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Norm attaining operators*

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Abstract. Let X and Y be Banach spaces. $L(X, Y)$ and $K(X, Y)$ denote respectively the bounded and compact linear operators from X to Y. $\mathrm{NA}(X, Y)$ denotes the set of operators $A\colon X \to Y$ such that $\|Ax_0\| = \|A\|$ for some $x_0 \in X$ with $\|x_0\| = 1$. The main results are: (1) If S and T are compact Hausdorff spaces, $\mathrm{NA}(C(S), C(T))$ is dense in $L(C(S), C(T))$. (2) If X or Y is $C(S)$ or $L^1(\mu)$ then $\mathrm{NA}(X, Y) \cap K(X, Y)$ is dense in $K(X, Y)$.

1. Introduction. We establish some notation to be used in the statement of the central problem and the main results of this paper. The letters X and Y will always denote real Banach spaces. A bounded linear operator $A\colon X \to Y$ is *norm attaining* if there is an $x_0 \in X$ with $\|x_0\| = 1$ such that

$$\|Ax_0\| = \|A\| = \sup\{\|Ax\|\colon x \in X \text{ and } \|x\| \leqslant 1\}.$$

The question considered in this paper is when an operator of a certain class can be approximated by a norm attaining operator of the same class.

The letters S and T will always denote compact Hausdorff spaces, and the (sup norm) Banach space of continuous real-valued functions on S will be denoted by $C(S)$. For a positive measure μ (on some set Ω), $L^1(\mu)$ will denote the Banach space of integrable real-valued functions $f\colon \Omega \to R$ with $\|f\| = \int |f|\,d\mu$.

The main results of this paper are the following:

THEOREM 1. *Let $A\colon C(T) \to C(S)$ be a bounded linear operator where S and T are compact Hausdorff spaces. Then for any $\varepsilon > 0$, there is a norm attaining operator $A'\colon C(T) \to C(S)$ such that $\|A - A'\| \leqslant \varepsilon$.*

An Asplund space is defined in § 2. To put the next theorem in perspective we note that X is an Asplund space iff X^* has the Radon–Nikodym property (one direction is in [12], the other direction is an unpublished result of Stegall).

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THEOREM 2. *Suppose X is an Asplund space and $A: X \to C(S)$ is a bounded linear operator. Then for every $\varepsilon > 0$ there is a norm attaining operator $A': X \to C(S)$ such that $\|A - A'\| \leqslant \varepsilon$.*

An operator is *finite rank* if its range is finite dimensional and *compact* if the image of the unit ball of the domain is precompact.

THEOREM 3. *Suppose A is a compact operator whose domain or range is a $C(S)$ space. Then for every $\varepsilon > 0$ there is a finite rank norm attaining operator A' such that $\|A - A'\| \leqslant \varepsilon$.*

THEOREM 4. *Suppose A is a compact operator whose domain or range is an $L^1(\mu)$ space. Then for every $\varepsilon > 0$ there is a finite rank norm attaining operator A' such that $\|A - A'\| \leqslant \varepsilon$.*

In Theorem 4, the case where the domain of A is an $L^1(\mu)$ space is a result of Diestel and Uhl ([4], p. 6).

These results are motivated by the papers [4] and [11]. Let $L(X, Y)$ and $\mathrm{NA}(X, Y)$ denote respectively all bounded and all norm attaining linear operators from X to Y. In [11] Lindenstrauss investigated the problem of deciding when $\mathrm{NA}(X, Y)$ is norm dense in $L(X, Y)$. Progress on this problem has recently been made by Uhl [14] and Bourgain [2]. Uhl [14] shows that if Y is strictly convex then $\mathrm{NA}(L^1, Y)$ is dense in $L(L^1, Y)$ if and only if Y has the Radon–Nikodym property ($L^1 = L^1[0,1]$) and raises the question(1) whether $\mathrm{NA}(L^1, L^1)$ is dense in $L(L^1, L^1)$. Bourgain [2] shows that if X has the Radon–Nikodym property, then for any Banach space Y, $\mathrm{NA}(X, Y)$ is dense in $L(X, Y)$. (He actually answers problem 11 in [4], p. 26, affirmatively.)

Theorems 1 and 2 come from an attempt to answer such density questions for $C(S)$ spaces. Previously the two main results concerning these spaces were the following from [11]:

(1) If S is any infinite compact metric space, then there is a Banach space Y such that $\mathrm{NA}(C(S), Y)$ is not dense in $L(C(S), Y)$.

