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Method of orthogonal projections and approximation of the spectrum of a bounded operator

by

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Abstract. For a given bounded operator A and a sequence $\{P_n\}$ of orthogonal projections converging strongly to the identity operator on a complex Hilbert space H we can define operators

$$A_n = P_n A|_{P_n H} : P_n H \to P_n H.$$

These operators are compressions of A and approximate it in some way. In this work the asymptotical behaviour of spectra of operators A_n is studied.

Notation. In the following H will denote a complex Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$, $L(H)$ denotes the space of linear bounded operators on H, $F(H)$, $LC(H)$ denote the sets of finite-dimensional and compact linear operators on H. By a *projection* (not necessary orthogonal) is meant an operator $P \in L(H)$ with $P^2 = P$. Two projections are said to be *ordered in their natural order* $P < Q$ if $PQ = QP = P$. $P_f(H) = \{P \in F(H);\ P = P^2 = P^*\}$.

The spectrum, resolvent set of an operator A are denoted by $\Sigma(A)$, $\varrho(A)$ respectively. $\partial\Omega$ means a boundary of a set Ω, $\Omega_+(\varepsilon)$ means the ε-neighbourhood of the set Ω.

If $\lambda \in \boldsymbol{C}$, $\Omega \subset \boldsymbol{C}$ ($\boldsymbol{C}$ — the complex plane) then

$$d(\lambda, \Omega) = \inf_{\omega\in\Omega} |\omega - \lambda|,$$

$\operatorname{dist}(\Omega', \Omega'')$ denotes the Hausdorff distance between the sets Ω', Ω''.

We define (following [1], VII) a spectral set for the operator A to be any set $\Omega \subset \boldsymbol{C}$ for which $\Omega \cap \Sigma(A)$ is open and closed in $\Sigma(A)$. For each spectral set Ω the projection $E(\Omega, A)$ is defined by the formula

$$E(\Omega, A) = \frac{-1}{2\pi i} \int_{\Gamma} R(\lambda, A)\, d\lambda,$$

where $R(\lambda, A) = (A - \lambda)^{-1}$, and $\Gamma \subset \varrho(A)$ is any rectifable Jordan curve containing $\Omega \cap \Sigma(A)$ but no other points of $\Sigma(A)$ in its interior. It is known that

$$E(\Omega, A)A = AE(\Omega, A),$$
$$E(\Omega', A)E(\Omega'', A) = E(\Omega'\cap\Omega'', A),$$
$$\Sigma(A|_{E(\Omega,A)H}) = \Omega\cap\Sigma(A),$$
$$E(\Sigma(A), A) = 1, \text{ and if } \Omega\subset\varrho(A) \text{ then } E(\Omega, A) = 0.$$

Basic lemmas. Two following lemmas are well known.

LEMMA 1 ([3], Lemma 3.7, p. 151). *If $\{P_n\}$ is a sequence of projections in H, $P_n\to 1$ strongly, $K\in LC(H)$ then $\|P_nK-K\|\to 0$.*

LEMMA 2 ([3], 4.24, p. 131, [1], VII.6, p. 585). *If $A, B\in L(H)$, $\lambda\in\varrho(A)$, $\|A-B\|\,\|R(\lambda, A)\| < \frac{1}{2}$ then $\lambda\in\varrho(B)$ and the following inequalities hold*

$$\|R(\lambda, B)\|\leqslant 2\|R(\lambda, A)\|,$$
$$\|R(\lambda, B)-R(\lambda, A)\|\leqslant 2\|A-B\|\,\|R(\lambda, A)\|^2.$$

It is also known that $\|R(\lambda, A)\|\geqslant \big(d(\lambda, \Sigma(A))\big)^{-1}$ for $\lambda\in\varrho(A)$.

