x)^{(2/p)-1+\varepsilon}\,d\mu(x).$$

By (7), the integral on the right-hand side of this inequality is smaller than or equal to $c\sigma^{\varepsilon}$. Therefore,

$$(20)\qquad \int_{B_i\sim B_{i-1}}|M(x)|^2\,d\mu(x)\leqslant c2^{(i-1)(-(2/p)+1-\varepsilon)}\,\sigma^{-(2/p)+1}.$$

For the case $i=0$, we get the same estimate, using (6) this time.

The estimates obtained in (17), (18), (19) and (20) imply that, if $p<1$, then the series in the last member of (16) is majorized by

$$c\Big(\sum_{i=0}^{\infty}2^{-i\varepsilon/2}\Big)\|\varphi\|_{\mathrm{Lip}(1/p-1)}$$

and if $p=1$, by

$$c\Big(\sum_{i=0}^{\infty}(i+1)2^{-i\varepsilon/2}\Big)\|\varphi\|_{\mathrm{Lip}(0)},$$

which show that

$$|L_M(\varphi)|\leqslant\int |M(x)|\,|\varphi(x)|\,d\mu(x)\leqslant c\,\|\varphi\|_{\mathrm{Lip}(1/p-1)}.\ ■$$

Proof of Theorem 3. As usual, if $E\subset X$, $\chi_E(x)$ stands for the characteristic function of E. Let $B_i = B(z, a^i\sigma)$, $a>(c_2/c_1)$, see (1), and $C_i = B_i\sim B_{i-1}$ for any positive integer i. C_0 is defined as B_0. Let $a_i(x)$ be the function

$$a_i(x) = \big(M(x)-m_i\big)\chi_{C_i}(x),$$

where

$$M_i = \mu(C_i)^{-1}\int_{C_i}M(x)\,d\mu(x).$$

Clearly, the integral of $a_i(x)$ is equal to zero and its support is contained in B_i. Then,

$$\sum_{i=0}^{m}a_i(x) = M(x)\chi_{B_m}(x)-\sum_{i=0}^{m}M_i\chi_{C_i}(x).$$

Now, let $\{\delta_i\}_{i=0}^{\infty}$ be the sequence defined as

$$\delta_i = \int_{X\sim B_i}M(x)\,d\mu(x),\qquad\text{if}\quad i>0,$$

and

$$\delta_0 = \int_X M(x)\,d\mu(x) = 0.$$

This sequence satisfies

$$\delta_i-\delta_{i+1} = \int_{C_i}M(x)\,d\mu(x) = \mu(C_i)M_i,$$

therefore,

$$\sum_{i=0}^{m}M_i\chi_{C_i}(x) = \sum_{i=0}^{m}(\delta_i-\delta_{i+1})\,\mu(C_i)^{-1}\chi_{C_i}(x)$$
$$= \sum_{i=1}^{m}\delta_i\cdot\big(\mu(C_i)^{-1}\chi_{C_i}(x)-\mu(C_{i-1})^{-1}\chi_{C_{i-1}}(x)\big)-$$
$$-\delta_{m+1}\,\mu(C_m)^{-1}\chi_{C_m}(x).$$

Define $\beta_i(x)$, $i>0$, as

$$\beta_i(x) = \delta_i\cdot\big(\mu(C_i)^{-1}\chi_{C_i}(x)-\mu(C_{i-1})^{-1}\chi_{C_{i-1}}(x)\big).$$

Clearly, the integral of $\beta_i(x)$ is equal to zero and its support is contained in B_i.

We can write:

$$(21)\qquad \sum_{i=0}^{m}a_i(x)+\sum_{i=1}^{m}\beta_i(x) = M(x)\chi_{B_m}(x)+\delta_{m+1}\mu(C_m)^{-1}\chi_{C_m}(x).$$

Let us estimate the normalized L^2-norm of $a_i(x)$. We have:

$$\Big(\mu(B_i)^{-1}\int|a_i(x)|^2d\mu(x)\Big)^{1/2}\leqslant\Big(\mu(B_i)^{-1}\int_{C_i}|M(x)|^2d\mu(x)\Big)^{1/2}+|M_i|.$$

