Bases for $C(K)$ spaces are not in general B-perturbable for any B. Warren [7] and Wojtaszczyk [8] have shown the existence of a normalized basis $\{f_n\}$ for $C([0,1])$ which is weakly convergent to 0. Warren's construction provides an example of a basis for $C([0,1])$ which is not B-perturbable for any $B > 0$.

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Subspaces of smooth sequence spaces

by

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Abstract. This work is concerned with subspaces of nuclear Fréchet smooth sequence spaces. Particular attention is paid to those subspaces which are isomorphic to power series spaces.

The investigation of all infinite-dimensional subspaces of nuclear power series spaces of finite and infinite types is the subject of two important papers of Dubinsky [6], [7]. The earlier works of Rolewicz [12] and Zahariuta [16] were concerned, to some extent, with subspaces of power series spaces. The concepts of smooth sequence spaces of finite and infinite types were introduced in [13] as a generalization of the notion of power series spaces and nuclearities based on such spaces were briefly studied in [4]. The present paper is basically concerned with subspaces of nuclear Fréchet smooth sequence spaces.

In Section 1 we collect the necessary definitions and in Section 2 obtain some properties of block basic sequences with respect to the canonical basis of nuclear Köthe spaces. Section 3 is on basic sequences in $\Lambda_1(\alpha)$ and G_∞-subspaces of $\Lambda_1(\alpha)$ and G_1-subspaces of $\Lambda_1(\alpha)$. In particular, it is proved that if a G_1-space is isomorphic to a subspace of $\Lambda_1(\alpha)$, then it is isomorphic to a power series space of finite type (Theorem 7). In Section 4 we study subspaces of G_∞-spaces; the subspaces considered are power series spaces of infinite type or $L_f(b, \infty)$ spaces of Dragilev [3] or G_∞-spaces. Zahariuta [16] showed earlier that an $L_f(b, \infty)$ space is either isomorphic to a power series space of infinite type or has no subspace isomorphic to a power series space. We show that this result does not extend to general G_∞-spaces. We also give examples of G_∞-spaces which do not contain subspaces isomorphic to power series spaces while these G_∞-spaces are themselves isomorphic to subspaces of each nuclear power series space $\Lambda_\infty(\beta)$ which is stable.

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1. Preliminaries. We refer the reader to [4], [9], [10], and [11] for terms that are not defined here.

Smooth sequence spaces were introduced in [13] as a generalization of the notion of power series spaces without the countability assumptions made in the definitions that follow. In this paper a countable Köthe set $A = \{(a_n^k)\}$ is called a G_∞*-set* and the corresponding Köthe space $\lambda(A)$ a G_∞*-space* if A satisfies:

(1) $a_n^1 = 1$ and $a_n^k \leqslant a_{n+1}^k$ for each k and n;

(2) $\forall k \exists j$ with $(a_n^k)^2 = O(a_n^j)$.

The nuclearity of such a space is equivalent to $\sum 1/a_n^2 < \infty$.

A countable Köthe set $B = \{(b_n^k)\}$ is called a G_1*-set* and $\lambda(B)$ a G_1-*space* if

(1) $0 < b_{n+1}^k \leqslant b_n^k < 1$ for each k and n;

(2) $\forall k \exists j$ with $b_n^k = O((b_n^j))$.

The nuclearity of $\lambda(B)$ is equivalent to $B \subset l_1$.

W. Robinson has proved that a Köthe space is a G_1-space (resp. G_∞-space) if and only if it is regular and (d_1) (resp. regular and (d_2)) in the terminology of Dragilev [3]. An indication of the proof for the case of G_∞-spaces is given in Section 4.

Let f be a positive, log-convex function on $[0, \infty)$ and let (b_n) be an increasing sequence of positive real numbers with $\lim b_n = \infty$. Then $L_f(b, \infty)$ is the Köthe space $\lambda(P)$, $P = \{(e^{f(kb_n)})\colon k = 1, 2, \ldots\}$. This is a G_∞-space and the particular choice of $f(u) = u$ gives the power series space $\Lambda_\infty(b_n)$ [3].

Considering only nuclear G_1 or G_∞-spaces $\lambda(A)$, the space $\lambda(A)$ is said to be *stable* if it is isomorphic to $\lambda(A) \times \lambda(A)$ and *multiplicatively stable* if $\lambda(A)$ is isomorphic to the complete (projective) tensor product $\lambda(A) \otimes \lambda(A)$. The G_∞-space $\lambda(A)$ is stable if and only if for each k there is a j such that $a_{2n}^k = O(a_n^j)$ and multiplicatively stable if and only if for each k there is a j such that $a_{n^2}^k = O(a_n^j)$ (see [14]). The power series space $\Lambda_\infty(a)$ is stable if and only if $a_{2n} = O(a_n)$ (see [4]).

A sequence (x_n) in a nuclear Fréchet space X is a *basis* if each $x \in X$ has a unique representation of the form $x = \sum t_n x_n$. A sequence in X is called a *basic sequence* if it is a basis for the closed subspace it generates. A sequence (y_n) is called a *block basic sequence* with respect to a basis (x_n) in X if there is an increasing sequence (p_n) of positive integers such that

$$y_n = \sum_{p_{n-1}+1}^{p_n} t_i^n x_i \neq 0$$

holds for all n.

