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On a singular integral

by

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Abstract. The commutator singular integral

$$\text{p.v.} \int_{\mathbf{R}^n} K(x-y)\{F(x)-F(y)\}\, g(y)\, dy$$

(where $K(x)$ is even, positively homogeneous of degree $-n-1$, integrable over the unit sphere of $\mathbf{R}^n$) is studied when

$$\operatorname{grad} F \in L^p(\mathbf{R}^n), \quad 1 \leqslant p < n, \quad g \in L^q(\mathbf{R}^n), \quad 1 \leqslant 1/p + 1/q \leqslant (n+1)/n.$$

0. Introduction. The purpose of this paper is to extend results in [6]. Let $k(x)$ be positively homogeneous of degree $-n-1$, even and locally integrable in $|x| > 0$. Let $F(x)$ have first order derivatives in the distributions sense in $L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$. Let $g(x)$ be a function in $L^q(\mathbf{R}^n)$, $1 \leqslant q \leqslant \infty$. Assume that r is given by $1/r = 1/p + 1/q$, p and q not infinity simultaneously. Consider now the operator

$$T(F, g) = \text{p.v.} \int_{\mathbf{R}^n} \{F(x) - F(y)\} K(x-y)\, g(y)\, dy. \tag{0.1}$$

It has been shown in [2] that, if $r > 1$, the above limit exits in L^r norm; furthermore, the principal value converges a.e. (see [1]). If $p = \infty$, $r = 1$, $q = 1$, it is shown in [1] that $T(F, g)$ converges a.e. provided that smoothness is assumed on $K(x)$ (for example C^1).

In the paper [6] it is shown that if $r = 1$, p is such that $1 < p < \infty$, then (0.1) exists a.e. and in $L^1(\mathbf{R}^n)$-norm; no smoothness condition is assumed on K. (1) In addition, if the following smoothness condition is assumed on K:

$$\int_{|x|>4|h|} |K(x+h) - K(x)|\, |x|\, dx < C. \tag{0.2}$$

(1) In a non-published paper *Pointwise estimates for commutator singular integrals* B. Bajsanski and R. Coifman have shown a very similar result, but weak type instead of strong type, and making the following smoothness assumption on the kernel:

$$\int_{|x|=1} |K(x+h) - K(x)|\, d\sigma < C \cdot |h|^{\delta}, \qquad 0 < \delta < 1.$$

(here C does not depend on h), then $T(F, g)$ exists in L^r-metric and a.e. provided that $p > n$, $q \geqslant 1$. Notice that in this case $r > n/(n+1)$. Our task throughout this paper is to extend and improve this result. In the first place, we are going to get rid of the smoothness condition (0.2) and also of the unnecessary restriction $p > n$. We are going to study the cases

$$1 < 1/p + 1/q \leqslant (n+1)/n, \qquad 1 \leqslant p < n$$

and also the limiting cases $p = 1$, $q \geqslant n$, $r = q/(q+1)$.

Our results are the best possible ones in the sense that for every pair p, q such that

$$(0.3) \qquad 1 \leqslant p < n,\ q \geqslant 1,\ n \geqslant 2,\ 1/p + 1/q > (n+1)/n,$$

there exist two functions F and g that satisfy

$$(0.4) \qquad \operatorname{grad} F \in L^p(\boldsymbol{R}^n), \qquad g \in L^q(\boldsymbol{R}^n)$$

and $T(F, g) = \infty$ on a set of positive measure. This is shown by using a very elementary example.

1. Definitions and statement of results. Given a function f, real valued and measurable on $\boldsymbol{R}^n$, the symbol $\|f\|_p$, $1 \leqslant p \leqslant \infty$, will denote the usual L^p norm with respect to the Lebesgue measure.

The symbols $\dfrac{\partial f}{\partial x_i}$, $i = 1, 2, \ldots, n$, denote the derivatives of f in the distributions sense. $\operatorname{grad} f$ will stand for the vector

$$(1.1) \qquad \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

and, whenever it makes sense,

$$(1.2) \qquad \|\operatorname{grad} f\|_p = \left(\int_{\boldsymbol{R}^n} \left(\sum_1^n \left|\frac{\partial f}{\partial x_i}\right|^2\right)^{p/2} dx\right)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty.$$

When $p = \infty$, we have instead

$$(1.3) \qquad \|\operatorname{grad} f\|_\infty = \operatorname{ess\,sup} \left(\sum_1^n \left|\frac{\partial f}{\partial x_i}\right|^2\right)^{1/2}.$$

If $f \geqslant 0$, $E(f > \lambda)$ will denote the set of points in $\boldsymbol{R}^n$ where f exceeds $\lambda > 0$; $|E(f > \lambda)|$ denotes its Lebesgue measure. Throughout this paper, C will denote a constant, not necessarily the same at each occurrence.

$K(x)$ will denote a positively homogeneous function of degree $-n-1$ satisfying either properties (P_1) or (P_2).

(P_1) *$K(x)$ is even and*

$$(1.4) \qquad \int_\Sigma |K(x)|\, d\sigma < \infty.$$

Here Σ denotes the unit sphere and $d\sigma$ is its "area" element.

(P_2) *$K(x)$ is odd and*

$$(1.5) \qquad \int_\Sigma |K(x)| \log^+ |K(x)|\, d\sigma < \infty,$$

$$(1.6) \qquad \int_\Sigma K(x) x_i\, d\sigma = 0, \quad i = 1, 2, \ldots, n.$$

We shall occasionally write $K(x) = \Omega(x)|x|^{-(n+1)}$, where $\Omega(x)$ is a homogeneous function of degree 0.

Assume that $\|\operatorname{grad} F\|_p < \infty$ for some $p \geqslant 1$ and $g \in L^q(\boldsymbol{R}^n)$, $q \geqslant 1$; then $\overset{*}{T}(F, g)$ is defined as

$$(1.7) \qquad \overset{*}{T}(F, g) = \sup_{\varepsilon > 0} |T_\varepsilon(F, g)|,$$

where

$$(1.8) \qquad T_\varepsilon(F, g) = \int_{|x-y| > \varepsilon} \{F(x) - F(y)\} K(x-y) g(y)\, dy.$$

THEOREM 1. *Suppose that $\|\operatorname{grad} F\|_p < \infty$ and $g \in L^q(\boldsymbol{R}^n)$, $1 < p < n$, $q > 1$, $1 \leqslant 1/p + 1/q \leqslant (n+1)/n$. Then, if K satisfies properties* (P_1) *or* (P_2), *we have*

(i) *$T_\varepsilon(F, g)(x)$ converges* a.e.

(ii) *If r is given by $1/r = 1/p + 1/q$, the following estimate holds:*

$$|E(\overset{*}{T}(F, g) > \lambda)| < \frac{C}{\lambda^r} \|\operatorname{grad} F\|_p^r \|g\|_q^r. \ (^2)$$

Here C does not depend on F or g.

The case $p = 1$ is covered by the more general result:

THEOREM 2. *Suppose that $\dfrac{\partial F}{\partial x_i} = \mu_i$, $i = 1, 2, \ldots, n$. The μ_i are finite Borel measures defined on $\boldsymbol{R}^n$. Let us denote by ν_i their respective variations. Assume that $q \geqslant n$ and $r = q/(q+1)$; then we have*

(i) *$T_\varepsilon(F, g)$ converges* a.e.

(ii) $|E(\overset{*}{T}(F, g) > \lambda)| < \dfrac{C}{\lambda^r} \left(\sum_1^n \nu_i(\boldsymbol{R}^n)\right)^r \|g\|_q^r.$

Here C does not depend on μ or g.