Since,

$$|M_i|\leqslant\mu(C_i)^{-1}\int_{C_i}|M(x)|\,d\mu(x)\leqslant\mu(B_i)^{1/2}\mu(C_i)^{-1/2}\Big(\mu(B_i)^{-1}\int_{C_i}|M(x)|^2d\mu(x)\Big)^{1/2},$$

then,

$$\Big(\mu(B_i)^{-1}\int|a_i(x)|^2d\mu(x)\Big)^{1/2} \leqslant\big(1+\mu(B_i)^{1/2}\mu(C_i)^{-1/2}\big)\Big(\mu(B_i)^{-1}\int_{C_i}|M(x)|^2d\mu(x)\Big)^{1/2}.$$

Recalling that $C_i = B_i\sim B_{i-1}$, we get, $\mu(C_i) = \mu(B_i)-\mu(B_{i-1})$. Then, by (1)

$$\mu(C_i)\geqslant c_1a^i\sigma-c_2a^{i-1}\sigma\geqslant(c_1a-c_2)a^{i-1}\sigma\geqslant(c_1a-c_2)a^{-1}c_2^{-1}\mu(B_i) = c\mu(B_i),$$

with $c>0$. This shows that

$$|M_i|\leqslant c\cdot\Big(\mu(B_i)^{-1}\int_{C_i}|M(x)|^2d\mu(x)\Big)^{1/2},$$

therefore,

$$(22)\qquad \Big(\mu(B_i)^{-1}\int|a_i(x)|^2d\mu(x)\Big)^{1/2}\leqslant c\cdot\Big(\mu(B_i)^{-1}\int_{C_i}|M(x)|^2d\mu(x)\Big)^{1/2}.$$

Arguing in the same way as it was done in the proof of Theorem 2 in order to prove (20), we can show that (22) is smaller than or equal to

$$c\sigma^{-1/p}a^{-i((1/p)+\varepsilon/2)},$$

which shows that $a_i(x)$ is equal to a constant s_i times a p-atom. The modulus of s_i is smaller than a given constant depending on A times $a^{-i((1/p)+\varepsilon/2)}$.

Next, we shall estimate the normalized L^2-norm of $\beta_i(x)$. We have,

$$\Big(\mu(B_i)^{-1}\int|\beta_i(x)|^2d\mu(x)\Big)^{1/2}\leqslant c|\delta_i|(a^i\sigma)^{-1}.$$

Now, $|\delta_i|$ can be estimated as

$$|\delta_i|\leqslant\int_{X\sim B_i}|M(x)|\,d\mu(x) \leqslant\Big(\int|M(x)|^2d(z,x)^{(2/p)-1+\varepsilon}d\mu(x)\Big)^{1/2}\Big(\int_{X\sim B_i}d(z,x)^{-(2/p)+1-\varepsilon}d\mu(x)\Big)^{1/2}.$$

Therefore, applying (2) and (7), we get

$$(23)\qquad |\delta_i|\leqslant c\sigma^{\varepsilon/2}(a^i\sigma)^{-(1/p)+1-\varepsilon/2}.$$

Thus,

$$\Big(\mu(B_i)^{-1}\int|\beta_i(x)|^2d\mu(x)\Big)^{1/2}\leqslant c\sigma^{-1/p}a^{-i((1/p)+\varepsilon/2)}.$$

Then, as in the case of the a_i's, we get that $\beta_i(x)$ is equal to a constant t_i times a p-atom. The modulus of t_i is smaller than or equal to a given constant depending on A times $a^{-i((1/p)+\varepsilon/2)}$. The representations obtained for the a_i's and the β_i's allows us to write the left-hand side of (21) as

$$\sum_{i=0}^{2m+1}\lambda_ia_i(x),$$

where the a_i's are p-atoms and $\sum_{i=0}^{2m+}|\lambda_i|^p\leqslant B$, for B a suitable chosen constant depending on A but not on m. Therefore, since the linear functional L_a on $Lip(1/p-1)$ associated to a p-atom has its norm bounded by 1, see (5), and since

$$\sum_{i=0}^{\infty}|\lambda_i|\leqslant\Big(\sum_{i=0}^{\infty}|\lambda_i|^p\Big)^{1/p}\leqslant B^{1/p},$$

we get that

$$\sum_{i=0}^{\infty}\lambda_i\int a_i(x)\varphi(x)\,d\mu(x)$$

is finite for every $\varphi\in\mathrm{Lip}(1/p-1)$.