LEMMA 3. *Let $Q, P_n, n = 1, 2, \ldots,$ be projections in H such that $Q\in F(H)$, $P_n\to 1$ strongly, $P_n^*\to 1$ strongly, then there exists a sequence $\{Q_n\}$ of projections in H such that*

$$Q_n < P_n, \qquad \|Q-Q_n\|\to 0.$$

Proof. Let $B_n = QP_n+(1-Q)$, note that

$$\|B_n-1\| = \|QP_n-Q\| = \|P_n^*Q^*-Q^*\|.$$

It follows from Lemmas 1 and 2 that for n large enough there exist operators B_n^{-1} and

$$\|1-B_n^{-1}\|\leqslant 2\|P_n^*Q^*-Q^*\|. \tag{1}$$

Note that

$$1 = B_n^{-1}B_n = B_n^{-1}QP_n+B_n^{-1}(1-Q) = QP_nB_n^{-1}+(1-Q)B_n^{-1}.$$

Multiplying this identity by $(1-Q)$ we obtain that

$$1-Q = (1-Q)B_n^{-1}.$$

We shall show that the operators $Q_n = P_nB_n^{-1}QP_n$ satisfy the thesis of our lemma:

$$\begin{aligned} Q_n-Q_n^2 &= P_n(1-B_n^{-1}QP_n)B_n^{-1}QP_n = P_nB_n^{-1}(1-Q)B_n^{-1}QP_n \\ &= P_nB_n^{-1}(1-Q)QP_n = 0. \end{aligned}$$

Thus Q_n is a projection and obviously $Q_n < P_n$.

The convergence $\|Q_n-Q\|\to 0$ follows from Lemma 1, (1) and the computation below

$$\begin{aligned} \|Q-Q_n\| &= \|Q-P_nB_n^{-1}QP_n\| \\ &\leqslant \|Q-QP_n\|+\|QP_n-P_nQP_n\|+\|P_nQP_n-P_nB_n^{-1}QP_n\| \\ &\leqslant \|Q-QP_n\|+\|Q-P_nQ\|\,\|P_n\|+\|P_n\|\,\|1-B_n^{-1}\|\,\|Q\|\,\|P_n\| \\ &\leqslant \|Q-P_nQ\|\,\|P_n\|+\|P_n^*Q^*-Q^*\|(1+2\|P_n\|^2\|Q\|)\to 0. \ \blacksquare \end{aligned}$$

The next lemma shows that the assumptions of Lemma 3 are necessary.

LEMMA 4. *Let $\{P_n\}$ be a sequence of projections in H such that $\|P_n\|\leqslant K$, $n = 1, 2, 3, \ldots$ If for any projection $Q\in F(H)$ there exists a sequence $\{Q_n\}$ of projections in H such that $Q_n < P_n$, $\|Q_n-Q\|\to 0$ then*

$$P_n\to 1 \text{ strongly} \quad\text{and}\quad P_n^*\to 1 \text{ strongly}.$$

Proof. For a given $x^*\in H$ ($\|x^*\| = 1$) take $y\in H$ such that $\langle y, x^*\rangle = 1$. Define the projection Q by the formula $Qz = \langle z, x^*\rangle y$ then

$$Q^*z = \langle z, y\rangle x^* \quad\text{and}\quad Q^*x^* = x^*.$$

Let $\{Q_n\}$ be a sequence of projections such that $\|Q-Q_n\|\to 0$, $Q_n < P_n$. Then

$$\begin{aligned} \|P_n^*x^*-x^*\| &= \|P_n^*Q^*x^*-Q^*x^*\|\leqslant\|P_n^*Q^*-Q^*\| = \|QP_n-Q\| \\ &\leqslant \|QP_n-Q_nP_n\|+\|Q_nP_n-Q\|\leqslant\|Q-Q_n\|(K+1)\to 0. \end{aligned}$$

This shows that $P_n^*\to 1$ strongly. Proof of the convergence $P_n\to 1$ strongly is alike so we omit it.

Lemmas 1–4 are also valid in any Banach space.