Two bases (x_n) and (y_n) of Fréchet spaces X and Y are said to be *equivalent* if there exists an isomorphism $T\colon X \to Y$ with $Tx_n = y_n$ for all n. The bases (x_n) and (y_n) are said to be *semi-equivalent* if there exist scalars $d_n \neq 0$ such that the map which sends each x_n to $d_n y_n$ is an isomorphism. The bases above are *quasi-equivalent* if there is a permutation π of N so that (x_n) and $(y_{\pi(n)})$ are semi-equivalent. All bases of a nuclear smooth sequence space are quasi-equivalent [1].

Throughout we will have the following conventions. The Köthe spaces considered are all *nuclear*. If (y_n) is a basic sequence in X, then Y denotes the closure of the span of (y_n). π is always a permutation of the set of positive integers N. All the Fréchet spaces considered are assumed to admit a continuous norm and therefore the space ω of all sequences of scalars is excluded from our considerations.

A basis (x_n) in a nuclear Fréchet space X is said to be *regular* if there is a fundamental system of norms $(\|\ \|_k)$ such that

$$\frac{\|x_n\|_{k+1}}{\|x_n\|_k} \leqslant \frac{\|x_{n+1}\|_{k+1}}{\|x_{n+1}\|_k} \qquad \forall n, k \in N.$$

(x_n) is said to be *of type* (d_3) if there is a fundamental system of norms $(\|\ \|_k)$ such that

$$\forall k \exists \varepsilon_k > 0 \text{ with } \varepsilon_k \frac{\|x_n\|_{k+1}}{\|x_n\|_k} \leqslant \frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}}, \qquad n \in N.$$

For details see Dragilev [3] and Dubinsky [8].

2. A fundamental inequality. As is well known, the basis theorem for nuclear spaces states that every basis of a nuclear Fréchet space is equivalent to the canonical basis of a suitable Köthe space (see [10]). If (y_n) is a basic sequence in a Köthe space $\lambda(A)$, then Y is isomorphic to the Köthe space generated by the Köthe set $\{(|y_n|_k)\colon k = 1, 2, \ldots\}$. Using the nuclearity of $\lambda(A)$, the norms

$$|y_n|_k = \sum |t_i^n| a_i^k$$

can be replaced by the equivalent system of norms

$$\|y_n\|_k = \max_i |t_i^n| a_i^k,$$

where t_i^n is the ith coordinate of y_n in its expansion in terms of the canonical basis (e_i) of $\lambda(A)$. Following Dubinski [5], [6] we define

$$q(k, n) = q^k(y_n) = \max\{q\colon \|y_n\|_k = |t_q^n| a_q^k\}.$$

We then have

$$\|y_n\|_k \geqslant |t_{q(k+j,n)}^n| a_{q(k+j,n)}^k = \|y_n\|_{k+j} (a_{q(k+j,n)}^k / a_{q(k+j,n)}^{k+j}).$$

Similarly, we have

$$\|y_n\|_{k+j} \geqslant |t_{q(k,n)}^n| a_{q(k,n)}^{k+j} = \|y_n\|_k (a_{q(k,n)}^{k+j} / a_{q(k,n)}^k).$$

Combining these two we obtain the following fundamental inequality:

$$\frac{a^k_{q(k+j,n)}}{a^{k+j}_{q(k+j,n)}} \leqslant \frac{\|y_n\|_k}{\|y_n\|_{k+j}} \leqslant \frac{a^k_{q(k,n)}}{a^{k+j}_{q(k,n)}}. \tag{1}$$

This inequality gives, for instance, the following result:

PROPOSITION 1. *If (y_n) is a block basic sequence with respect to a regular basis of a nuclear Fréchet space, then (y_n) is also a regular basis of Y.*

Proof. We can assume that the nuclear Fréchet space is $\lambda(A)$ and (a^k_n/a^{k+1}_n) is a decreasing sequence of n, because of the basis theorem and regularity of the basis. Since

$$y_n = \sum_{p_{n-1}+1}^{p_n} t^n_i e_i,$$

where $0 = p_0 < p_1 < p_2 < \ldots$, we have $p_{n-1} < q(k, n) \leqslant p_n$ and so $q(k+1, n) \leqslant q(k, n+1)$. Therefore, from the fundamental inequality and regularity we obtain

$$\frac{\|y_n\|_k}{\|y_n\|_{k+1}} \geqslant \frac{a^k_{q(k+1,n)}}{a^{k+1}_{q(k+1,n)}} \geqslant \frac{a^k_{q(k,n+1)}}{a^{k+1}_{q(k,n+1)}} \geqslant \frac{\|y_{n+1}\|_k}{\|y_{n+1}\|_{k+1}}.$$

The characterization of subspaces of (s), the space of rapidly decreasing sequences, has been the subject of two recent important papers by Dubinsky [8] and Vogt [15]. Vogt (Satz 2.1., [15]) has shown that a nuclear Fréchet space X is isomorphic to a subspace of (s) if and only if there exists a sequence $(\|\ \|_k)$ of norms defining the topology of X such that there is a $C > 0$ with

$$\forall k \exists j \text{ with } \|x\|_k^2 \leqslant C\|x\|_1\|x\|_j, \quad x \in X.$$

Dubinsky's characterization is more restrictive in the sense that the subspaces have bases. Vogt (loc. cit.) has shown that the above mentioned condition is equivalent to (d_3)-condition when X has a basis. When X has a basis, considering $X = \lambda(A)$ and dividing each a^k by a^1 we have that $\lambda(A)$ satisfies (d_3) if $a^1_n = 1$ for each n and

$$\forall k \exists j \text{ with } (a^k_n)^2 = O(a^j_n).$$

If we assume in addition that $a^k_n \leqslant a^k_{n+1}$ for all k and n, then $\lambda(A)$ is a G_∞-space.