Multiplying by $\varphi(x)$ on the right-hand side of (21) and integrating, we obtain

$$(24)\qquad \int_{B_m}M(x)\varphi(x)\,d\mu(x)+\delta_{m+1}\mu(C_m)^{-1}\int_{C_m}\varphi(x)\,d\mu(x).$$

Since, as we have shown in Theorem 2, $M(x)\varphi(x)$ is integrable, then the limit of the first term on (24), for m tending to infinity, is equal to

$$\int M(x)\varphi(x)\,d\mu(x).$$

Now, we shall prove that the second term in (24) goes to zero for m tending to infinity. We have

$$\Big|\mu(C_m)^{-1}\int_{C_m}\varphi(x)\,d\mu(x)\Big| \leqslant c\cdot(a^m\sigma)^{-1}\int_{B_m}|\varphi(x)-m_{B_m}(\varphi)|\,d\mu(x)+|m_{B_m}(\varphi)|.$$

Applying Schwarz inequality, the definition of $\mathrm{Lip}(1/p-1)$ and Lemma 1 (we can assume without loss of generality that $m_{B_0}(\varphi)=0$), we obtain that if $p<1$,

$$\Big|\mu(C_m)^{-1}\int_{C_m}\varphi(x)\,d\mu(x)\Big| \leqslant c\cdot(a^m\sigma)^{(1/p)-1}\|\varphi\|_{\mathrm{Lip}(1/p-1)}$$

and

$$\Big|\mu(C_m)^{-1}\int_{C_m}\varphi(x)\,d\mu(x)\Big| \leqslant cm\|\varphi\|_{\mathrm{Lip}(0)},$$

for $p=1$. Therefore, since by (23),

$$|\delta_{m+1}| \leqslant c\,\sigma^{\varepsilon/2}\cdot(a^{m+1}\sigma)^{-(1/p)+1-\varepsilon/2},$$

we get

$$\Big|\delta_{m+1}\mu(C_m)^{-1}\int_{C_m}\varphi(x)\,d\mu(x)\Big|$$

$$\leqslant\begin{cases} c\sigma^{\varepsilon/2}(a^{m+1}\sigma)^{-(1/p)+1-\varepsilon/2}(a^m\sigma)^{(1/p)-1}\|\varphi\|_{\mathrm{Lip}(1/p-1)}, & \text{for } p<1,\\ ca^{-(m+1)\varepsilon/2}m\|\varphi\|_{\mathrm{Lip}(0)}, & \text{for } p=1,\end{cases}$$

which clearly go to zero for m tending to infinity. ■

Before continuing with the proofs of Theorems 4 and 5, we have to show that Definitions 3 and 4 make sense. We will consider first Definition 3. $K_B^{\#}(\varphi)(y)$ is the sum of two terms. In order to see that the first term exists, take $h(y)\in L^2(B,\mu)$, then, we have

$$\int_{2kB}K(h)(x)\varphi(x)\,d\mu(x) = \lim_{\eta\to 0}\int_{2kB}K_\eta(h)(x)\varphi(x)\,d\mu(x)$$

$$=\lim_{\eta\to 0}\int_{2kB}\Big(\int_{X\sim B(x,\eta)}K(x,y)h(y)\,d\mu(y)\Big)\varphi(x)\,d\mu(x).$$

By Fubini's Theorem, this is equal to

$$\lim_{\eta\to 0}\int\Big(\int_{2kB\sim B(y,\eta)}K(x,y)\varphi(x)\,d\mu(x)\Big)h(y)\,d\mu(y),$$

therefore, the weak-L^2 limit on B

$$\lim_{\eta\to 0}\int_{2kB\sim B(y,\eta)}K(x,y)\varphi(x)\,d\mu(x)$$

exists. As for the second term in the definition of $K_B^{\#}(\varphi)(y)$, Lemma 2 shows that it is bounded for $y\in B$.

The following lemma provides the properties of $K_B^{\#}(\varphi)(y)$ that will be needed in order to prove that Definition 4 is meaningful.