(2) In the case $1 < p < n$ actually strong type holds, nevertheless, we have not included here these result because it follows from the weak type estimates by using techniques very similar to the ones in [6] or [8].

Observation. In Theorem 1, the limiting cases occur when $1/p+1/q = (n+1)/n$, while in Theorem 2, the limiting case is $q = n$.

THEOREM 3. *Let p and q be such that*

$$1 \leqslant p < n/(1+a),\ n \geqslant 2,\ 0 < a < 1,\ q > 1,\ 1/p+1/q > (n+1)/n.$$

Then there exist two functions F and g such that

(i) $\|\operatorname{grad} F\|_p < \infty$, $g \in L^q(\boldsymbol{R}^n)$.

(ii) $T(F, g) = \infty$ *on a ball.*

(iii) K *can be chosen to be C^∞ in $\boldsymbol{R}^n - \{0\}$.*

2. Proof of Theorem 3. Consider K satisfying properties (P_1) or (P_2), being C^∞ in $\boldsymbol{R}^n - \{0\}$ and having the value

$$K(x) = \frac{1}{|x|^{n+1}} \tag{2.1}$$

for x belonging to the cone

$$k\sum_2^n x_i^2 < x_1^2, \tag{2.2}$$

large k and $0 < x_1$.

Let $\beta > 0$ be given by $\beta = n-a$ and define g to be

$$g(x) = \frac{1}{|x|^\beta} \tag{2.3}$$

if x belongs to the truncated cone,

$$k\sum_2^n x_i^2 < x_1^2, \quad 0 < x_1 < A \tag{2.4}$$

and zero otherwise.

$F(x)$ is going to be chosen C^∞ in $\boldsymbol{R}^n - \{0\}$ and satisfying

$$F(x) = \frac{1}{|x|^a} \tag{2.5}$$

if x belongs to the truncated cone

$$k\sum_2^n x_i^2 < x_1^2, \quad 0 < x_1 < A, \tag{2.6}$$

and $F(x) = 0$ outside the truncated cone

$$k\sum_2^n x_i^2 < (x_1+\varepsilon)^2, \quad -\varepsilon < x_1 < A+\varepsilon. \tag{2.7}$$

Let now x be a point in a neighborhood of $x_0 = (-2\varepsilon, 0, 0, \ldots, 0)$ such that we have

$$-\int \{F(x)-F(y)\}K(x-y)g(y)\,dy = \int F(y)K(x-y)g(y)\,dy \tag{2.8}$$
$$> C\frac{1}{(A+4\varepsilon)^{n+1}}\int_0^A \frac{dr}{r} = \infty.$$

Here the neighborhood of x_0 is chosen small enough so that $F(x) = 0$ and

$$|K(x-y)| > \frac{C}{|A+4\varepsilon|^{n+1}} \quad \text{for} \quad k\sum_2^n y_i^2 < y_1^2,\ 0 < y_1 < A.$$

Finally, it is very easy to check that

$$\begin{aligned} &\|\operatorname{grad} F\|_p < \infty \quad \text{for} \quad 1 \leqslant p < n/(1+a),\\ &\|g\|_q < \infty \quad \text{for} \quad 1 \leqslant q < n/\beta. \end{aligned} \tag{2.9}$$

This finishes the proof.

3. Auxiliary lemmas.

3.1. LEMMA. *Let $F(x)$ be given by the integral*

$$F(x) = \int_Q K(x-y)\,d\mu, \tag{3.1.1}$$

where Q is a cube, μ a finite Borel measure defined on $\boldsymbol{R}^n$ and such that $\int_Q d\mu = 0$. $K(x)$ is a homogeneous function of degree $-(n-1)$, C^2 on $\boldsymbol{R}^n - \{0\}$. Call δ the diameter of Q, ν the variation of μ, y_0 the center of Q. Then, if $d(x_i, Q) > 5\delta$, $i = 1, 2$, we have

(i) $|F(x_1)-F(x_2)|$

$$< C|x_1-x_2|\left\{\frac{\delta}{\delta^{n+1}+|x_1-y_0|^{n+1}} + \frac{\delta}{\delta^{n+1}+|x_2-y_0|^{n+1}}\right\}\nu(Q).$$

Here C does not depend on δ, Q or μ.

Proof. Without loss of generality we may assume that $|x_1-y_0| \leqslant |x_2-y_0|$. Consider also a third point Z, selected so that

$$|x_i - Z| < 5|x_1-x_2|, \quad i = 1, 2, \tag{3.1.2}$$

and also

$$|x-y_0| > \tfrac{1}{5}|x_1-y_0| \tag{3.1.3}$$

for any x belonging to the polygonal (x_1, Z, x_2).

It is very easy to check that such a point Z always exists, provided that $d(x_i, Q) > 5\delta$, $i = 1, 2$.

Consider now $F(x_2)-F(x_1) = F(x_2)-F(Z)+F(Z)-F(x_1)$. By using the mean value theorem we get

$$(3.1.4)\qquad \begin{aligned} F(x_2)-F(Z) &= |x_2-Z|\int_Q K_2(s_2-y)\,d\mu,\\ F(Z)-F(x_1) &= |Z-x_1|\int_Q K_1(s_1-y)\,d\mu, \end{aligned}$$

where $s_1\in\overline{x_1Z}$ and $s_2\in\overline{Zx_2}$ and K_1 and K_2 are homogeneous functions of degree n (directional derivative of s the kernel along the directions of $\overline{x_1Z}$ and $\overline{Zx_2}$, respectively).

Now, using the fact that $d\mu$ has vanishing integral over Q, we obtain

$$(3.1.5)\qquad |F(x_i)-F(Z)| < |x_i-Z|\max_{y\in Q}|K_i(s_i-y)-K_i(s_i-y_0)\nu(Q), \qquad i=1,2.$$

In turn, the right-hand member above can be dominated in the following way:

$$(3.1.6)\qquad |x_i-Z|\max_{y\in Q}|K_i(s_i-y)-K_i(s_i-y_0)|\nu(Q)\leqslant C|x_i-Z|\frac{\delta}{|s_i-y_0|^{n+1}}.$$

By using (3.1.2) and (3.1.3) one obtains immediately (i). This finishes the proof.

3.2. Lemma. *Let F be such that $\|\operatorname{grad}F\|_p<\infty$, where $1<p<n$. Let x be any point in $\mathbf{R}^n$ and Q an arbitrary cube centered at x having edges parallel to the coordinate axes. Then we have the following inequality:*

$$(\mathrm{i})\qquad \left(\frac{1}{|Q|}\int_Q\left|\frac{F(x)-F(y)}{\delta}\right|^s dy\right)^{1/s}\leqslant C_{p,n}\sup_{I_x}\left(\frac{1}{|I_x|}\int_{I_x}|\operatorname{grad}F|^p\,dy\right)^{1/p},$$

where $1/s=1/p-1/n$, $\operatorname{diam}(Q)=\delta$ and the supremum is taken over all cubes I_x centered at x and having edges parallel to the coordinate axes. The constant $C_{p,n}$ depends on p and n only.

Here $|\operatorname{grad}F|$ *means* $\left(\sum_{j=1}^n\left(\frac{\partial F}{\partial y_i}\right)^2\right)^{1/2}$.