Spectra and numerical ranges. The set $\Sigma_d(A)$ of all those $\lambda\in\Sigma(A)$ such that λ is an isolated point of $\Sigma(A)$ and $E(\lambda, A) = E(\{\lambda\}, A)\in F(H)$ is called the *discrete spectrum* of the operator A.

The set $\Sigma_e(A) = \Sigma(A)\setminus\Sigma_d(A)$ is called (Browder) *essential spectrum.* N. Salinas proved in [6]:

LEMMA 5.

$$\Sigma_e(A) = \bigcap_{1-P\in P_f(H)}\Sigma(PA|_{PH}).$$

The *numerical range* $W(A)$ of an $A\in L(H)$ is defined as

$$W(A) = \{\langle Ax, x\rangle;\ \|x\| = 1\},$$

the *essential numerical range* is given by the formula

$$W_e(A) = \bigcap_{K\in LC(H)}\overline{W(A+K)}.$$

It is known that $W(A)$, $W_e(A)$ are convex sets, that $\Sigma(A) \subset \overline{W(A)}$ and

(2) $$R(\lambda, A) \leqslant \big(d(\lambda, W(A))\big)^{-1} \quad \text{for} \quad \lambda \notin W(A)$$

([3], V, Th. 3.2). Then next lemma gives a usefull charâcterization of $W_e(A)$.

Lemma 6. *The following conditions are equivalent:*

(i) $\lambda \in W_e(A)$,

(ii) $\langle Ax_n, x_n\rangle \to \lambda$ *for some sequence of unit vectors such that* $x_n \to 0$ *weakly,*

(iii) $\lambda \in \bigcap_{1-P\in P_f(H)} \overline{W(PA|_{PH})}$.

For a proof we send reader to [2] and [5].

These lemmas imply that $\Sigma_e(A) \subset W_e(A)$ so

(3) $$\Sigma(A)\setminus W_e(A) \subset \Sigma_d(A).$$

$W_e(A)$ is a convex set, so it is obvious that the convex hull of $\Sigma_e(A)$ is contained in $W_e(A)$. N. Salinas proved in [4] that if $A \in L(H)$ is a hyponormal operator then $\operatorname{conv}\Sigma_e(A) = W_e(A)$.

Projectional methods. For a given operator $A \in L(H)$, and a given sequence $\{P_n\}$ of projections in H, we define operators

$$\overline{A}_n = P_n A|_{H_n} \in L(H_n) \quad \text{where} \quad H_n = P_n H.$$

The following theorem holds:

Theorem 1. *If* $A \in L(H)$, Ω *is a subset of the complex plane such that* $\Omega \cap W_e(A) = \emptyset = \partial\Omega\cap\Sigma(A)$, $\{P_n\}$ *is a sequence of orthogonal projections in* H, $P_n \to 1$ *strongly then*

$$\|E(\Omega, A) - E(\Omega, \overline{A}_n)P_n\| \to 0 \quad \textit{with} \quad n \to \infty.$$

It is known, that if P, Q are projections such that $\|P-Q\| < 1$ then $\dim PH = \dim QH$ ([3], p. 33).

Using this result to projections $E(\Omega, A)$ and $E(\Omega, \overline{A}_n)P_n$, we obtain under the assumptions of Theorem 1 the following

Corollary 1. (i) $\operatorname{dist}\big(\Omega\cap\Sigma(A), \Omega\cap\Sigma(\overline{A}_n)\big) \to 0$.

(ii) *If* $\lambda \notin W_e(A)$ *then* $\lambda \in \Sigma(A)$ *if and only if* $d\big(\lambda, \Sigma(\overline{A}_n)\big) \to 0$.

Proof of Theorem 1.