LEMMA 3. *Let B_1 and B_2 be two balls such that $B_1\subset B_2$ and let φ_1 and φ_2 belong to $\mathrm{Lip}(1/p-1)$, $\varepsilon>(1/p-1)$, satisfying $\varphi_1(y)-\varphi_2(y)=\text{constant}$, almost everywhere. Then,*

(1) *$K_{B_2}^{\#}(\varphi)(y)-K_{B_1}^{\#}(\varphi)(y)$ is equal to a constant for almost every $y\in B_1$.*

(2) *For every ball B, $K_B^{\#}(\varphi_1)(y)-K_B^{\#}(\varphi_2)(y)$ is equal to a constant for almost every $y\in B$.*

Proof. The first claim follows from identity

$$K_{B_2}^{\#}(\varphi)(y)-K_{B_1}^{\#}(\varphi)(y) = \int_{X\sim 2kB_2}\big(K(x,z_1)-K(x,z_2)\big)\varphi(x)\,d\mu(x)+$$

$$+\int_{2kB_2\sim 2kB_1}K(x,z_1)\varphi(x)\,d\mu(x),$$

where $y\in B_1$ and z_1 and z_2 denote the centers of B_1 and B_2, respectively.

As for the second claim, we can assume that $\varphi_1-\varphi_2=1$. Then if z is the center of B, we have

$$K_B^{\#}(\varphi_1)(y)-K_B^{\#}(\varphi_2)(y) = K_B^{\#}(1)(y)$$

$$=\lim_{\eta\to 0}\int_{2kB\sim B(y,\eta)}K(x,y)\,d\mu(x)+\int_{X\sim 2kB}\big(K(x,y)-K(x,z)\big)\,d\mu(x).$$

Let $h(y)$ be any function in $L^2(B,\mu)$ with integral equal to zero. We shall show that

$$\int K_B^{\#}(1)(y)h(y)\,d\mu(y)=0,$$

which will prove claim (2). It is easy to check that, if $R>2k$, then

$$\int_{2kB\sim B(y,\eta)}K(x,y)\,d\mu(x)+\int_{X\sim 2kB}\big(K(x,y)-K(x,z)\big)\,d\mu(x)$$

$$=\int_{RB\sim B(y,\eta)}K(x,y)\,d\mu(x)+\int_{X\sim RB}\big(K(x,y)-K(x,z)\big)\,d\mu(x)-$$

$$-\int_{RB\sim 2kB}K(x,z)\,d\mu(x).$$

By Lemma 2, the second integral on the right-hand side goes to zero uniformly on $y\in B$ as R goes to infinity. Therefore,

$$\int K_B^{\#}(1)(y)h(y)\,d\mu(y)$$

$$=\lim_{R\to 0}\Big[\lim_{\eta\to 0}\int\Big(\int_{RB\sim B(y,\eta)}K(x,y)\,d\mu(x)\Big)h(y)\,d\mu(y)-$$

$$-\int\Big(\int_{RB\sim 2kB}K(x,z)\,d\mu(x)\Big)h(y)\,d\mu(y)\Big].$$

The last integral is equal to zero, since the expression in parentheses does not depend on y. On the other hand,

$$\lim_{\eta\to 0}\int_{RB\sim B(y,\eta)}K(x,y)\,d\mu(x) = K^*(\chi_{RB}),$$

then, by (v) in Definition 1, we get

$$\int K_B^{\#}(1)(y)h(y)\,d\mu(y) = \lim_{R\to\infty}\int K^*(\chi_{RB})(y)h(y)\,d\mu(y)$$
$$= \mathrm{const}\int h(y)\,d\mu(y) = 0. \ \blacksquare$$

Part (1) of Lemma 3 allows us to construct a function $\psi(x)$ on X satisfying the requirements of Definition 4 for a given $\varphi(x)$. Part (2) of Lemma 3 shows that the class $K^{\#}(\bar\varphi)$ does not depend on the representative of $\bar\varphi$ chosen.

Proof of Theorem 4. Let B be a ball in X. Let $\bar\varphi$ belong to $\mathit{Lip}(1/p-1)$. We can assume that $m_{2kB}(\varphi)=0$. Let $\psi\in K^{\#}(\bar\varphi)$. By definition of $K^{\#}(\bar\varphi)$, there exists a constant c_B such that

$$\psi(y) = K_B^{\#}(\varphi)(y)+c_B,$$

for almost every $y\in B$. Then,

$$\Big(\mu(B)^{-1}\int_B|\psi(y)-m_B(\psi)|^2d\mu(y)\Big)^{1/2}$$
$$= \Big(\mu(B)^{-1}\int_B\big|K_B^{\#}(\varphi)(y)-m_B\big(K_B^{\#}(\varphi)\big)\big|^2d\mu(y)\Big)^{1/2}$$
$$\leqslant \Big(\mu(B)^{-1}\int_B|K_B^{\#}(\varphi)(y)|^2d\mu(y)\Big)^{1/2}+\big|m_B\big(K_B^{\#}(\varphi)\big)\big|.$$