The case $p=1$ is covered by the more general result:

Suppose that F is such that $\dfrac{\partial F}{\partial y_i}=\mu_i$, $i=1,2,\ldots,n$, where the μ_i are finite Borel measures defined on $\mathbf{R}^n$. The ν_i's are their respective variations. Denote by $\overset{*}{M}F(x)$ the following maximal operator

$$\overset{*}{M}F=\operatorname*{Sup}_{I_x}\left[\frac{1}{|I_x|}\int_{I_x}\left|\frac{F(x)-F(y)}{d(I_x)}\right|^{n/(n-1)}dy\right]^{(n-1)/n},$$

where the I_x have the same meaning as above and $d(I_x)$ stands for diameter of I_x. Then the following estimate holds:

$$(\mathrm{ii})\qquad |E(\overset{*}{M}F>\lambda)|<\frac{C}{\lambda}\sum_{j=1}^n\nu_j(\mathbf{R}^n).$$

Here the constant C depends on the dimension only.

Proof. Let $\varphi(y)$ be a C_0^∞ function equal to 1 over Q and 0 in the complement of $2Q$ (dilation of Q by the factor 2 about its center) and such that $\|\partial\varphi/\partial y_i\|_\infty<4\delta^{-1}$, $i=1,2,\ldots,n$. Here δ denotes diameter of Q. Consider now the auxiliary function

$$(3.2.1)\qquad \varphi(y)\{F(x)-F(y)\},$$

where x = center of Q.

Apply Sobolev's inequality to (3.2.1) if $p>1$ or Gagliardo–Nirenberg's one if $p=1$ (see [9], p. 129) and get

$$(3.2.2)\qquad \begin{aligned}&\Big(\int_Q|F(x)-F(y)|^s dy\Big)^{1/s}\\ &\quad\leqslant C\Big(\int_{2Q}|\operatorname{grad}F|^p dy\Big)^{1/p}+C\Big(\delta^{-p}\int_{\delta/2<|x-y|<3\delta}|F(x)-F(y)|^p dy\Big)^{1/p}.\end{aligned}$$

Here C does not depend on F, δ or Q and s is given by $1/s=1/p-1/n$, $1\leqslant p<n$. Let us turn our attention to $|F(x)-F(y)|$ and write y as $y=x+ra$, where ra is the polar expression for $y-x$.

Consider now the inequalities

$$(3.2.3)\qquad \begin{aligned}|F(x)-F(x+ra)|^p&\leqslant C^p\Big|\int_0^r|\operatorname{grad}F(x+sa)|\,ds\Big|^p\\ &\leqslant C(3\delta)^{p-1}\int_0^{3\delta}|\operatorname{grad}F(x+sa)|^p ds,\qquad p\geqslant 1.\end{aligned}$$

Thus, we have the estimate

$$(3.2.4)\qquad \begin{aligned}&\delta^{-p}\int_{\delta/2<|x-y|<3\delta}|F(x)-F(y)|^p dy\\ &\quad=\delta^{-p}\int_\Sigma d\sigma\int_{\delta/2}^{3\delta}|F(x+ra)-F(x)|^p r^{n-1}dr\\ &\quad\leqslant C\delta^{-1}\int_\Sigma d\sigma\int_{\delta/2}^{3\delta}r^{n-1}dr\int_0^{3\delta}|\operatorname{grad}F(x+sa)|^p ds.\end{aligned}$$

Interchanging the order of integration in the last integral above, we get

$$\text{(3.2.5)}\qquad C\,\delta^n\,\delta^{-1}\int_{\Sigma} d\sigma\int_0^{3\delta}|\operatorname{grad}F(x+s\sigma)|^p\,ds$$

which, in turn, equals

$$\text{(3.2.6)}\qquad C\,\delta^n\,\delta^{-1}\int_{|x-y|<3\delta}\frac{1}{|x-y|^{n-1}}|\operatorname{grad}F(y)|^p\,dy$$

and, consequently, it is dominated by

$$\text{(3.2.7)}\qquad C\,\delta^n\,\operatorname{Sup}_{I_x}\frac{1}{|I_x|}\int_{I_x}|\operatorname{grad}F|^p\,dy.$$

Bringing back this estimate to (3.2.2) we get (i) and (ii) for the case of absolutely continuous measures.

In order to face the general case, consider a modified definition of $\overset{*}{M}F$, namely

$$\text{(3.2.8)}\qquad \overset{*}{M}_N F(x)=\operatorname*{Sup}_{1\leqslant i\leqslant N}\left(\frac{1}{|I_x^i|}\int_{I_x^i}\left|\frac{F(x)-F(y)}{\delta_i}\right|^{n/(n-1)}dy\right)^{(n-1)/n},$$

where the I_x^i are cubes centered at x, having edges parallel to coordinate axes and their diameters $\operatorname{diam}(I_x^i)=\delta_i$, $i=1,2,\ldots,N$, are rational numbers. When we let N go to infinity, the δ_i take all the rational values; thus $\overset{*}{M}_N F\uparrow\overset{*}{M}F$. We have for $\overset{*}{M}_N F$ the following inequality:

$$\text{(3.2.9)}\qquad |E(\overset{*}{M}_N F>\lambda)|<\frac{C}{\lambda}\sum_{j=1}^{n}\int_{R^n}\left|\frac{\partial F}{\partial y_j}\right|dy$$

provided that $\dfrac{\partial F}{\partial y_i}\in L^1(\boldsymbol{R}^n)$, $i=1,2,\ldots,n$. Here C does not depend on λ, N or F. Let $\psi_\varepsilon(y)$ be a C_0^∞ aproximating unit, that is $\psi_\varepsilon(y)=\varepsilon^{-n}\psi(\varepsilon^{-1}y)$, where $\psi\in C_0^\infty$ and

$$\int_{R^n}\psi(y)\,dy=1.$$

Call $F_\varepsilon(y)$ the convolution $\psi_\varepsilon * F$, where F is such that

$$\text{(3.2.10)}\qquad \frac{\partial F}{\partial y_j}=\mu_j,\qquad j=1,2,\ldots,n,$$

where the μ_j are finite Borel measures and ν_j are their respective variations.

From Young's inequality we have

$$\text{(3.2.11)}\qquad \left\|\frac{\partial F_\varepsilon}{\partial y_j}\right\|_1\leqslant\|\psi\|_1\,\nu_j(\boldsymbol{R}^n).$$

Also, from the very definition of the $F_\varepsilon(y)$ it follows that

$$\text{(3.2.12)}\qquad F_\varepsilon(x)\to F(x)\ \text{a.e.},$$

$$\text{(3.2.13)}\qquad \int_I g(y)F_\varepsilon(y)\,dy\to\int_I g(y)F(y)\,dy$$

for every cube I and for every $g\in L^n(I)$; consequently

$$\text{(3.2.14)}\qquad \left(\frac{1}{|I_x^i|}\int_{I_x^i}\left|\frac{F_\varepsilon(x)-F_\varepsilon(y)}{\delta_i}\right|^{n/(n-1)}dy\right)^{(n-1)/n}$$

$$\text{tends to}\quad \left(\frac{1}{|I_x^i|}\int_{I_x^i}\left|\frac{F(x)-F(y)}{\delta_i}\right|^{n/(n-1)}dy\right)^{(n-1)/n}$$

for $1\leqslant i\leqslant N$ and for a.e. x in $\boldsymbol{R}^n$. Now, using (3.2.9), (3.2.10), (3.2.11) and (3.2.14) we obtain

$$\text{(3.2.15)}\qquad |E(\overset{*}{M}_N F>\lambda)|<\frac{C}{\lambda}\sum_{j=1}^{n}\nu_j(\boldsymbol{R}^n).$$

Here C does not depend on N, λ or μ. From (3.2.15) and the fact that $\overset{*}{M}_N F\uparrow\overset{*}{M}F$ we have (ii) in the general case.