PROPOSITION 2. *Let (y_n) be a basic sequence in $\lambda(A)$ and assume $\lambda(A)$ satisfies (d_3)-condition. Then (y_n) is semi-equivalent to the canonical basis of $\lambda(A_0)$, where*

$$A_0 = \{(a^k_{q(k,n)}) \colon k = 1, 2, \ldots\}.$$

Proof. For a given k find j with $(a^k_n)^2 = O(a^{k+j}_n)$. Then, by (1),

$$a^k_{q(k,n)}\|y_n\|_1 \leqslant a^k_{q(k,n)}\|y_n\|_k \leqslant \varrho' \frac{a^{k+j}_{q(k,n)}}{a^k_{q(k,n)}} \|y_n\|_k = O(\|y_n\|_{k+j}).$$

Thus for each k there exists a j with

$$a^k_{q(k,n)} = O\left(\frac{\|y_n\|_{k+j}}{\|y_n\|_1}\right).$$

On the other hand, for each k we have again from (1)

$$\frac{\|y_n\|_1}{\|y_n\|_k} \geqslant \frac{1}{a^k_{q(k,n)}}$$

and so

$$\|y_n\|_k = O(\|y_n\|_1 a^k_{q(k,n)}).$$

This completes the proof.

Note that if (y_n) is a block basic sequence in $\lambda(A)$, then $q(k, n+1) \geqslant q(k, n)$ and so we obtain

COROLLARY 3. *If (y_n) is a block basic sequence with respect to the canonical basis in a G_∞-space, then Y is also a G_∞-space.*

Thus we see that the subspace spanned by a block basic sequence with respect to the canonical basis in a power series space $\Lambda_\infty(\alpha)$ is isomorphic to a G_∞-space. Dubinsky ([8], Problem 3) has given an example of a G_∞-space which is not isomorphic to a subspace of (s) spanned by a block basic sequence. The following result illuminates this phenomenon. $\boldsymbol{K}$ denotes the field.

THEOREM 4. *Let Y be a subspace of $\Lambda_\infty(\alpha)$ spanned by a block basic sequence. If $Y \times \boldsymbol{K}$ is isomorphic to Y, then Y is isomorphic to a power series space of infinite type.*

Proof. By Proposition 2 we may assume that Y is equal to the G_∞-space $\lambda(A_0)$, where

$$A_0 = \{(k^{\alpha_{q(k,n)}}) \colon k = 1, 2, \ldots\}.$$

Since $\lambda(A_0)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$, from consideration of diametral dimensions ([13]) we have $\Lambda_\infty(\alpha)' \subset \lambda(A_0)'$ and hence

$$\alpha_n = O(\alpha_{q(k_0,n)})$$

for some integer k_0. The assumption that $\lambda(A_0)$ is isomorphic to $\lambda(A_0) \times \boldsymbol{K}$ implies the existence of a j_0 satisfying

$$\alpha_{q(k_0,n+1)} = O(\alpha_{q(j_0,n)}).$$

Since $q(j,n) \leqslant q(k_0, n+1)$ for all j and n, we have for each m, $a_{q(m,n)} \leqslant a_{q(k_0,n+1)} = O(a_{q(j_0,n)})$. If we now pick $\beta_n = a_{q(j_0,n)}$, we see that $\lambda(A_0) \cong \Lambda_\infty(\beta_n)$.

Remark. The above result shows that if $\lambda(P)$ is a G_∞-space which is *not* a power series space and if $\lambda(P)\times \boldsymbol{K} \cong \lambda(P)$, then $\lambda(P)$ cannot be isomorphic to a subspace Y of any power series space of infinite type spanned by a block basic sequence. The assumption that $Y\times\boldsymbol{K}$ is isomorphic to Y is essential in this, cf. Proposition 13. On the other hand, it is known that $\lambda(P)$ is isomorphic to a subspace of (s) spanned by a block basic sequence with respect to a *permutation* of the canonical basis of (s); for details see Dubinsky [8].

3. Basic sequences in $\Lambda_1(a)$. Let (a_n) be a nuclear exponent sequence of finite type and let $\lambda_n = e^{-a_n}$. Then the Köthe set $\{(\lambda_n)^{1/k}\colon k = 1, 2, \ldots\}$ generates the power series space $\Lambda_1(a)$. So if (y_n) is a basic sequence in $\Lambda_1(a)$ and $q(k,n) = q^k(y_n)$ as before, then our fundamental inequality becomes

$$(\lambda_{q(k+j,n)})^{j/k(k+j)} \leqslant \frac{\|y_n\|_k}{\|y_n\|_{k+j}} \leqslant (\lambda_{q(k,n)})^{j/k(k+j)}. \tag{2}$$

We first discuss briefly the problem of embedding a G_∞-space in $\Lambda_1(a)$ and start with a somewhat technical necessary condition for this.