Since

$$\big|m_B\big(K_B^{\#}(\varphi)\big)\big| \leqslant \mu(B)^{-1}\int_B|K_B^{\#}(\varphi)(y)|\,d\mu(y),$$

then, by Schwarz inequality,

$$\big|m_B\big(K_B^{\#}(\varphi)\big)\big| \leqslant \Big(\mu(B)^{-1}\int_B|K_B^{\#}(\varphi)(y)|^2d\mu(y)\Big)^{1/2},$$

therefore,

$$(25)\qquad \Big(\mu(B)^{-1}\int_B|\psi(y)-m_B(\psi)|^2d\mu(y)\Big)^{1/2}\leqslant c\cdot\Big(\mu(B)^{-1}\int_B|K_B^{\#}(\varphi)(y)|^2d\mu(y)\Big)^{1/2}.$$

In order to estimate the L^2-norm on B of $K_B^{\#}(\varphi)$, let us take any function $h(y)\in L^2(B,\mu)$ and consider

$$\int_B h(y)K_B^{\#}(\varphi)(y)\,d\mu(y) = \lim_{\eta\to0}\int_B h(y)\int_{2kB\sim B(y,\eta)}K(x,y)\varphi(x)\,d\mu(x)\,d\mu(y)+$$
$$+\int_B h(y)\Big(\int_{X\sim2kB}\big(K(x,y)-K(x,z)\big)\varphi(x)\,d\mu(x)\Big)d\mu(y)$$
$$= I_1+I_2.$$

For I_1, we have

$$I_1 = \lim_{\eta\to0}\int_B h(y)\Big(\int_{2kB\sim B(y,\eta)}K(x,y)\varphi(x)\,d\mu(x)\Big)d\mu(y)$$
$$= \lim_{\eta\to0}\int_{2kB}\Big(\int_{X\sim B(x,\eta)}K(x,y)h(y)\,d\mu(y)\Big)\varphi(x)\,d\mu(x)$$
$$= \int_{2kB}K(h)(x)\varphi(x)\,d\mu(x),$$

thus,

$$|I_1|\leqslant\|K(h)\|_2\Big(\int_{2kB}|\varphi(x)|^2d\mu(x)\Big)^{1/2}\leqslant\|K\|\Big(\int_{2kB}|\varphi(x)|^2d\mu(x)\Big)^{1/2}\|h\|_2$$

and since $m_{2kB}(\varphi)=0$, we also get that

$$|I_1|\leqslant\|K\|\Big(\int_{2kB}|\varphi(x)-m_{2kB}(\varphi)|\,d\mu(x)\Big)^{1/2}\|h\|_2$$
$$\leqslant c\|K\|\,\|\varphi\|_{\mathrm{Lip}(1/p-1)}\,\mu(B)^{(1/p)-1/2}\|h\|_2.$$

As for I_2, from Lemma 2, we have the estimate

$$|I_2|\leqslant\int_B|h(y)|\Big(\int_{X\sim2kB}|K(x,y)-K(x,z)|\,|\varphi(x)|\,d\mu(x)\Big)d\mu(y)$$
$$\leqslant c\int_B|h(y)|\Big(d(y,z)^{\varepsilon}\mu(B)^{(1/p)-1-\varepsilon}\|\varphi\|_{\mathrm{Lip}(1/p-1)}\Big)d\mu(y)$$
$$\leqslant c\,\mu(B)^{(1/p)-1}\|\varphi\|_{\mathrm{Lip}(1/p-1)}\int_B|h(y)|\,d\mu(y)$$
$$\leqslant c\,\mu(B)^{(1/p)-1/2}\|\varphi\|_{\mathrm{Lip}(1/p-1)}\|h\|_2,$$

therefore, by the estimates obtained for I_1 and I_2, we get

$$\Big|\int_B h(y)K_B^{\#}(\varphi)(y)\,d\mu(y)\Big|\leqslant c\|\varphi\|_{\mathrm{Lip}(1/p-1)}\mu(B)^{(1/p)-1/2}\|h\|_2,$$

which implies that

$$\Big(\int_B|K_B^{\#}(\varphi)(y)|^2d\mu(y)\Big)^{1/2}\leqslant c\|\varphi\|_{\mathrm{Lip}(1/p-1)}\mu(B)^{(1/p)-1/2}.$$