(2) If S is a compact Hausdorff space having a dense set of isolated points, then $\mathrm{NA}(X, C(S))$ is dense in $L(X, C(S))$ for any Banach space X.

It still appears to be unknown whether (1) and (2) remain true if S is an arbitrary compact Hausdorff space. Also the density question still seems to be unanswered for certain pairs of classical Banach spaces, e.g., $C[0,1]$ to L^1, and L^1 to $C[0,1]$.

Let $K(X, Y)$, $\mathrm{KNA}(X, Y)$ and $\mathrm{FRNA}(X, Y)$ denote respectively all compact, compact norm attaining and finite rank norm attaining linear operators from X to Y. In [4], p. 6, the problem is raised of deciding if $\mathrm{KNA}(X, Y)$ is dense in $K(X, Y)$ and, in particular, the problem is posed

(1) Added in proof: It was recently shown by A. Iwanik that $\overline{\mathrm{NA}(L', L')} = L(L', L')$ (to appear in Pacific J. Math.).

if X is a $C(S)$ space. Theorems 3 and 4 answer this question affirmatively (in fact we show that $\mathrm{FRNA}(X, Y)$ is dense) if either X or Y is either a $C(S)$ space or an $L^1(\mu)$ space. It still seems to be open whether the answer to this problem is affirmative for any pair of Banach spaces X and Y or, in particular, if X is arbitrary and $Y = L^p$ for $1 < p < \infty$.

In addition to the papers mentioned above, norm attaining operators are discussed in [1], [6] and [7].

In spite of recent progress, quite a number of density questions about norm attaining operators appear to be unresolved. Several such questions are listed in § 4 which contains a further discussion of these problems.

Theorems 1 and 2 are proved in § 2. Theorems 3 and 4 are proved in § 3.

2. Proof of Theorems 1 and 2. In the sequel we identify the dual of $C(K)$ with the space $M(K)$ of regular Borel measures on K. We also use the standard representation theorem for operators into a $C(K)$ space (cf. [5], Theorem 1, p. 490):

LEMMA 2.1. *Given an operator $A: X \to C(S)$ define $\mu: S \to X^*$ by $\mu(s) = A^*(\delta_s)$ where δ_s is the point measure at $s \in S$. Then, for $x \in X$, the relationship $Ax(s) = \langle x, \mu(s)\rangle$ defines an isometric isomorphism between $L(X, C(S))$ and the space of weak* continuous functions from S to X^* with the supremum norm $\|\mu\| = \sup\{\|\mu(s)\|: s \in S\}$. The compact operators correspond to the norm continuous functions.*

For $\nu \in M(T)$, $|\nu|$ denotes the total variation measure. The next lemma is a consequence of the fact that if V is an open subset of the compact Hausdorff space T, then the function from $M(T)$ to the reals defined by $\nu \to |\nu|(V)$ is weak* lower semicontinuous. We omit the routine proof.

LEMMA 2.2. *Let $\mu: S \to M(T)$ be weak* continuous. Let $\varepsilon > 0$, $s_0 \in S$ and an open set V in T be given. Then there is an open neighborhood U of s_0 such that if $s \in U$, then $|\mu(s)|(V) \geqslant |\mu(s_0)|(V) - \varepsilon$.*

LEMMA 2.3. *Let $\mu: S \to M(T)$ be weak* continuous. Then for any $\delta > 0$ there is a weak* continuous $\mu': S \to M(T)$, an open set U in S, an open set V in T and an $h \in C(T)$ with $\|h\| = 1$ such that*

(a) $s \in U \Rightarrow |\mu'(s)|(V) = 0$,

(b) $\int h\mu'(s) \geqslant \|\mu'\| - \delta$ *for* $s \in U$,

(c) $|h(t)| = 1$ *if* $t \in T \setminus V$,

(d) $\|\mu - \mu'\| \leqslant \delta$.

Proof. Choose $s_0 \in S$ such that $\|\mu(s_0)\| \geqslant \|\mu\| - \delta/3$. By the Hahn decomposition theorem for measures and the regularity of $\mu(s_0)$ we can choose disjoint closed sets F^+ and F^- such that $\mu(s_0)$ is positive (negative) on F^+ (F^-) and

$$(*) \qquad \|\mu(s_0)\| \leqslant \mu(s_0)(F^+) - \mu(s_0)(F^-) + \delta/3.$$

CLAIM. There is an open set V in T and an $h \in C(T)$ with $\|h\| = 1$ such that (c) holds, $h(F^+) = 1$ and $h(F^-) = -1$; in addition, there is an open set W in T and a $\psi \in C(T)$ with $\|\psi\| = 1$ such that $F^+ \cup F^- \subset W$, $\psi(W) = 0$ and $\psi(V) = 1$.