3.3. Lemma. *Let $K_\varepsilon(Y)$ be defined by*

$$K_\varepsilon(y)=\frac{\varepsilon}{\varepsilon^{n+1}+|y|^{n+1}}\,\Omega(y),$$

where $\Omega(y)=\Omega(|y|)$ is a homogeneous function of degree 0 absolutely integrable over the unit sphere of $\boldsymbol{R}^n$. Consider the operator

$$\overline{K}(f)=\sum_k\int_{Q_k}K_{\delta_k}(x-y)f(y)\,dy,$$

where the Q_k are pairwise disjoint n dimensional cubes, with edges parallel to the coordinate axes, δ_k denotes, as usual, the diameter of Q_k. Then, we have the following estimate:

$$\text{(i)}\qquad \|\overline{K}(f)\|_p^p<C_p\sum_1^\infty\int_{Q_k}|f|^p\,dy,$$

$1\leqslant p<\infty$; C_p does not depend on f.

Proof. The case $p=1$ follows from Fubini's theorem after taking absolute values. For the case $p>1$ we shall use the following auxiliary maximal operator:

(3.3.1) $$\overset{*}{f}(x) = \operatorname*{Sup}_{\varepsilon>0} \int |K_\varepsilon(x-y)|\,|f(y)|\,dy.$$

An application of the method of "rotation" (see [5]) gives the following inequalities:

(3.3.2) $$\|\overset{*}{f}\|_q < C_q \|f\|_q, \qquad 1<q<\infty;$$

C_q depending on q only.

Let f belong to $L^p(\boldsymbol{R}^n)$ and $g \in L^{p/(p-1)}(\boldsymbol{R}^n)$ and consider the expression

(3.3.3) $$\Big|\int_{\boldsymbol{R}^n} g\bar{K}(f)\,dx\Big|.$$

The above integral is dominated by

(3.3.4) $$\int_{\boldsymbol{R}^n} \Big(\sum_k \int_{Q_k} |K_{\delta_k}(x-y)|\,|f(y)|\,dy\Big)\,|g(x)|\,dx.$$

Interchanging the order of integration, we get

(3.3.5) $$\sum_k \int_{Q_k} |f(y)| \Big(\int |K_{\delta_k}(x-y)|\,|g(x)|\,dx\Big)\,dy \leqslant \sum_k \int_{Q_k} |f(y)|\,\overset{*}{g}(y)\,dy$$

which, in turn, is dominated by

(3.3.6) $$\Big(\sum_k \int_{Q_k} |f|^p\,dy\Big)^{1/p} C_{p/(p-1)} \Big(\int |g|^{p/(p-1)}\,dy\Big)^{(p-1)/p}.$$

The above inequality yields the desired result for $1<p<\infty$.

4. Proof of Theorem 1. Construction of the set G_λ and the partition of F. Let $f \in L^p(\boldsymbol{R}^n)$, $1<p<n$, $f \geqslant 0$, $\|f\|_p = 1$. Let F be given by

(4.0.1) $$F(x) = \int_{\boldsymbol{R}^n} \frac{1}{|x-y|^{n-1}} f(y)\,dy.$$

4.1. *The set A_λ.* Let us fix $\lambda>0$ from now on and consider the family of cubes $\{I_k\}$ associated with f and λ satisfying the properties:

(a_0) The I_k have edges parallel to the coordinate axes.

(a_1) $f^p(x) < \lambda^r$ a.e. in $\boldsymbol{R}^n - \bigcup_1^\infty I_k$.

(a_2) $\mathring{I}_i \cap \mathring{I}_j = \emptyset$, $i \neq j$.

(a_3) $\lambda^r \leqslant \dfrac{1}{|I_k|}\displaystyle\int_{I_k} f^p\,dt < 2^n\lambda^r$.

Here $1 > r \geqslant n/(n+1)$ (for details see [6]). Write $f = f_1 + f_2$, where

(4.1.1) $$f_1 = \begin{cases} f & \text{in } \boldsymbol{R}^n - \bigcup_1^\infty I_k, \\ \sum_1^\infty m_k\varphi_k(x) & \text{over } \bigcup_1^\infty I_k. \end{cases}$$

Here $m_k = \dfrac{1}{|I_k|}\displaystyle\int_{I_k} f\,dt$, $k=1,2,\ldots$ and $\varphi_k(x)$ is the characteristic function of I_k.

f_2 is defined in the following way:

(4.1.2) $$f_2 = \sum_1^\infty \big(f(x) - m_k\big)\varphi_k(x).$$

$F_1(x)$ and $F_2(x)$ are the potentials:

(4.1.3) $$F_i(x) = \int_{\boldsymbol{R}^n} \frac{1}{|x-y|^{n-1}} f_i(y)\,dy, \qquad i=1,2.$$

From (a_3) it follows

(4.1.4) $$\Big|\bigcup_1^\infty I_k\Big| < \frac{1}{\lambda^r}\int_{\boldsymbol{R}^n} f^p\,dy = \frac{1}{\lambda^r}.$$

Let us denote by $10I_k$ the dilation of I_k by the factor 10 about its center. The set A_λ is defined to be the union:

(4.1.5) $$\bigcup_1^\infty 10 I_k.$$

From (4.1.4) we get

(4.1.6) $$|A_\lambda| < \frac{10^n}{\lambda^r}.$$

4.2. *The set S_λ.* Given the family $\{I_k\}$ defined in (4.1), associated with it we define the function $\Delta(x)$:

(4.2.1) $$\Delta(x) = \sum_1^\infty \frac{\varepsilon_k}{\varepsilon_k^{n+1} + |x-t_k|^{n+1}}\,|I_k|.$$

Here t_k and ε_k denote respectively center and diameter of I_k.

S_λ is defined to be the set

(4.2.2) $$\{x;\ \Delta(x) > 1\}.$$

From Chebyshev's inequality we have

(4.2.3) $$|S_\lambda| < \int_{\boldsymbol{R}} \Delta(x)\,dx = \Big(\int \frac{1}{1+|y|^{n+1}}\,dy\Big) \sum_1^\infty |I_k| < \frac{C}{\lambda^r}.$$

4.3. *The set* J_λ. Consider the family of cubes $\{B_k\}$ satisfying:

(b_0) The B_k have edges parallel to the coordinate axes.

(b_1) $\mathring{B}_i \cap \mathring{B}_j = \emptyset$, $i \neq j$.

(b_2) $\frac{1}{|B_k|} \int_{B_k} |\mathrm{grad}\, F_2|^p dy < C_n \lambda^r$, C_n depends on n only.

(b_3) In $\mathbf{R}^n - \bigcup_1^\infty B_k$ we have

$$\sup_{I_x} \frac{1}{|I_x|} \int_{I_x} |\mathrm{grad}\, F_2|^p dy \leqslant \lambda^r \text{ a.e.}$$

(Here the I_x have the same meaning as in Lemma (3.2).)

(b_4) $\sum_1^\infty |B_k| < \frac{C}{\lambda^r} \int_{\mathbf{R}_n} |\mathrm{grad}\, F_2|^p dy$.

In order to construct the family $\{B_k\}$, apply ([9], p. 19, paragraph 3.5) to $|\mathrm{grad}\, F_2|^p$ with $\alpha = \lambda^r$.