Using this in (25), we get

$$\Big(\mu(B)^{-1}\int_B|\psi(y)-m_B(\psi)|^2d\mu(y)\Big)^{1/2}\leqslant c\|\varphi\|_{\mathrm{Lip}(1/p-1)}\mu(B)^{(1/p-1)},$$

which shows that $\psi\in\mathrm{Lip}(1/p-1)$ and that

$$\|\psi\|_{\mathrm{Lip}(1/p-1)}\leqslant c\|\varphi\|_{\mathrm{Lip}(1/p-1)},$$

therefore,

$$\|K^{\#}(\bar{\varphi})\|_{Lip(1/p-1)} \leqslant c\,\|\bar{\varphi}\|_{Lip(1/p-1)}. \ \blacksquare$$

The proof of Theorem 5 depends heavily on the following lemma.

LEMMA 4. *Let $a(x)$ be a p-atom and $K(x, y)$ a singular integral kernel with $\varepsilon > (1/p)-1$. For every $\bar{\varphi} \in Lip(1/p-1)$,*

$$\langle K(a), \bar{\varphi}\rangle = \langle a, K^{\#}(\bar{\varphi})\rangle$$

holds.

Proof. We proved in Theorem 1 that $K(a)(x)$ is a (p, ε)-molecule. Moreover, by Theorem 2, we have that $K(a)(x)\cdot\varphi(x)$ is an integrable function and that the integral

$$\int K(a)(x)\varphi(x)\,d\mu(x) \tag{26}$$

defines a bounded linear functional on $Lip(1/p-1)$. Then

$$\int K(a)(x)\varphi(x)\,d\mu(x) = \lim_{R\to\infty}\int_{B(z,R)} K(a)(x)\varphi(x)\,d\mu(x).$$

By definition of $K(a)$, we get that

$$\begin{aligned}\int_{B(z,R)} K(a)(x)\varphi(x)\,d\mu(x) &= \lim_{\eta\to 0}\int_{B(z,R)} K_\eta(a)(x)\varphi(x)\,d\mu(x)\\ &= \lim_{\eta\to 0}\int_{B(z,R)}\Big(\int_{X\sim B(x,\eta)} K(x, y)\,a(y)\,d\mu(y)\Big)\varphi(x)\,d\mu(x).\end{aligned}$$

Since the support of $a(y)$ is bounded and $\eta > 0$, by part (i) of Definition 1, we can change the order of integration, obtaining

$$\lim_{\eta\to 0}\int\Big(\int_{B(z,R)\sim B(y,\eta)} K(x, y)\varphi(x)\,d\mu(x)\Big)\,a(y)\,d\mu(y).$$

Let $a(x)$ be supported on $B(z, \sigma)$, then, if $0 < \eta < \sigma$, we have $B(y, \eta) \subset B(z, 2k\sigma)$ for every y in the support of $a(y)$. Thus, for $R > 2k\sigma$,

$$\begin{aligned}\int\Big(\int_{B(z,R)\sim B(y,\eta)} & K(x,y)\,\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y)\\ &= \int\Big(\int_{B(z,2k\sigma)\sim B(y,\eta)} K(x, y)\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y)+\\ &\quad+\int\Big(\int_{B(z,R)\sim B(z,2k\sigma)} \big(K(x, y)-K(x, z)\big)\,\varphi(x)\,d\mu(x)\Big)\,a(y)\,d\mu(y)+\\ &\quad+\int\Big(\int_{B(z,R)\sim B(z,2k\sigma)} K(x, z)\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y).\end{aligned}$$

We observe that the last integral is equal to zero since the innermost integral does not depend on y and $\int a(y)\,d\mu(y) = 0$. Thus, for the integral in (26), we have

$$\begin{aligned}\int K(a)(x)\varphi(x)\,d\mu(x) &= \lim_{\eta\to 0}\int\Big(\int_{B(z,2k\sigma)\sim B(y,\eta)} K(x, y)\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y)+\\ &\quad+\lim_{R\to\infty}\int\Big(\int_{B(z,R)\sim B(z,2k\sigma)} \big(K(x, y)-K(x, z)\big)\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y).\end{aligned} \tag{27}$$