PROPOSITION 5. *Let (y_n) be a basic sequence in $\Lambda_1(a)$ such that Y is isomorphic to a G_∞-space $\lambda(P)$. Then there exists a permutation π such that*

$$\forall k \exists j \text{ with } \exp(k\,a_{q(k,n)}) = O(p^j_{\pi(n)})$$

and

$$\forall j \exists s \text{ with } p^j_{\pi(n)} = O\left(\exp\left(\frac{s-1}{s}\,a_{q(s,n)}\right)\right).$$

Proof. Since all bases of a G_∞-space are quasi-equivalent ([1]), there is a permutation π and scalars $d_n > 0$ so that the Köthe sets $\{(d_n\|y_n\|_k)\colon k = 1, 2, \ldots\}$ and $\{(p^k_{\pi(n)})\colon k = 1, 2, \ldots\}$ are equivalent. So, find k_0 with

$$1 = p^1_{\pi(n)} = O(d_n\|y_n\|_{k_0}).$$

Now, using inequality (1) with $k = j$, we get for a given k

$$d_n\|y_n\|_k \exp\left(\frac{1}{2k}\,a_{q(k,n)}\right) = O(d_n\|y_n\|_{2k}).$$

We find m with $d_n\|y_n\|_{2k} = O(p^m_{\pi(n)})$ and then j such that $(p^m_n)^{2k^2} = O(p^j_n)$, this being possible since $\lambda(P)$ is a G_∞-space. Then

$$d_n^{2k^2}(\|y_n\|_k)^{2k^2}\exp(ka_{q(k,n)}) = O(p^j_{\pi(n)}).$$

So, if $k \geqslant k_0$, we have

$$\exp(ka_{q(k,n)}) = O(p^j_{\pi(n)}).$$

Since it follows from (2) that for $r \geqslant s$ one has $q(r,n) \geqslant q(s,n)$, we have that the above boundedness condition holds for all k.

Now for the second assertion we first determine j_0 with

$$d_n\|y_n\|_1 = O(p^{j_0}_{\pi(n)}).$$

Let $j \geqslant j_0$ and for this j find m so that $(p^j_n)^2 = O(p^m_n)$ and determine s so that $p^m_{\pi(n)} = O(d_n\|y_n\|_s)$. Then we have

$$p^j_{\pi(n)} = O\left(\frac{\|y_n\|_s}{\|y_n\|_1}\right).$$

Appealing now to (2), we have

$$p^j_{\pi(n)} = O\left(\exp\left((s-1/s)\,a_{q(s,n)}\right)\right).$$

Finally recognizing that there is no loss of generality in assuming $j \geqslant j_0$ we see that the proof is complete.

A consequence of the necessary condition just proved is the following result which exhibits a limitation on how simple the embedding of a G_∞-space into a power series space of finite type can be. This result is an extension of a result of Dubinsky ([6], Theorem 2).

THEOREM 6. *Let $\lambda(P)$ be a G_∞-space which is isomorphic to $\lambda(P)\times\boldsymbol{K}$. Then $\lambda(P)$ cannot be isomorphic to a subspace of any $\Lambda_1(a)$ spanned by a block basic sequence with respect to the canonical basis of $\Lambda_1(a)$.*

Proof. Assume, if possible, that there is a block basic sequence (y_n) in $\Lambda_1(a)$ such that $\lambda(P)$ is isomorphic to Y. Using Proposition 5 and the nuclearity of $\lambda(P)$ we determine j with

$$\exp a_{q(1,n)} = o(p^j_{\pi(n)}),$$

where π is the permutation whose existence was obtained in Proposition 5. Now find m with

$$p^j_{n+1} = o(p^m_n)$$

this being a consequence of the hypothesis $\lambda(P)\times\boldsymbol{K} \cong \lambda(P)$ and of nuclearity. Let

$$N_0 = \{n\colon \pi(n+1) \leqslant \pi(n)+1\}.$$

Then N_0 is an infinite set (cf. [6], p. 270) and so

$$\lim_{n\in N_0} p^j_{\pi(n+1)}/p^m_{\pi(n)} = 0.$$

Using again the previous proposition and nuclearity, determine k so that

$$p^m_{\pi(n)} = o\left(\exp\left(\frac{k-1}{k}\,a_{q(k,n)}\right)\right).$$

We find $n_0 \in N_0$ such that for all n in N_0 and exceeding n_0 we have

$$\log p^m_{\pi(n)} \leqslant \frac{k-1}{k} a_{q(k,n)}$$

and

$$\log p^j_{\pi(n+1)} \leqslant \log p^m_{\pi(n)}.$$

Since (y_n) is a block basic sequence, we also have $q(k,n) \leqslant q(1, n+1)$ and hence combining these we obtain

$$\log p^j_{\pi(n+1)} \leqslant \frac{k-1}{k} a_{q(k,n)} \leqslant \frac{k-1}{k} a_{q(1,n+1)} \leqslant \frac{k-1}{k} \log p^j_{\pi(n+1)}.$$

This contradiction completes the proof.