I. We can choose a number $\varepsilon > 0$ in such a way that

$$\inf\{|x-y|: x\in\Omega,\ y\in W_e(A)\} > \varepsilon \quad \text{and} \quad \Sigma(A)\cap\partial\big(W_e(A)+(\varepsilon)\big) = \emptyset.$$

Now put $G = W_e(A)+(\varepsilon)$, note that the set $\Sigma(A)\setminus G$ is a finite subset of $\Sigma_d(A)$ thus the projection $Q = E(C\setminus G, A)$ is of a finite dimension. Because $G\cap\Omega = \emptyset$ so $E(\Omega, A) < Q$, Lemma 3 implies that there exists a sequence $\{Q_n\}$ of projections in H such that $Q_n < P_n$, $\|Q_n - Q\| \to 0$.

Using the projections Q, Q_n we define some new operators:

$$\begin{aligned}
C &= A|_{(1-Q)H}\colon (1-Q)H \to (1-Q)H \subset H,\\
D &= A|_{QH}\colon QH \to QH \subset H,\\
C_n &= (1-Q_n)A|_{(1-Q_n)H}\colon (1-Q_n)H \to (1-Q_n)H \subset H,\\
D_n &= Q_nA|_{Q_nH}\colon Q_nH \to Q_nH \subset H_n \subset H,\\
\overline{C}_n &= P_nC_n|_{(1-Q_n)H_n}\colon (1-Q_n)H_n \to (1-Q_n)H_n \subset H_n \subset H,\\
B_n &= C_n(1-Q_n)+D_nQ_n\colon H\to H,\\
\overline{B}_n &= P_nB_n|_{H_n}\colon H_n \to H_n \subset H.
\end{aligned}$$

I have written the inclusions such as for example $H_n \subset H$ because the operators from $L(H_n)$ are sometimes understood as the operators from $L(H_n, H)$.

Note that

$$A = C(1-Q)+DQ, \qquad \overline{B}_n = \big(\overline{C}_n(1-Q_n)+D_nQ_n\big)|_{H_n},$$

$$A - B_n = (Q_n-Q)A(1-Q)+(1-Q_n)A(Q_n-Q)+QA(Q-Q_n)+ +(Q-Q_n)AQ_n;$$

this implies that

$$\|A-B_n\| \leqslant \|Q_n-Q\|\,\|A\|(\|1-Q\|+\|Q\|+\|Q_n\|+\|1-Q_n\|)$$

so

(4) $$\|A-B_n\| \to 0.$$

This with the definitions of $\overline{A}_n$ and $\overline{B}_n$ implies

(5) $$\|\overline{A}_n-\overline{B}_n\| \leqslant \|A-B_n\| \to 0.$$

Note also that

$$\Sigma(C) = \Sigma(A)\cap(G\setminus\partial G), \qquad \Sigma(D) = \Sigma(A)\setminus G.$$

II. In this part I shall show that for n large enough $\Sigma(\overline{C}_n) \subset G$ and that there exists such number M that for n large enough

$$\|R(\lambda, \overline{C}_n)\| \leqslant M \quad \text{for} \quad \lambda\notin G.$$

If this is not true we could find sequences $x_n \in H_n$, $\lambda_n \in C$ such that: $\|x_n\| = 1$, $\lambda_n \notin G$, and zero is a cluster point of the sequence $\|(\overline{C}_n-\lambda_n)x_n\|$.

The norms of the operators $\overline{C}_n$ are bounded by a constant r independent of n, thus

$$\|(\overline{C}_n-\lambda)x_n\| \geqslant |\lambda| - \|C_n\| \geqslant |\lambda| - r > 1 \quad \text{when} \quad |\lambda| > r+1.$$

This shows that the sequence λ_n is bounded. Choosing a subsequence and changing indices we can assume that $\lambda_n \to \lambda_0$, $x_n \to x_0$ weakly, $\|(\overline{C}_n-\lambda_n)x_n\| \to 0$. Because $\lambda_n \notin G$ so