By Lemma 2, integral

$$\int_{B(z,R)\sim B(z,2k\sigma)} \big(K(x, y)-K(x, z)\big)\varphi(x)\,d\mu(x)$$

is uniformly bounded for $y \in B(z, \sigma)$. Then, by the Lebesgue bounded convergence theorem, we can take the limit in R under the integration sign in (27). As for the limit in η in the same expression (27), we observe that

$$\int\Big(\int_{B(z,2k\sigma)\sim B(y,\eta)} K(x, y)\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y) = \big(\varphi\chi_{B(z,2k\sigma)}, K_\eta(a)\big),$$

and since, by hypothesis, $K_\eta(a)(x)$ converges in $L^2(X, \mu)$ to $K(a)(x)$ for any $a(x)$ in $L^2(X, \mu)$ with support in $B(z, \sigma)$, it follows that

$$\lim_{\eta\to 0}\int_{B(z,2k\sigma)\sim B(y,\eta)} K(x, y)\varphi(x)\,d\mu(x)$$

exists weakly in L^2 on bounded sets. Therefore,

$$\begin{aligned}\langle K(a), \bar{\varphi}\rangle &= \int K(a)(x)\varphi(x)\,d\mu(x)\\ &= \int\Big(\lim_{\eta\to 0}\int_{B(z,2k\sigma)\sim B(y,\eta)} K(x, y)\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y)+\\ &\quad+\int\Big(\int_{X\sim B(z,2k\sigma)} \big(K(x, y)-K(x, z)\big)\varphi(x)\,d\mu(x)\Big)a(y)\,d\mu(y)\\ &= \langle a, K^{\#}(\bar{\varphi})\rangle. \ \blacksquare\end{aligned}$$

Proof of Theorem 5. Let $f \in \mathscr{H}^p$ be represented by

$$f = \sum_{i=1}^{\infty}\lambda_i a_i$$

where $\sum_{i=1}^{\infty}|\lambda_i|^p < \infty$. Consider the series

$$\sum_{i=1}^{\infty}\lambda_i K(a_i). \tag{28}$$

Theorems 1, 2 and 3 show that for any i, $K(a_i)$ belongs to $\mathscr{H}^p$ and $\|K(a_i)\|_{\mathscr{H}^p} < c$. Then, since $p \leqslant 1$, we have

$$\sum_{i=1}^{\infty} |\lambda_i| \, \|K(a_i)\|_{\mathscr{H}^p} \leqslant c \sum_{i=1}^{\infty} |\lambda_i| \leqslant c \Big(\sum_{i=1}^{\infty} |\lambda_i|^p \Big)^{1/p} < \infty.$$

This proves that the series in (28) converges in $\mathscr{H}^p$ to an element $g \in \mathscr{H}^p$. Moreover,

$$\|g\|_{\mathscr{H}^p} \leqslant c \Big(\sum |\lambda_i|^p \Big)^{1/p}. \tag{29}$$

Consider now $\bar{\varphi} \in Lip(1/p-1)$. By Lemma 4, we have

$$\langle g, \bar{\varphi} \rangle = \Big\langle \sum_{i=1}^{\infty} \lambda_i K(a_i), \bar{\varphi} \Big\rangle = \sum_{i=1}^{\infty} \lambda_i \langle K(a_i), \bar{\varphi} \rangle$$

$$= \sum_{i=1}^{\infty} \lambda_i \langle a_i, K^{\#}(\bar{\varphi}) \rangle = \Big\langle \sum_{i=1}^{\infty} \lambda_i a_i, K^{\#}(\bar{\varphi}) \Big\rangle = \langle f, K^{\#}(\bar{\varphi}) \rangle.$$

Thus, $\langle g, \bar{\varphi} \rangle = \langle f, K^{\#}(\bar{\varphi}) \rangle$, which shows that g does not depend on the representation of f as a series of multiples of p-atoms but on f itself. Therefore, we can define

$$Kf = g,$$

moreover, from (29), we see that

$$\|Kf\|_{\mathscr{H}^p} = \|g\|_{\mathscr{H}^p} \leqslant c \|f\|_{\mathscr{H}^p}. \ \blacksquare$$

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