The claim follows easily by (1) choosing an $h_0 \in C(T)$ with $h_0(F^+) = 1$ and $h_0(F^-) = -1$, (2) defining $V = \{t\colon -\frac{1}{2} < h_0(t) < \frac{1}{2}\}$ and $W = \{t\colon h_0(t) < -\frac{3}{4}$ or $h_0(t) > \frac{3}{4}\}$ and (3) constructing h and ψ by normality.

Condition (*) implies that

$$\int h(1-\psi)\mu(s_0) \geqslant \|\mu\| - 2\delta/3 \quad \text{and} \quad |\mu(s_0)|(W) \geqslant \|\mu\| - 2\delta/3.$$

So by weak* continuity and Lemma 2.2 choose an open neighborhood U_0 of s_0 such that if $s \in U_0$ then

$$\int h(1-\psi)\mu(s) > \|\mu\| - \delta \tag{**}$$

and

$$|\mu(s)|(W) > \|\mu\| - \delta. \tag{***}$$

Choose $\varphi \in C(S)$ such that $\varphi(U) = 1$ for some neighborhood U of s_0, $\|\varphi\| = 1$ and $\varphi(S \setminus U_0) = 0$. Then $s_0 \in U \subset U_0$. Define $\mu'\colon S \to M(T)$ by $\mu'(s) = \mu(s) - \varphi(s)\psi\mu(s)$. This μ' works. For if $s \in U$, then $\varphi(s) = 1$ and so $\mu'(s) = (1-\psi)\mu(s)$. Since $(1-\psi)(V) = 0$, (a) holds and (b) follows from (**). For (d) note that $\|\mu - \mu'\| = \sup\{|\varphi(s)|\,\|\psi\mu(s)\|\colon s \in S\}$. But if $s \in S \setminus U_0$, then $\varphi(s) = 0$ and if $s \in U_0$, then

$$\|\psi\mu(s)\| \leqslant |\mu(s)|(T \setminus W) \leqslant \|\mu\| - |\mu(s)|(W) \leqslant \delta$$

since ψ vanishes on W. This finishes the lemma.

LEMMA 2.4. *Let $\mu\colon S \to M(T)$ be weak* continuous. Suppose there is an open set $U \subset S$, an open set $V \subset T$, an $s_0 \in U$ and an $h \in C(T)$ with $\|h\| = 1$ such that*

(a) *if $s \in U$, then $|\mu(s)|(V) = 0$,*

(b) $\int h\mu(s_0) \geqslant \|\mu\| - \varepsilon$,

(c) $|h(t)| = 1$ *for* $t \in T \setminus V$.

Then for any $r > \frac{2}{3}$ there is a weak continuous function $\mu'\colon S \to M(T)$ and a point $s_1 \in U$ such that*

(d) $\|\mu - \mu'\| \leqslant r\varepsilon$,

(e) $\|\mu'\| - \int h\mu'(s_1) \leqslant r\varepsilon$,

(f) *if $s \in U$, $|\mu'(s)|(V) = 0$.*

Proof. Fix $\delta > 0$ so that $\frac{2}{3}\varepsilon + \delta \leqslant r\varepsilon$. Choose an open set U^* such that $s_0 \in U^* \subset U$ and if $s \in U^*$ then

$$\int h\mu(s) \geqslant \|\mu\| - \varepsilon - \delta. \tag{*}$$

Let $M = \sup\{\|\mu(s)\|\colon s \in U^*\}$.

Case 1: Suppose $M \leqslant \|\mu\| - \varepsilon/3$. Choose $a \in T \setminus V$ such that $|h(a)| = 1$ and choose $\varphi \in C(S)$ with $\varphi(s_0) = h(a)$, $\|\varphi\| = 1$ and $\varphi(s) = 0$ for $s \in S \setminus U^*$. Define $\mu'(s) = \mu(s) + (\varepsilon/3)\varphi(s)\delta_a$. Then letting $s_1 = s_0$,

$$\|\mu'\| = \|\mu\| \quad \text{and} \quad \|\mu - \mu'\| = \varepsilon/3 < r\varepsilon$$

and

$$\int h\mu'(s_0) = \int h\mu(s_0) + \varepsilon/3 \geqslant \|\mu'\| - \tfrac{2}{3}\varepsilon,$$