Our final result in this section is on G_1-spaces isomorphic to subspaces of $\Lambda_1(\alpha)$. As remarked earlier, every nuclear Köthe space satisfying (d_3)-condition is isomorphic to a subspace of the power series space (s); in contrast to this, G_1-subspaces of any power series space of finite type are much more restricted.

THEOREM 7. *If a G_1-space $\lambda(Q)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$, then $\lambda(Q)$ is isomorphic to a power series space of finite type.*

Proof. We assume the existence of a basis sequence (y_n) in $\Lambda_1(\alpha)$ such that Y is isomorphic to $\lambda(Q)$. This means, however, that (y_n) is quasi-equivalent to the canonical basis of $\lambda(Q)$ ([1]). Hence there is a permutation π and $d_n > 0$ such that $(d_n \|y_n\|_k)$ and $(q^j_{\pi(n)})$ are equivalent. Hence we find m and j_0 with

$$d_n \|y_n\|_1 = O(q^m_{\pi(n)}) \quad \text{and} \quad q^m_{n]} = O\big((q^{j_0}_n)^2\big).$$

If k_0 is chosen so that $q^{j_0}_{\pi(n)} = O(d_n \|y_n\|_{k_0})$, then

$$\frac{\|y_n\|_1}{\|y_n\|_{k_0}} = O(q^{j_0}_{\pi(n)}).$$

Now, from (2) and the fact that $q^j_n < 1$ for all n and j, we have

$$\text{(i)} \qquad \exp(-\alpha_{q(k_0,n)}) = O(q^{j_0}_{\pi(n)}).$$

For each fixed j find r and k_j such that

$$q^j_n = O\big((q^r_n)^2\big) \quad \text{and} \quad q^r_{\pi(n)} = O(d_n \|y_n\|_k)$$

for each $k \geqslant k_j$. Then, by (2), we have

$$q^r_{\pi(n)} = O\big(d_n \|y_n\|_{2k} \exp(-\alpha_{q(k,n)}/2k)\big).$$

Since $(d_n \|y_n\|_{2k})$ is dominated by some $(q^s_{\pi(n)})$, it is a bounded sequence. Hence we have shown that for each j we can find k_j such that

$$\text{(ii)} \qquad q^j_{\pi(n)} = O\big(\exp(-\alpha_{q(k,n)}/k)\big)$$

for each $k \geqslant k_j$. We are now ready to show that $\lambda(Q)$ is isomorphic to $\Lambda_1(\beta)$ where $\beta_n = -\log q^{j_0}_n$, j_0 as in (i).

Given $j \geqslant j_0$ we find $k \geqslant k_j$ and $k \geqslant k_0$ so that by (ii) we have

$$(q^j_{\pi(n)})^k = O\big(\exp(-\alpha_{q(k,n)})\big).$$

Since by (2) $\alpha_{q(k,n)} \geqslant \alpha_{q(k_0,n)}$, we get from (i) and the above boundedness condition

$$(q^j_{\pi(n)})^k = O(q^{j_0}_{\pi(n)}).$$

Hence we have shown that for each j there is an integer k with

$$q^j_n = O\big(\exp(-\beta_n)^{1/k}\big)$$

and this completes the proof.

4. G_∞-subspaces of G_∞-spaces. We first point out two useful facts about G_∞-spaces. Assume that $\lambda(A)$ is a G_∞-space and $A = \{(a^k)\}$. If we set $a^{(k)}_n = \log a^k_n$, then $\lambda(A)$ is equal to the intersection of the power series spaces $\Lambda_\infty(a^k)$, $k = 1, 2, \ldots$ Define $b^k_n = a^1_n a^2_n \ldots a^k_n$. Then $\lambda(B)$ is also a G_∞-space. Since $a^k_n \leqslant b^k_n$ and $b^k_n \leqslant (a^k_n)^k = O(a^j_n)$ for some j we have that B is equivalent to A. Moreover, since

$$\frac{b^k_n}{b^{k+1}_n} = \frac{1}{a^{k+1}_n} \geqslant \frac{1}{a^{k+1}_{n+1}} = \frac{b^k_{n+1}}{b^{k+1}_{n+1}},$$

the basis of $\lambda(B)$ is regular. This argument is due to W. Robinson. In view of this we shall assume that the canonical basis of a G_∞-space is also regular.

We first derive a necessary condition for a G_∞-space to be isomorphic to a subspace of another given G_∞-space and later consider special versions of this.

PROPOSITION 8. *If a G_∞-space $\lambda(P)$ is isomorphic to a subspace Y spanned by a basic sequence in a G_∞-space $\lambda(A)$, then there exists a permutation π such that*

$$\forall k \exists j \text{ with } a^k_{q(k,n)} = O(p^j_{\pi(n)})$$

and

$$\forall j \exists s \text{ with } p^j_{\pi(n)} = O(a^s_{q(s,n)}).$$

Proof. By our assumption there is a basic sequence (y_n) in $\lambda(A)$ such that Y is isomorphic to $\lambda(P)$. This means that (y_n) is quasi-equivalent to the canonical basis of $\lambda(P)$. By Proposition 2 we have that $(p^j_{\pi(n)})$ is equivalent to $(d_n a^k_{q(k,n)})$ for some permutation π and $d_n > 0$. Here $q(k,n)$ stands for $q^k(y_n)$ as usual.