(6) $$\lambda_0 \notin G\setminus\partial G.$$

An easy computation shows that

$$\|(\bar{C}_n-\lambda_0)x_n\|\to 0.$$

Because $Q_nx_n = 0$ so $\|Qx_n\| = \|(Q-Q_n)x_n\|\to 0$ and for any $y\in H$

$$\langle Qx_0, y\rangle = \lim_n \langle Qx_n, y\rangle = 0;$$

hence

(7) $$Qx_0 = 0.$$

The following identity holds:

$$\begin{aligned}\langle Ax_n, x_n\rangle-\lambda_0 &= \langle Ax_n, x_n\rangle-\langle P_n(1-Q_n)Ax_n, x_n\rangle+\langle(\bar{C}_n-\lambda_0)x_n, x_n\rangle\\ &= \langle Ax_n, x_n\rangle-\langle(1-Q_n)Ax_n, x_n\rangle+\langle(\bar{C}_n-\lambda_0)x_n, x_n\rangle\\ &= \langle(Q_n-Q)Ax_n, x_n\rangle+\langle A(Q-Q_n)x_n, x_n\rangle+\langle(\bar{C}_n-\lambda_0)x_n, x_n\rangle.\end{aligned}$$

To obtain the last equality we use the relations $Q_nx_n = 0$ and $AQ = QA$. This identity implies that

$$|\langle Ax_n, x_n\rangle-\lambda_0|\leqslant 2\|A\|\|Q-Q_n\|+\|(\bar{C}_n-\lambda_0)x_n\|\to 0$$

but $\lambda_0\notin W_e(A)$, this and Lemma 6 imply that $x_0\neq 0$.

For any $y\in H$ the following identity holds:

$$\begin{aligned}\langle(1-Q)(A-\lambda_0)x_n, y\rangle &= \langle(1-Q_n)(A-\lambda_0)x_n, P_ny\rangle+\\ &+\langle(Q_n-Q)(A-\lambda_0)x_n, P_ny\rangle+\langle(1-Q)(A-\lambda_0)x_n, y-P_ny\rangle\end{aligned}$$

and because

$$\langle(1-Q_n)(A-\lambda_0)x_n, P_ny\rangle = \langle(\bar{C}_n-\lambda_0)x_n, y\rangle$$

so

$$\begin{aligned}&|\langle(1-Q)(A-\lambda_0)x_n, y\rangle|\\ &\leqslant\|(\bar{C}_n-\lambda_0)x_n\|\|y\|+\|Q_n-Q\|\|A-\lambda_0\|\|y\|+\|1-Q\|\|A-\lambda_0\|\|y-P_ny\|.\end{aligned}$$

Hence in the limit we obtain that

$$\langle(1-Q)(A-\lambda_0)x_0, y\rangle = 0,$$

but $y\in H$ is arbitrary, $x_0\in(1-Q)H$ so $Cx_0 = \lambda_0x_0$. But $x_0\neq 0$ so $\lambda_0\in\Sigma(C)\subset G\setminus\partial G$, this contradicts (6) and proves the statement of part II.

III. For n large enough $\Sigma(C_n)\subset G$. This statement may be proved like part II, but note that in the special case when $P_n\equiv 1$, the projections Q_n may be defined to be the previous ones. Then $C_n = \bar{C}_n$, so this part is a simple corollary from part II.

IV. It is enough to prove the theorem in the case when the boundary $\partial\Omega$ of the set Ω is a regular Jordan curve with a finite length $|\partial\Omega|$.

Let $M_1 = \sup_{\lambda\in\partial\Omega}\|R(\lambda, A)\|$. (4) implies that for n large enough

$$\|A-B_n\|\|R(\lambda, A)\|\leqslant\|A-B_n\|M_1<\tfrac{1}{2}\quad\text{for}\quad\lambda\in\partial\Omega.$$

Using Lemma 2 we see that for such n $\partial\Omega\subset\varrho(B_n)$ and for $\lambda\in\partial\Omega$ $\|R(\lambda, B_n)\|\leqslant 2M_1$, $\|R(\lambda, A)-R(\lambda, B_n)\|\leqslant 2M_1^2\|A-B_n\|$. Hence