Now, $\mathfrak{J}_\lambda$ is defined to be the union $\bigcup_1^\infty B_k$. Thus

(4.3.1) $$|\mathfrak{J}_\lambda| < \frac{C}{\lambda^r} \int_{\mathbf{R}^n} |\mathrm{grad}\, F_2|^p dy.$$

Using the definition of F_2, we have

(4.3.2) $$\|F_2\|_p^p \leqslant C_p \|f_2\|_p^p \leqslant C_p' \|f\|_p = C_p'.$$

Here C_p and C_p' depend on p only.

Therefore

(4.3.3) $$|\mathfrak{J}_\lambda| < \frac{\bar{C}_p}{\lambda^r},$$

where $\bar{C}_p$ depends on p only.

4.4. *The set* G_λ *and some properties*. G_λ is chosen to be open and satisfying:

(1) $G_\lambda \supset A_\lambda \cup S_\lambda \cup J_\lambda$.

(2) $|G_\lambda| < 2(|A_\lambda| + |S_\lambda| + |\mathfrak{J}_\lambda|) = \frac{C_p}{\lambda^r}$.

Here C_p depends on p only.

Consider for G_λ a covering by cubes of Whitney's type (see [9], pages 16 and 167). That is,

(4.4.1) $$G_\lambda = \bigcup_1^\infty Q_k, \quad \mathring{Q}_i \cap \mathring{Q} = \emptyset, \quad i \neq j.$$

Calling δ_k the diameter of Q_k and $C(G_\lambda)$ the complement of G_λ we have

(4.4.2) $$\delta_k \leqslant \mathrm{dist}(Q_k, C(G_\lambda)) \leqslant 4\delta_k, \quad k = 1, 2, \ldots$$

For each cube Q_k, we construct a larger cube $\overset{*}{Q}_k$ such that

(4.4.3) $$\overset{*}{Q}_k \supset Q_k, \quad k = 1, 2, \ldots,$$

(4.4.4) $$\overset{*}{Q}_k \text{ is centered at } y_k, \quad y_k \in C(G_\lambda), \quad k = 1, 2, \ldots$$

(4.4.5) $$|\overset{*}{Q}_k| < C_n |Q_k|, \quad k = 1, 2, \ldots$$

Here C_n depends on the dimension only.

4.5. *Behavior of* $F_2(y)$ *on* Q_k. From the definition of Q_k and $\overset{*}{Q}_k$ we have

(4.5.1) $$\left(\frac{1}{|Q_k|} \int_{Q_k} \left|\frac{F_2(y) - F_2(y_k)}{\delta_k}\right|^s dy\right)^{1/s} \leqslant C \left(\frac{1}{|\overset{*}{Q}_k|} \int_{\overset{*}{Q}_k} \left|\frac{F(y) - F(y_k)}{\delta_k}\right|^s dy\right)^{1/s}.$$

Now, using the fact that $y_k \in C(G_\lambda)$ and Lemma 3.2, we have

(4.5.2) $$\left(\frac{1}{|Q_k|} \int_{Q_k} \left|\frac{F_2(y) - F_2(y_k)}{\delta_k}\right|^s dy\right)^{1/s} \leqslant C\lambda^{r/p},$$

where $1/s = 1/p - 1/n$, $1 < p < n$.

4.6. *Behavior of* $\Delta(x)$ *on* $C(G_\lambda)$. Since $S_\lambda \subset G_\lambda$, we have for $x \in C(G_\lambda)$

(4.6.1) $$\Delta(x) < 1.$$

4.7. *Behavior of* F_2 *on* $C(G_\lambda)$. Recall the definition of F_2:

(4.7.1) $$F_2(x) = \sum_{k=1}^\infty \int_{I_k} \frac{(f - m_k)}{|x-y|^{n-1}} dy.$$

We are interested in estimating $|F_2(x_1) - F_2(x_2)|$ for $x_i \in C(G)$, $i = 1, 2$. Apply Lemma 3.1 to the terms

(4.7.2) $$F_2^{(k)}(x) = \int_{I_k} \frac{f(y) - m_k}{|x-y|^{n-1}} dy$$

and get the estimate

(4.7.3) $$\begin{aligned} &|F_2(x_1) - F_2(x_2)| \\ &\leqslant \sum_1^\infty |F_2^{(k)}(x_1) - F_2^{(k)}(x_2)| \\ &\leqslant C|x_1 - x_2| \sum_1^\infty \left(\frac{\varepsilon_k}{\varepsilon_k^{n+1} + |x_1 - t_k|^{n+1}} + \frac{\varepsilon_k}{\varepsilon_k^{n+1} + |x_2 - t_k|^{n+1}}\right)^2 \int_{I_k} f\, dt \\ &\leqslant \lambda^{r/p}(\Delta(x_1) + \Delta(x_2))\, C|x_1 - x_2|. \end{aligned}$$

From (4.7.3) we conclude that

(4.7.4) $|F_2(x_1)-F_2(x_2)| < C\lambda^{r/p}|x_1-x_2|, \quad x_i \in C(G_\lambda),\ i=1,2.$

4.8. *Proof of the basic result.* We have fixed $f \geqslant 0$ such that $\|f\|_p = 1$, $1 < p < n$. We have fixed $\lambda > 0$ and $1 > r \geqslant n/(n+1)$. Take now g belonging to $L^q(\boldsymbol{R}^n)$; $1/r = 1/p+1/q$; and $\|g\|_q = 1$.

To begin with, we have

(4.8.1) $$\overset{*}{T}(F, g) \leqslant \overset{*}{T}(F_1, g) + \overset{*}{T}(F_2, g).$$

From Theorem C in [6] (p. 162) it follows that

(4.8.2) $$\big|E\big(\overset{*}{T}(F_1, g) > \lambda\big)\big| < \frac{C_q}{\lambda}\|f_1\|_{q/(q-1)}\|g\|_q.$$

Let us dominate f_1 in the following way:

(4.8.3) $$f_1^{q/(q-1)} = f_1^{q/(q-1)-p} f_1^p < C_{p,q}\lambda^{\frac{r}{p}(q/(q-1)-p)} f_1^p.$$

Using the above estimate in (4.8.2), we get

(4.8.4) $$\big|E\big(\overset{*}{T}(F_1, g) > \lambda\big)\big| < \frac{C}{\lambda^r}.$$

Here C does not depend on λ, F or g.

Our next step will be to get analogous estimates for $\overset{*}{T}(F_2, g)$. We are going to define an exceptional set where x has to be away from. Our exceptional set is going to be defined as

(4.8.5) $$6G_\lambda = \bigcup_1^\infty 6Q_k.$$

That is

(4.8.6) $$|6G_\lambda| < \frac{C}{\lambda^r}.$$

Consider now $x \in \boldsymbol{R}^n - G_\lambda$ and $\varepsilon > 0$; decompose g as g_1+g_2, where $g_1 = g$ over $C(G_\lambda)$ and zero otherwise and $g_2 = g - g_1$. Let $\tilde{F}_2(y)$ be the Lipschitz extension of F_2 from $C(G_\lambda)$ to the whole space (see [9], p. 174). The above remark and 4.7 give

(4.8.7) $|\tilde{F}_2(y_1) - \tilde{F}_2(y_2)| < C\lambda^{r/p}|y_1-y_2|, \quad y_i \in \boldsymbol{R}^n,\ i = 1, 2.$

On account of the definitions of g_1, g_2 and $\tilde{F}_2$, we have

(4.8.8) $$T_\varepsilon(F_2, g) = T_\varepsilon(\tilde{F}_2, g_1) + T_\varepsilon(F_2, g_2).$$

Since $\|\mathrm{grad}\,\tilde{F}_2\|_\infty < C\lambda^{r,p}$, the following estimate holds (see [1]):

(4.8.9) $$\big|E\big(\overset{*}{T}(\tilde{F}_2, g_1) > \lambda\big)\big| < \frac{C_q}{\lambda^q}\|\mathrm{grad}\,\tilde{F}_2\|_\infty^q \|g_1\|_q^q \leqslant \frac{C_q}{\lambda^r}.$$

Here C_q does not depend on F, g or λ.