First find j_0 so that

$$d_n = d_n a^1_{q(1,n)} = O(p^{j_0}_{\pi(n)}).$$

Now, for a given j, find k so that $p_n^j p_n^{j_0} = O(p_n^k)$ and then find s with $p_{\pi(n)}^k = O(d_n a_{q(s,n)}^s)$. Then

$$d_n p_{\pi(n)}^j \leqslant p_{\pi(n)}^{j_0} p_{\pi(n)}^j = O(d_n a_{q(s,n)}^s).$$

This proves the second assertion.

To prove the first assertion, we set

$$\gamma_n^{k,j} = a_{q(k,n)}^k / p_{\pi(n)}^j$$

and

$$\mu_n^{k,j} = d_n \gamma_n^{k,j}.$$

We assume, contrary to the first assertion, that there exists a k_0 such that for each j $\sup_n \gamma_n^{k_0,j} = \infty$. Let

$$N_j = \{n:\ \gamma_n^{k,j} > j\}.$$

Then each N_j is an infinite set and since $p_{\pi(n)}^j \leqslant p_{\pi(n)}^{j+1}$, $N_{j+1} \subset N_j$. Let n_j be a subsequence of integers with $n_j \in N_j$ and $N_0 = \{n_j:\ j = 1, 2, \ldots\}$. If $j > s$, then

$$\gamma_{n_j}^{k_0,s} \geqslant \gamma_{n_j}^{k_0,j} > j \quad \text{and so} \quad \lim_{n\in N_0} \gamma_n^{k_0,s} = \infty$$

for every s. From regularity of the canonical basis of $\lambda(A)$ we have $q(k+1, n) \geqslant q(k, n)$ ([6]; Lemma 1) and therefore

$$\frac{\gamma_n^{k+1,j}}{\gamma_n^{k,j}} = \frac{a_{q(k+1,n)}^{k+1}}{a_{q(k,n)}^k} \geqslant \frac{a_{q(k,n)}^{k+1}}{a_{q(k,n)}^k} \geqslant 1.$$

Hence

$$\lim_{n\in N_0} \gamma_n^{k,j} = \infty$$

for each $k \geqslant k_0$ and each j. For each k we find $r_k > k$ with $(a_n^k)^2 = O(a_n^{r_k})$. Then

$$a_{q(k,n)}^k \leqslant \varrho \frac{a_{q(k,n)}^{r_k}}{a_{q(k,n)}^k} \leqslant \varrho \frac{a_{q(r_k,n)}^{r_k}}{a_{q(k,n)}^k} = \varrho \frac{\mu_n^{r_k,j}}{\mu_n^{k,j}}$$

and so we obtain

$$\gamma_n^{k,j} = O\left(\frac{\mu_n^{r_k,j}}{p_{\pi(n)}^j \mu_n^{k,j}}\right).$$

For each $k \geqslant k_0$ we find j with $d_n a_{q(r_k,n)}^{r_k} = O(p_{\pi(n)}^j)$. That is $\mu_n^{r_k,j} \leqslant \sigma$ for some σ and all n. Thus

$$\gamma_n^{k,j} = O\left(\frac{1}{p_{\pi(n)}^j \mu_n^{k,j}}\right)$$

for this j depending on k. So

$$\lim_{n\in N_0} p_{\pi(n)}^j \mu_n^{jk} = \lim_{n\in N_0} d_n a_{q(k,n)}^k = 0$$

for each $k \geqslant k_0$. But, on the other hand, we can find s with

$$1 = p_{\pi(n)}^1 = O(d_n a_{q(s,n)}^s).$$

Since for $k \geqslant s$ we have $q(k, n) \geqslant q(s, n)$, this sets up a contradiction.

We now give specialized versions of the above result.

Proposition 9. *If $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of a G_∞-space $\lambda(A)$, then there exists a permutation π, integers i_n and k_0 such that $(\alpha_{\pi(n)})$ is asymptotically equivalent to $(\log a_{i_n}^k)$ for each $k \geqslant k_0$.*

Proof. Preserve the notation in Proposition 8; find k_0 with $2^{\alpha_{\pi(n)}} = O(a_{q(k_0,n)}^{k_0})$ and set $i_n = q(k_0, n)$. For $k \geqslant k_0$ we have $\alpha_{\pi(n)} = O(\log a_{i_n}^k)$. For $k \geqslant k_0$ we find $R > 1$ with $a_{q(k,n)}^k = O(R^{\alpha_{\pi(n)}})$. Since $i_n = q(k_0, n) \leqslant q(k, n)$, we have $\log a_{i_n}^k = O(\alpha_{\pi(n)})$.