(8) $$\begin{aligned}\|E(\Omega, A)-E(\Omega, B_n)\| &= \left\|\frac{-1}{2\pi i}\int_{\partial\Omega}\big(R(\lambda, A)-R(\lambda, B_n)\big)d\lambda\right\|\\ &\leqslant\frac{|\partial\Omega|}{2\pi}2M_1^2\|A-B_n\|.\end{aligned}$$

Because $R(\lambda, B_n) = R(\lambda, C_n)(1-Q_n)+R(\lambda, D_n)Q_n$ so

(9) $$E(\Omega, B_n) = E(\Omega, C_n)(1-Q_n)+E(\Omega, D_n)Q_n.$$

We have shown in part III that for n large enough $\Omega\subset C\setminus G\subset\varrho(C_n)$ then $E(\Omega, C_n) = 0$, this with (8) and (9) imply that for n sufficiently large

(10) $$\|E(\Omega, A)-E(\Omega, D_n)Q_n\|\leqslant\frac{|\partial\Omega|}{\pi}M_1^2\|A-B_n\|.$$

Because for n large enough $\partial\Omega\subset\varrho(B_n) = \varrho(C_n)\cap\varrho(D_n)$, $\Sigma(\bar{C}_n)\subset G\subset C\setminus\Omega$ and $\varrho(\bar{B}_n) = \varrho(\bar{C}_n)\cap\varrho(D_n)$ so $\bar{\Omega}\subset\varrho(\bar{C}_n)$, $\partial\Omega\subset\varrho(\bar{B}_n)$. Hence

(11) $$E(\Omega, \bar{B}_n) = \big(E(\Omega, \bar{C}_n)(1-Q_n)+E(\Omega, D_n)Q_n\big)|_{H_n} = E(\Omega, D_n)Q_n|_{H_n}.$$

The identity $R(\lambda, \bar{B}_n) = R(\lambda, \bar{C}_n)(1-Q_n)+R(\lambda, D_n)Q_n$ together with part II of this proof implies that for $\lambda\in\partial\Omega$ and sufficiently large n

$$\|R(\lambda, \bar{B}_n)\|\leqslant\|Q_n\|\big(\|R(\lambda, \bar{C}_n)\|+\|R(\lambda, D_n)\|\big)\leqslant K(M+2M_1)\overset{\mathrm{df}}{=}M_2,$$

where $K = \sup_n\|Q_n\|$, because $\|R(\lambda, D_n)\|\leqslant\|R(\lambda, B_n)\|\leqslant 2M_1$. So for n large enough

$$\|\bar{A}_n-\bar{B}_n\|\|R(\lambda, \bar{B}_n)\|\leqslant\|A-B_n\|M_2<\tfrac{1}{2}\quad\text{for}\quad\lambda\in\partial\Omega$$

and from Lemma 2 the following relations hold:

$$\partial\Omega\subset\varrho(\bar{A}_n),$$

$$\|R(\lambda, \bar{A}_n)-R(\lambda, \bar{B}_n)\|\leqslant 2\|A-B_n\|M_2^2\quad\text{for}\quad\lambda\in\partial\Omega.$$

Integrating this inequality along the curve $\partial\Omega$ we obtain

$$\|E(\Omega, \bar{A}_n)-E(\Omega, \bar{B}_n)\|\leqslant\frac{|\partial\Omega|}{\pi}M_2^2\|A-B_n\|.$$

This with (10) and (11) imply

$$\|E(\Omega, A)-E(\Omega, \bar{A}_n)P_n\|\leqslant\frac{|\partial\Omega|}{\pi}(M_1^2+M_2^2)\|A-B_n\|\to 0. \blacksquare$$

LEMMA 7. *If H is an infinite dimensional Hilbert space, $A \in L(H)$, $P_0 \in P_f(H)$, $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $\lambda_n \in W_e(A)$, $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence of positive numbers then there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of orthogonal projections such that:*

(i) $P_{n+1}z = P_n z + \langle z, x_{n+1}\rangle x_{n+1}$ *where* $P_n x_{n+1} = 0$, $\|x_{n+1}\| = 1$;