Let us return to $T_\varepsilon(F_2, g_2)$, whose expression is

(4.8.10) $$T_\varepsilon(F_2, g_2) = \sum_k \int_{Q_k \cap C(B_\varepsilon(x))} \{F_2(x) - F_2(y)\}K(x-y)g_2(y)\,dy.$$

Here $C(B_\varepsilon(x))$ stands for the complement of the ball of radius ε centered at x.

Let the y_k be the points defined in (4.4.4) and consider

(4.8.11) $$F_2(x) - F_2(y) = F_2(x) - F_2(y_k) + F_2(y_k) - F_2(y)$$

In turn, we have

(4.8.12) $$F_2(x) - F_2(y_k) = \tilde{F}_2(x) - \tilde{F}_2(y_k)$$
$$= \tilde{F}_2(x) - \tilde{F}_2(y) + \tilde{F}_2(y) - F_2(y_k).$$

Call $M(\theta, g)$ the following expression:

(4.8.13) $$M(\theta, g) = \sum_k \int_{Q_k} |\theta(y) - \theta(y_k)|\,|K(x-y)|\,|g(y)|\,dy.$$

Taking into account (4.8.11), (4.8.12) and (4.8.13), we have

(4.8.14) $$\overset{*}{T}(F_2, g_2)(x) \leqslant \overset{*}{T}(\tilde{F}_2, g_2) + M(\tilde{F}_2, g) + M(F_2, g).$$

We handle $\overset{*}{T}(\tilde{F}_2, g_2)$ in the same way as we did with $\overset{*}{T}(\tilde{F}_2, g_1)$ and get

(4.8.15) $$\big|E\big(\overset{*}{T}(\tilde{F}_2, g_2) > \lambda\big)\big| < \frac{C}{\lambda^r}.$$

Call now

$$h(y) = \sum_1^\infty \left|\frac{F_2(y) - F_2(y_k)}{\delta_k}\right| |g(y)|\,\psi_k(y)$$

and

$$\tilde{h}(y) = \sum_1^\infty \left|\frac{\tilde{F}_2(y) - \tilde{F}_2(y_k)}{\delta_k}\right| |g(y)|\,\psi_k(y),$$

where $\psi_k(y)$ are the characteristic functions of the cubes Q_k. Notice that $\left|\dfrac{\tilde{F}_2(y) - \tilde{F}_2(y_k)}{\delta_k}\right| \psi_k(y) \leqslant C\lambda^{r/p}$. Keeping the notation of Lemma 3.3, we have

(4.8.16) $$M(\tilde{F}_2, g)(x) \leqslant C\cdot \overline{K}(\tilde{h})(x),$$
$$M(F_2, g)(x) \leqslant C\cdot \overline{K}(h)(x)$$

for $x \in \boldsymbol{R}^n - 6G_\lambda$. Here C denotes a constant independent of λ, F and g.

From Chebyshev's inequality and Lemma 3.3, we have

$$(4.8.17)\qquad |E\{\overset{*}{T}(\tilde{F}_2, g_2) > \lambda\}| \leqslant |6G_\lambda| + \frac{C}{\lambda^q}\|\bar{K}(\tilde{h})\|_q$$

$$\leqslant \frac{C_1}{\lambda^r} + C_2\frac{1}{\lambda^q}\lambda^{\frac{r}{p}q}\int_{\mathbf{R}^n}|g|^q dy \leqslant \frac{C}{\lambda^r}.$$

Here C does not depend on λ, F or G.

Let us turn our attention to $h(y)$. In the first place $h(y) \in L^l(\mathbf{R}^n)$, where $1/l = 1/s + 1/q$ and $1/s = 1/p - 1/n$. Notice that $l = 1$ when $r = n/(n+1)$.

Our second task is to estimate $\|h\|_l^l$.

From the definition of h it follows

$$(4.8.18)\qquad \|h\|_l^l = \sum_1^\infty \int_{Q_k}\left|\frac{F_2(y) - F_2(y_k)}{\delta_k}\right|^l |g(y)|^l dy.$$

Applying Hölder's inequality with exponents $(s/l, q/l)$, we get

$$(4.8.19)\qquad \sum_{k=1}^\infty \left(\int_{Q_k}\left|\frac{F_2(y) - F_2(y_k)}{\delta_k}\right|^s dy\right)^{l/s}\left(\int_{Q_k}|g|^q dy\right)^{l/q}.$$

From (4.5.2) it follows

$$(4.8.20)\qquad \left(\int_{Q_k}\left|\frac{F_2(y) - F_2(y_k)}{\delta_k}\right|^s dy\right)^{l/s} \leqslant C\lambda^{\frac{r}{p}l}|Q_k|^{l/s}.$$

Bringing (4.8.20) to (4.8.19) and applying Hölder's inequality to the series, we get

$$(4.8.21)\qquad C\lambda^{\frac{r}{p}l}|G_\lambda|^{l/s}\left(\sum_1^\infty \int_{Q_k}|g|^q dt\right)^{l/s}.$$

Using the fact that $|G_\lambda| < \dfrac{C}{\lambda^r}$, we have

$$(4.8.22)\qquad \|h\|_l^l \leqslant C\lambda^{\frac{r}{p}l}\lambda^{-r\frac{l}{s}}.$$

Applying Chebyshev's inequality and Lemma 3.3 with exponent l, we have

$$(4.8.23)\qquad \left|E\left(\overset{*}{T}(F_2, g_2) > \lambda\right)\right| < |5G_\lambda| + \frac{C}{\lambda_l}\|\bar{K}(h)\|_l^l \leqslant \frac{C}{\lambda^r} + \frac{C_2}{\lambda^l}\|h\|_l^l$$

$$\leqslant C_3\lambda^{-l}\lambda^{\frac{r}{p}l}\lambda^{-r\frac{l}{s}} = \frac{C_3}{\lambda^r}.$$

Here C_3 does not depend on F, g or λ.

Collecting estimates, we have

$$(4.8.24)\qquad \left|E\left(\overset{*}{T}(F_1, g) > \lambda\right)\right| < \frac{C}{\lambda^r},$$

where C does not depend on F, g or λ. The case $f \geqslant 0$, $\|f\|_p \neq 1$ and $\|g\|_q \neq 1$, follows by applying (4.8.24) to

$$(4.8.25)\qquad \overset{*}{T}(\|f\|_p^{-1}F,\ \|g\|_q^{-1}g)$$

and the general case follows by decomposing $f = f_+ - f_-$.

The pointwise convergence follows from the maximal inequalities and from the fact that the operator converges everywhere for $F \in C_0^\infty$ and $g \in C_0^\infty$.