so

$$\|\mu'\| - \int h\mu'(s_0) \leqslant \tfrac{2}{3}\varepsilon < r\varepsilon.$$

Case 2: Suppose $M > \|\mu\| - \varepsilon/3$. Then choose $s_1 \in U^*$ so that

$$\|\mu(s_1)\| > \|\mu\| - \varepsilon/3. \tag{**}$$

This case follows from the

CLAIM. There is a $\psi \in C(T)$ with $0 \leqslant \|\psi\| \leqslant 1$ and an open set $W \subset T$ such that

$$|\mu(s_1)|(W) > \|\mu\| - \tfrac{2}{3}\varepsilon - \delta, \qquad \psi(W) = 0$$

and

$$\int (h - h\psi)\mu(s_1) \geqslant \|\mu\| - \tfrac{2}{3}\varepsilon - \delta.$$

We show the lemma follows from this claim. To define $\mu'\colon S \to M(T)$ use Lemma 2.2 to choose an open neighborhood U_0 of s_1 such that if $s \in U_0$, then $|\mu(s)|(W) > \|\mu\| - \frac{2}{3}\varepsilon - \delta$. Choose $\varphi \in C(S)$ with $\varphi(s_1) = 1$, $0 \leqslant \|\varphi\| \leqslant 1$ and $\varphi = 0$ on $S \setminus U_0$. Define

$$\mu'(s) = \mu(s) - \varphi(s)\psi\mu(s) = [1 - \varphi(s)\psi]\mu(s).$$

Then $\|\mu'(s)\| \leqslant \|\mu(s)\|$ and

$$\|\mu' - \mu\| = \sup\{|\varphi(s)|\,\|\psi\mu(s)\|\colon s \in S\} \leqslant \sup\{|\mu(s)|(\{\psi \neq 0\})\colon s \in U_0\}$$
$$\leqslant \sup\{|\mu(s)|(T \setminus W)\colon s \in U_0\} \leqslant \tfrac{2}{3}\varepsilon + \delta \leqslant r\varepsilon,$$

since $|\mu(s)|(W) \geqslant \|\mu\| - \frac{2}{3}\varepsilon - \delta$. Also

$$\|\mu'\| - \int h\mu'(s_1) \leqslant \tfrac{2}{3}\varepsilon + \delta \leqslant r\varepsilon,$$

since $\int h\mu'(s_1) = \int (h - h\psi)\mu(s_1) \geqslant \|\mu\| - \frac{2}{3}\varepsilon - \delta$. Condition (f) is trivial and the lemma follows from the claim.

Proof of claim. By the Hahn Decomposition Theorem there are disjoint measurable sets E^+ and E^- whose union is T such that $\mu(s_1)$ is positive on E^+ and negative on E^-. Let

$$P = \{h = 1\} \cap E^+ = \{t \in T\colon h(t) = 1 \text{ and } t \in E^+\}$$

$$\text{and} \quad N = \{h = -1\} \cap E^-.$$

Since $|h| = 1$ on $T\setminus V$, the two sets

$$(P\cup N)\setminus V \quad \text{and} \quad [(\{h=1\}\cap E^-)\cup(\{h=-1\}\cap E^+)]\setminus V$$

partition $T\setminus V$ into two disjoint measurable sets (we ignore V since $|\mu(s_1)|(V) = 0$). Since $\mu(s_1)$ is regular, choose two closed sets $F_1 \subset (P\cup N)\setminus\setminus V$ and F_2 contained in the other member of the partition such that if $F = F_1\cup F_2$ then $|\mu(s_1)|((T\setminus V)\setminus F) < \delta/6$. Then $h-\chi_{F_2}h = \chi_P-\chi_N$ on F. Choose the open set W with $F_1 \subset W$ and choose $\psi \in C(T)$ with $0 \leqslant \|\psi\| \leqslant 1$ such that $\psi = 1$ on F_2 and 0 on W. Then $h-\psi h = \chi_P-\chi_N$ on F and thus

$$\int (h-\psi h)\mu(s_1) \geqslant \int (\chi_P-\chi_N)\mu(s_1) - \delta/6 = |\mu(s_1)|(P\cup N) - \delta/6.$$

Thus we are done if we show that

$$(***)\qquad |\mu(s_1)|(P\cup N) \geqslant \|\mu\| - \tfrac{2}{3}\varepsilon - \tfrac{1}{2}\delta,$$

since we get the desired inequality on $\int (h-\psi h)\mu(s_1)$ and also

$$|\mu(s_1)|(W) \geqslant |\mu(s_1)|(F_1) \geqslant |\mu(s_1)|(P\cup N) - \delta/6 \geqslant \|\mu\| - \tfrac{2}{3}\varepsilon - \delta.$$