(ii) $\Sigma(A_{n+1}) = \Sigma(A_n) \cup \hat{\lambda}_{n+1}$ *where* $A_n = P_n A|_{P_nH}$, $|\lambda_n - \hat{\lambda}_n| \leqslant \varepsilon_n$;

(iii) $A_n x_m = \hat{\lambda}_m x_m$, $0 < m \leqslant n$;

(iv) *if λ_n is an interior point of $W_e(A)$ then $\lambda_n = \hat{\lambda}_n$.*

Proof. Suppose the projection P_n is just defined. Then let Q_n be the orthogonal projection onto the subspace $P_nH + AP_nH + A^*P_nH$. Because $\lambda_{n+1} \in W_e(A) \subset \overline{W\big((1-Q_n)A|_{(1-Q_n)H}\big)}$ so there exists a unit vector x_{n+1} such that: $|\langle Ax_{n+1}, x_{n+1}\rangle - \lambda_{n+1}| < \varepsilon_{n+1}$, $Q_n x_{n+1} = 0$. In this way we define one by one the projections P_n by the formula $P_{n+1}z = P_n z + \langle z, x_{n+1}\rangle x_{n+1}$ and the numbers $\hat{\lambda}_n = \langle Ax_n, x_n\rangle$.

Note that if λ_n is an interior point of $W_e(A)$ then for any $Q \in P_f(H)$ $\lambda_n \in W\big((1-Q)A|_{(1-Q)H}\big)$, hence in this case we may choose x_n in such a way that $\lambda_n = \hat{\lambda}_n = \langle Ax_n, x_n\rangle$.

Note that if $z = P_n z$ then $Az \perp x_{n+1}$, so

$$A_{n+1}z = P_{n+1}AP_n z = P_n AP_n z + \langle Az, x_{n+1}\rangle x_{n+1} = A_n z. \tag{12}$$

Note that $A^*P_n = Q_nA^*P_n$ so $P_nAQ_n = P_nA$, this implies that

$$\begin{aligned} A_{n+1}x_{n+1} &= P_{n+1}Ax_{n+1} = P_nAx_{n+1} + \langle Ax_{n+1}, x_{n+1}\rangle x_{n+1} \\ &= P_nAQ_nx_{n+1} + \hat{\lambda}_{n+1}x_{n+1} = \hat{\lambda}_{n+1}x_{n+1}. \end{aligned}$$

This with (12) shows that $A_{n+1} = A_n \oplus (\hat{\lambda}_{n+1}|_{(P_{n+1}-P_n)H})$.

This by induction gives the thesis of the lemma.

COROLLARY 2. *If P_n is a sequence of finite dimensional orthogonal projections in H converging strongly to 1_H, S is any subset of $W_e(A)$ then there exists a sequence Q_n such that $P_n < Q_n \in P_f(H)$ (so $Q_n \to 1$ strongly) and*

$$\operatorname{dist}\big(\Sigma(A_n) \cup S, \Sigma(\bar{A}_n)\big) \to 0 \quad \textit{with } n \to \infty,$$

where

$$A_n = P_nA|_{P_nH} \in L(P_nH)|, \quad \bar{A}_n = Q_nA|_{Q_nH} \in L(Q_nH).$$

Proof. For any $\varepsilon > 0$ there exists a finite subset S_ε of S such that $\operatorname{dist}(S_\varepsilon, S) < \varepsilon$. It follows from the lemma that there exists a projection $Q_n > P_n$ such that $\operatorname{dist}\big(\Sigma(A_n) \cup S_{1/n}, \Sigma(\bar{A}_n)\big) \leqslant 1/n$ hence

$$\operatorname{dist}\big(\Sigma(A_n) \cup S, \Sigma(\bar{A}_n)\big) \leqslant 2/n. \ \blacksquare$$

This corollary explains why in Theorem 1 and Corollary 1 the set $W_e(A)$ cannot be substituted by any smaller set.

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