5. Proof of Theorem 2. Construction of the set G_λ and the partitions of F. Consider the following representation for F:

$$(5.0.1)\qquad F(x) = C_n\sum_{j=1}^n \int \frac{x_j - y_j}{|x-y|^n}\, d\mu_j \ \text{a.e.}$$

(see [4], p. 110). Here C_n depends on n only; $\dfrac{\partial F}{\partial x_j} = \mu_j$, where the μ_j are Borel measures. Call $F_j(x)$ the integral:

$$(5.0.2)\qquad F_j(x) = C_n\int \frac{x_j - y_j}{|x-y|^n}\, d\mu_j.$$

5.1. *The sets A_λ^j and the functions $F_1^{(j)}$ and $F_2^{(j)}$.* Call ν_j the variation of μ_j and let

$$(5.1.1)\qquad \overset{*}{\nu}_j(x) = \operatorname*{Sup}_{I_x}\frac{1}{|I_x|}\nu_j(I_x),$$

where the I_x have the meaning of Lemma 3.2. Assume in addition that $\sum_{j=1}^n \nu_j(\mathbf{R}^n) = 1$, $j = 1, 2, \ldots, n$.

The exponent r satisfies

$$(5.1.2)\qquad r = q/(q+1),$$

where $q \geqslant n$; consequently, $r \geqslant n/(n+1)$. Let us fix $\lambda > 0$; the open set A_λ^j is going to be defined to satisfy the following conditions:

$$(5.1.3)\qquad \overset{*}{\nu}_j(x) < \lambda^r \quad \text{in} \quad \mathbf{R}^n - A_\lambda^j.$$

(5.1.4) The singular part of μ_j lives in A_λ^j.

(5.1.5) $\bigcup_{k=1}^{\infty} I_k^j$ is a Whitney covering of A_λ^j (see [9], page 19).

(5.1.6) $$|A_\lambda^j| < \frac{C}{\lambda^r}.$$

(5.1.7) $$\frac{1}{|I_k^j|}\int_{I_k^j} d\nu_j < C\lambda^r.$$

The constants involved depend only on the dimension. Associated with A_λ^j, we have the decomposition

(5.1.8) $$d\mu_j = f_j(x)\,dx + d\tau_j,$$

where $\tau_j(E) = \mu_j(A_\lambda^j \cap E)$ for any Borel subset E and

$$|f_j(x)| < \lambda^r \text{ a.e.}, \quad f_j(x) = 0 \quad \text{on} \quad A_\lambda^j.$$

Using the above properties, we decompose μ_j in the following way:

(5.1.9) $$d\mu_j = f_j^{(1)}(x)\,dx + d\mu_j^{(2)}, \quad j = 1, 2, \ldots, n,$$

where

(5.1.10) $$f_j^{(1)}(x) = \begin{cases} f_j(x) & \text{if} \quad x \in \boldsymbol{R}^n - A_\lambda^j, \\ \dfrac{1}{|I_k^j|}\displaystyle\int_{I_k^j} d\tau_j = m_k^j & \text{over} \quad I_k^j. \end{cases}$$

Call $\varphi_k^j(x)$ the characteristic function of I_k^j. Then we define $d\mu_j^{(2)}$ in the following way:

(5.1.11) $$d\mu_j^{(2)} = d\mu_j - \Big(\sum_{k=1}^{\infty} m_k^j \varphi_k^j(x)\Big)dx.$$

Altogether we have the following properties for $f_j^{(1)}(x)$:

(5.1.12) $$\begin{aligned} &|f_j^{(1)}| \leqslant C\lambda^r \text{ a.e.}, \\ &\|f_j^{(1)}\|_1 < C, \end{aligned} \qquad j = 1, 2, \ldots, n.$$

Here C depends only on the dimension, once we know that $\sum_{j=1}^{n} \nu_j(\boldsymbol{R}^n) = 1$, $j = 1, 2, \ldots, n$.

For $\mu_j^{(2)}$, the following holds:

(5.1.13) $\mu_j^{(2)}$ lives on A_λ^j.

(5.1.14) $$\int_{I_k^j} d\mu_j^{(2)} = 0, \quad k = 1, 2, \ldots, \quad j = 1, 2, \ldots, n.$$

(5.1.15) $$\int_{I_k^j} d\nu_j^{(2)} \leqslant C\lambda^r |I_k^j|, \quad k = 1, 2, \ldots, \quad j = 1, 2, \ldots, n.$$

Here $d\nu_j^{(2)}$ stands for the variation of $d\mu_j^{(2)}$.

We are going to decompose now $F_j(x) = F_j^{(1)}(x) + F_j^{(2)}(x)$.

(5.1.16) $$\begin{aligned} F_j^{(1)}(x) &= C_n \int_{\boldsymbol{R}^n} \frac{x_j - y_j}{|x-y|^n} f_j^{(1)}(y)\,dy, \\ F_j^{(2)}(x) &= C_n \int_{\boldsymbol{R}^n} \frac{x_j - y_j}{|x-y|^n}\, d\mu_j^{(2)}. \end{aligned}$$

The set A_λ is going to be defined as

(5.1.17) $$A_\lambda = \bigcup_{j=1}^{n}\bigcup_{k=1}^{\infty} 10 I_k^j,$$

where $10 I_k^j$ denotes the dilation of I_k^j about its center by the factor 10.

It follows that

(5.1.18) $$|A_\lambda| < \frac{C}{\lambda^r}.$$

5.2. *The sets S_λ^j and the functions $\Delta^j(x)$.* Let us denote by ε_k^j and t_k^j, respectively, the diameter and the center of I_k^j. The functions $\Delta^j(x)$ are defined in the following way:

(5.2.1) $$\Delta^j(x) = \sum_{k=1}^{\infty} \frac{\varepsilon_k^j}{(\varepsilon_k^j)^{n+1} + |x - t_k^j|^{n+1}} |I_k^j|.$$

Call S_λ^j the set $\{x;\ \Delta^j(x) > 1\}$. As in Theorem 1, it follows that

(5.2.2) $$|S_\lambda^j| < \frac{C}{\lambda^r}.$$

Calling $S_\lambda = \bigcup_{j=1}^{n} S_\lambda^j$, we have

(5.2.3) $$|S_\lambda| < \frac{C}{\lambda^r}.$$

5.3. *The set B_λ.* The set B_λ is defined in the following way:

(1) B_λ is open, and in $\boldsymbol{R}^n - B_\lambda$ we have

(5.3.1) $$\operatorname*{Sup}_{I_x}\left(\frac{1}{|I_x|}\int_{I_x}\left|\frac{F(x)-F(y)}{\delta(I)}\right|^{n/(n-1)} dy\right)^{(n-1)/n} < \lambda^r.$$

Here $\delta(I)$ denotes diam (I_x), the I_x have the same meaning as in Lemma 3.2.

(2) The measure of B_λ does not exceed

(5.3.2) $$\frac{C}{\lambda^r}.$$

Whitney's covering lemma ensures the existence of B_λ with C in (5.3.2) depending on the dimension only.

5.4. *The sets J_λ^j.* Let us return to the functions $F_j^{(1)}, j = 1, 2, \ldots, n$ given by the convolution operators

(5.4.1) $$F_j^{(1)}(x) = C_n \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^n} f_j^{(1)}(y)\,dy.$$

Notice that $\left(\frac{\partial}{\partial x_i} F_j^{(1)}\right)^\wedge = C\frac{x_i x_j}{|x|^2}\hat{f}_j^{(i)}$, where C depends on n only. Since $\frac{x_i x_j}{|x|^2}$ is C^∞ in $\mathbf{R}^n - \{0\}$ and homogeneous of degree 0, it is a multiplier for p such that $1 < p < \infty$.