Corollary 10. *If the G_∞-space $\lambda(A)$ is such that for each k there exists a $j > k$ with $\lim_n (\log a_n^j / \log a_n^k) = \infty$, then $\lambda(A)$ has no subspace isomorphic to a power series space $\Lambda_\infty(\alpha)$.*

Spaces $\mu(\alpha)$ and $\nu(\alpha)$. Let (α_n) be an infinite type exponent sequence. We define $\mu(\alpha)$ to be the G_∞-space obtained from the Köthe set $\{(\exp k^{\alpha_n}):\ k = 1, 2, \ldots\}$ and $\nu(\alpha)$ to be the G_∞-space obtained from $\{(\exp \alpha_n^k):\ k = 1, 2, \ldots\}$. It follows from the above corollary that neither $\mu(\alpha)$ nor $\nu(\alpha)$ contains any subspace isomorphic to a power series space. However, the spaces $\mu(\alpha)$, $\nu(\alpha)$ have the following interesting property:

Proposition 11. *$\mu(\alpha)$ and $\nu(\alpha)$ are isomorphic to subspaces of every stable power series space $\Lambda_\infty(\beta)$.*

Proof. $\Lambda_\infty(\beta)$ is stable implies that $\beta_n = O(n^k)$ for some $k \geqslant 1$. Now we show that $\mu(\alpha)$ and $\nu(\alpha)$ are both $\Lambda_N(n^k)$-nuclear for each $k \geqslant 1$. Since the diametral dimension of $\nu(\alpha)$ is contained in the diametral dimension of $\mu(\alpha)$, it is sufficient to show this only for $\nu(\alpha)$. By nuclearity of $\Lambda_\infty(\alpha)$ we have $\log n = O(\alpha_n)$ and so $n = O(\exp(j\alpha_n))$ for some $j \geqslant 1$. Thus $n^k = O(\exp(\alpha_n)^m)$ for some $m \geqslant 1$ and this means that the G_∞-space $\nu(\alpha)$ is $\Lambda_N(n^k)$-nuclear and so $\Lambda_N(\beta)$-nuclear [11]. M. Alpseymen [2] has proved that a $\Lambda_N(\beta)$-nuclear Köthe space which satisfies (d_3)-condition is isomorphic to a subspace of $\Lambda_\infty(\beta)$.

We now give an application of Proposition 9.

Theorem 12. *Let $\Lambda_\infty(\alpha)$ be isomorphic to $\Lambda_\infty(\alpha) \times K$. If a G_∞-space $\lambda(A)$ has a subspace isomorphic to $\Lambda_\infty(\alpha)$, then $\lambda(A)$ is itself isomorphic to a power series space of infinite type.*

Proof. By Proposition 9 we have π, k_0 and (i_n) with $\log a_{i_n}^k \sim \alpha_{\pi(n)}$ for each $k \geqslant k_0$. Define $r_n = \max\{i_{\pi^{-1}(j)}:\ 1 \leqslant j \leqslant n\}$. (r_n) increases to infinity and further $\log a^k_{r_n} \sim a_n$ if $k \geqslant k_0$. Let j_n be the smallest integer with $n \leqslant r_{j_n}$. Then

$$\frac{\log a_n^{k+1}}{\log a_n^k} \leqslant \frac{\log a^{k+1}_{r_{j_n}}}{\log a^k_{r_{j_n-1}}}.$$

Let $\varrho > 0$ and $\sigma > 0$ be such that

$$\log a^{k+1}_{r_{j_n}} \leqslant \varrho a_{j_n}$$

and

$$\log a^k_{r_{j_n-1}} \geqslant \sigma a_{j_n-1}.$$

Then

$$\frac{\log a_n^{k+1}}{\log a_n^k} = O\left(\frac{a_{j_n}}{a_{j_n-1}}\right).$$

Since $\Lambda_\infty(\alpha)$ is isomorphic to $\Lambda_\infty(\alpha)\times \boldsymbol{K}$ means $a_{n+1}/a_n = O(1)$, we have shown $\log a_n^{k+1} = O(\log a_n^k)$ for $k \geqslant k_0$. So if we let $\beta_n = \log a_n^{k_0}$, we see that $\lambda(A)$ is isomorphic to $\Lambda_\infty(\beta)$.

Finally, we give an application of Corollary 10. It is known [16] that an $L_f(b, \infty)$-space is either isomorphic to a power series space or it has no subspace isomorphic to a power series space. In view of this and Theorem 12 one may ask whether the above result is also true for G_∞-spaces. We now construct *an example of a proper G_∞-space which contains a complemented power series subspace.*

We start with an exponent sequence (a_n) of infinite type with $(a_{n+1}/a_n)\uparrow\infty$ and consider the following infinite matrix:

$$\begin{array}{cccccccc}
a_1 & a_2 & a_4 & a_5 & a_8 & a_9 & a_{13} & \cdots \\
a_1 & a_3 & a_4 & a_6 & a_8 & a_{10} & a_{13} & \cdots \\
\cdot & a_3 & \cdot & a_7 & \cdot & a_{11} & \cdot & \\
\cdot & \cdot & \cdot & a_7 & \cdot & a_{12} & \cdot & \cdots \\
 & & \cdot & \cdot & \cdot & a_{12} & \cdot & \cdots \\
 & & & & & \cdot & \cdot & \cdots
\end{array}$$

Let a_n^k denote the positive integer k raised to the nth entry of the kth row and $A = \{(a_n^k)\colon k = 1, 2, \ldots\}$. It is easy to verify that $\lambda(A)$ is a G_∞-space. The subspace of odd indexed entries of $\lambda(A)$ is isomorphic to the power series space $\Lambda_\infty(\beta)$, where $\beta_1 = a_1$, $\beta_2 = a_4$, $\beta_3 = a_8$ e.t.c. Also $(\log a_{2n}^{k+1}/\log a_{2n}^k)$ is increasing for almost all n and its limit is infinity. Thus the even indexed entries of $\lambda(A)$ form a subspace isomorphic to a G_∞-space $\lambda(A_0)$ which, by Corollary 10, has no power series subspace. Since $\lambda(A)$ is isomorphic to $\Lambda_\infty(\beta)\times\lambda(A_0)$, it is an example of a G_∞-space, which is not a power series space and yet it has a complemented subspace isomorphic to a power series space. In view of Zahariuta's result mentioned above, $\lambda(A)$ is not an $L_f(b, \infty)$-space.