Hence

(5.4.2) $$\left\|\frac{\partial F_i^{(1)}}{\partial x_j}\right\|_p < C_p \|f_i^{(i)}\|_p, \quad \text{for} \quad 1 < p < \infty,\ i, j = 1, 2, \ldots, n.$$

Consider now the Mary Weiss maximal operators:

(5.4.3) $$\overset{*}{f}{}_j^{(1)}(x) = \operatorname*{Sup}_{y \in \mathbf{R}^n} \frac{|F_j^{(1)}(x) - F_j^{(1)}(y)|}{|x-y|}.$$

The sets J_λ^j are defined to be:

(5.4.4) $$J_\lambda^j = \{x;\ \overset{*}{f}{}_j^{(1)}(x) > \lambda^r).$$

It follows from [6] (Lemma 1.4, p. 144) that

(5.4.5) $$|J_\lambda^j| < \frac{C_m}{\lambda^{rm}} \int_{\mathbf{R}^n} |f_j^{(1)}|^m dy \quad \text{for} \quad m > n.$$

Using now the fact that $f_j^{(1)} < C\lambda^r$, we get from (5.4.5)

(5.4.6) $$|J_\lambda^j| < \frac{C}{\lambda^r},$$

where C does not depend on λ or F. Let us define J_λ to be the union

(5.4.7) $$J_\lambda = \bigcup_{j=1}^{n} J_\lambda^j.$$

Thus

(5.4.8) $$|J_\lambda| < \frac{C}{\lambda^r}.$$

5.5. *The set G_λ.* The set G_λ is defined to be open and satisfying

(5.5.1) $$G_\lambda \supset A_\lambda \cup S_\lambda \cup B_\lambda \cup J_\lambda$$

and also

(5.5.2) $$|G_\lambda| < 2(|A_\lambda| + |S_\lambda| + |B_\lambda| + |\mathfrak{J}_\lambda|).$$

Consequently,

(5.5.3) $$|G_\lambda| < \frac{C}{\lambda^r},$$

where C does not depend on F or λ.

5.6. *Properties associated with G_λ.* Call $C(G_\lambda)$ the complement of G_λ; then if $x_i \in C(G_\lambda)$, $i = 1, 2$, we have

(5.6.1) $$|F_j^{(1)}(x_1) - F_j^{(1)}(x_2)| < C\lambda^r |x_1 - x_2|.$$

The above inequality follows from the definition of J_λ.

On the other hand, from Lemma 3.1 and the definitions of $\Delta^j(x)$, we have

(5.6.2) $$|F_j^{(2)}(x_1) - F_j^{(2)}(x_2)| \leqslant C\lambda^r\left(\Delta^j(x_1) + \Delta^j(x_2)\right)|x_1 - x_2| < C\lambda^r |x_1 - x_2|.$$

From (5.6.1) and (5.6.2) we get

(5.6.3) $$|F(x_1) - F(x_2)| < C\lambda^r |x_1 - x_2|,$$

where C does not depend on λ or F.

5.7. *Then behavior of F on G_λ.* Let $\bigcup_1^\infty Q_k$ be a Whitney's covering for G_λ. Let y_k, δ_k have the same meaning as in 4.4.

Since $B_\lambda \subset G_\lambda$, we have

(5.7.1) $$\left(\frac{1}{|Q_k|}\int_{Q_k} \left|\frac{F(y_k) - F(y)}{\delta_k}\right|^{n/(n-1)} dy\right)^{(n-1)/n} \leqslant C\lambda^r, \quad k = 1, 2, \ldots$$

The argument is the same as in 4.5.

5.8. *Outline of the proof.* The exceptional set is $6G_\lambda$; has the usual meaning. Thus

(5.8.1) $$|6G_\lambda| < \frac{C}{\lambda^r}.$$

Consider $g \in L^q$ and $\|g\|_q = 1$. As in Theorem 1, we decompose $g = g_1 + g_2$, where $g = g_1$ over $C(G_\lambda)$ and zero otherwise and $g_2 = g - g_1$. $\tilde{F}(y)$ stands for the Lipschitz extension of $F(y)$ from $C(G_\lambda)$ to the whole space. Clearly, we have

(5.8.2) $$|\tilde{F}(y_1) - \tilde{F}(y_2)| < C\lambda^r |y_1 - y_2|.$$

Now, for $x \in \boldsymbol{R}^n - 6G_\lambda$, we have

(5.8.3) $$T_\varepsilon(F, g)(x) = T_\varepsilon(\tilde{F}, g_1)(x) + T_\varepsilon(F, g_2)(x).$$

Consequently

(5.8.4) $$\overset{*}{T}(F, g) \leqslant \overset{*}{T}(\tilde{F}, g_1) + \overset{*}{T}(F, g_2).$$

$\overset{*}{T}(\tilde{F}, g)$ is handled in the same way as $\overset{*}{T}(\tilde{F}_2, g_1)$ in Theorem 1. As in Theorem 1, we have for $x \in \boldsymbol{R}^n - 6G_\lambda$

(5.8.5) $$\overset{*}{T}(F, g_2) \leqslant \overset{*}{T}(\tilde{F}, g_2) + M(\tilde{F}, g) + M(F, g).$$

The operators M are handled in the same way as in Theorem 1. The only difference is that $s = n/(n-1)$ and l is given by $1/l = (n-1)/n + 1/q$. On the other hand, $l = 1$ if $q = n$.

The general case when $\nu_j(\boldsymbol{R}^n) \neq 1$ and $\|g\|_q \neq 1$, is obtained from the preceeding one applied to $\dfrac{1}{\sum \nu_j(\boldsymbol{R}^n)} F(y)$ and $\|g\|_q^{-1} g$. This finishes the proof for the part (ii).

5.9. *The pointwise convergence.* Let us choose λ large and define

(5.9.1) $$H(y) = \sum_1^\infty \{F(y) - F(y_k)\} g(y) \psi_k(y),$$

(5.9.2) $$\tilde{H}(y) = \sum_1^\infty \{\tilde{F}(y) - \tilde{F}(y_k)\} g(y) \psi_k(y),$$

where the $\psi_k(y)$'s are the characteristic functions of cubes Q_k. Repeating the construction of the preceding paragraph, we have for $x \in \boldsymbol{R}^n - 6G_\lambda$

(5.9.3) $$T_\varepsilon(F, g) = T_\varepsilon(\tilde{F}, g_1) + T_\varepsilon(\tilde{F}, g_2) + \int\limits_{|x-y|>\varepsilon} K(x-y) H(y)\, dy + \int\limits_{|x-y|>\varepsilon} K(x-y) \tilde{H}(y)\, dy.$$

$T_\varepsilon(\tilde{F}, g_1)$ and $T_\varepsilon(F, g_2)$ converge a.e. as $\varepsilon \to 0$. On the other hand, keeping the notation of Theorem 1, we have for $x \in \boldsymbol{R}^n - 6G_\lambda$

(5.9.4) $$\int\limits_{\boldsymbol{R}^n} |K(x-y)|\,|H(y)|\, dy \leqslant C\overline{K}(h)(x), \qquad \int\limits_{\boldsymbol{R}^n} |K(x-y)|\,|\tilde{H}(y)|\, dy \leqslant C\overline{K}(\tilde{h})(x).$$

Thus, the third and fourth terms of (5.9.3) are absolutely convergent integrals for a.e. x in $\boldsymbol{R}^n - 6G_\lambda$. Since λ could be chosen arbitrarily large, the pointwise convergence a.e. follows. This finishes the proof of Theorem 2.

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