We will supplement the phenomenon described above by showing that such a G_∞-space can be constructed in any power series space of infinite type.

Proposition 1. *Let $\Lambda_\infty(\gamma)$ be any nuclear power series space of infinite type. Then there exists a block basic sequence (y_n) in $\Lambda_\infty(\gamma)$ which is semi-equivalent to the canonical basis of a G_∞-space $\lambda(A)$ which has the property $\lambda(A) \cong \lambda(A_0)\times\Lambda_\infty(\beta)$, where $\lambda(A_0)$ is a G_∞-space which contains no subspace isomorphic to a power series space.*

Proof. Let (a_n) be a subsequence of (γ_n) such that $a_n < a_{n+1}$ and $(a_{n+1}/a_n)\uparrow\infty$. Then $\Lambda_\infty(a)$ is a "Stückraum" of $\Lambda_\infty(\gamma)$ (cf. [9], § 31). So if (y_n) is a block basic sequence with respect to the canonical basis of $\Lambda_\infty(a)$, then it is also a block basic sequence with respect to the canonical basis of $\Lambda_\infty(\gamma)$. Hence, without any loss of generality, we may work in $\Lambda_\infty(a)$. In the notation preceeding this proposition we have $\lambda(A) \cong \lambda(A_0)\times \times\Lambda_\infty(\beta)$. Now, let

$$y_1 = e_1,\quad y_2 = t_2^2 e_2 + t_3^2 e_3,\quad y_3 = e_4,$$
$$y_4 = t_5^4 e_5 + t_6^4 e_6 + t_7^4 e_7,\quad y_5 = e_8\quad \text{etc.}$$

where we shall choose the (t_j^i). Using a lemma of Dubinsky ([6], Lemma 2) choose t_2^2, t_3^2 such that $q^1(y_2) = 2$ and $q^2(y_2) = 3$. If $k \geqslant 2$, then $q^k(y_2) \geqslant q^2(y_2)$ ([6], Lemma 1); also $q^k(y_2) \leqslant 3$ by definition of y_2 and so $q^k(y_2) = 3$ for all $k \geqslant 2$. Therefore $(a_{q^k(y_2)})$ is exactly the second column in the matrix. We can apply Dubinsky's result and choose t_5^4, t_6^4, t_7^4 etc. so that $(a_{q^k(y_{2n})})$ is equal to the $2n$th column of the matrix. By Proposition 2 we have that (y_n) is semi-equivalent to the canonical basis of $\lambda(A)$ constructed before.

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On a singular integral

by

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Abstract. The commutator singular integral

$$\text{p.v.} \int_{\mathbf{R}^n} K(x-y)\{F(x)-F(y)\}\, g(y)\, dy$$

(where $K(x)$ is even, positively homogeneous of degree $-n-1$, integrable over the unit sphere of $\mathbf{R}^n$) is studied when

$$\operatorname{grad} F \in L^p(\mathbf{R}^n), \quad 1 \leqslant p < n, \quad g \in L^q(\mathbf{R}^n), \quad 1 \leqslant 1/p + 1/q \leqslant (n+1)/n.$$

0. Introduction. The purpose of this paper is to extend results in [6]. Let $k(x)$ be positively homogeneous of degree $-n-1$, even and locally integrable in $|x| > 0$. Let $F(x)$ have first order derivatives in the distributions sense in $L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$. Let $g(x)$ be a function in $L^q(\mathbf{R}^n)$, $1 \leqslant q \leqslant \infty$. Assume that r is given by $1/r = 1/p + 1/q$, p and q not infinity simultaneously. Consider now the operator

$$T(F, g) = \text{p.v.} \int_{\mathbf{R}^n} \{F(x) - F(y)\} K(x-y)\, g(y)\, dy. \tag{0.1}$$

It has been shown in [2] that, if $r > 1$, the above limit exits in L^r norm; furthermore, the principal value converges a.e. (see [1]). If $p = \infty$, $r = 1$, $q = 1$, it is shown in [1] that $T(F, g)$ converges a.e. provided that smoothness is assumed on $K(x)$ (for example C^1).

In the paper [6] it is shown that if $r = 1$, p is such that $1 < p < \infty$, then (0.1) exists a.e. and in $L^1(\mathbf{R}^n)$-norm; no smoothness condition is assumed on K. (1) In addition, if the following smoothness condition is assumed on K:

$$\int_{|x| > 4|h|} |K(x+h) - K(x)|\, |x|\, dx < C. \tag{0.2}$$

(1) In a non-published paper *Pointwise estimates for commutator singular integrals* B. Bajsanski and R. Coifman have shown a very similar result, but weak type instead of strong type, and making the following smoothness assumption on the kernel:

$$\int_{|x|=1} |K(x+h) - K(x)|\, d\sigma < C \cdot |h|^{\delta}, \qquad 0 < \delta < 1.$$