For (***) note that $|\mu(s_1)|(V) = 0$ and $|h(t)| = 1$ for $t \notin V$ imply

$$\int (\chi_{\{h=1\}} - \chi_{\{h=-1\}})\mu(s_1) = \int h\mu(s_1)$$

and thus by (*) and (**) we get

$$\begin{aligned}
|\mu(s_1)|(P\cup N) &= \int (\chi_P-\chi_N)\mu(s_1)\\
&= \int h\mu(s_1) - \int (\chi_{\{h=1\}\cap E^-} - \chi_{\{h=-1\}\cap E^+})\mu(s_1)\\
&= \int h\mu(s_1) + |\mu(s_1)|\big((E^-\cap\{h=1\})\cup(E^+\cap\{h=-1\})\big)\\
&= \int h\mu(s_1) + |\mu(s_1)|\big(T\setminus(P\cup N)\big)\\
&= \int h\mu(s_1) + |\mu(s_1)|(T) - |\mu(s_1)|(P\cup N)\\
&\geqslant 2\|\mu\| - \tfrac{4}{3}\varepsilon - \delta - |\mu(s_1)|(P\cup N).
\end{aligned}$$

Now (***) follows from this inequality and the lemma is proved.

Proof of Theorem 1. Let $\varepsilon_0 > 0$ and a bounded linear operator $A\colon C(T)\to C(S)$ be given. Let $\frac{2}{3} < r < 1$. Suppose $\mu\colon S\to M(T)$ represents A as in Lemma 2.1. Apply Lemma 2.3 (with $\delta = \varepsilon_0$) to obtain $\mu_0\colon S\to M(T)$, $h \in C(T)$ and open sets $U \subset S$ and $V \subset T$ satisfying (a)–(d) of Lemma 2.3. Choose $s_0 \in U$. Let $\varepsilon = \|\mu_0\| - \int h\mu_0(s_0)$. Then $0 < \varepsilon < \varepsilon_0$ and Lemma 2.4 provides μ_1 and s_1 such that

$$\|\mu_0-\mu_1\| \leqslant r\varepsilon \leqslant r\varepsilon_0 \quad\text{and}\quad \|\mu_1\| - \int h\mu_1(s_1) \leqslant r\varepsilon_0.$$

Proceeding inductively we obtain a sequence $\mu_n\colon S\to M(T)$ of representations of operators and points $s_n \in S$ such that

$$\|\mu_{n-1}-\mu_n\| \leqslant r^n\varepsilon_0 \quad\text{and}\quad \|\mu_n\| - \int h\mu_n(s_n) \leqslant r^n\varepsilon_0.$$

These operators form a Cauchy sequence. Let $\mu' = \lim\mu_n$ and suppose $A'\colon C(T)\to C(S)$ corresponds to μ'. Then

$$A'(h)(s_n) = \int h\mu'(s_n) \geqslant \int h\mu_n(s_n) - \|\mu'-\mu_n\| \geqslant \|\mu_n\| - r^n\varepsilon_0 - \|\mu'-\mu_n\|.$$

Since $r^n\varepsilon_0 + \|\mu'-\mu_n\| \to 0$,

$$\|A'h\| \geqslant \sup Ah(s_n) = \sup\|\mu_n\| = \|\mu'\| = \|A'\|$$

and A' attains its norm at h. Also

$$\|A'-A\| = \|\mu'-\mu\| \leqslant \|\mu-\mu_0\| + \sum_{n=1}^{\infty}\|\mu_n-\mu_{n-1}\| \leqslant \varepsilon_0(1-r)^{-1}.$$

Thus A' can be made as close to A as desired.

Our next objective is to prove Theorem 2 regarding operators from an Asplund space to a $C(S)$ space. Although Asplund spaces were initially defined in terms of Fréchet differentiability of convex functions, for our purposes we will take the following equivalent definition (cf. [12], Lemma 3). A Banach space X is an *Asplund space* if, for every $\varepsilon > 0$ and every non-empty bounded subset $B \subset X^*$, there is a weak* open set $U \subset X^*$ such that $B\cap U \neq \emptyset$ and $\operatorname{diam}(B\cap U) = \sup\{\|x-y\|\colon x, y \in B\cap U\} \leqslant \varepsilon$. Theorem 2 will follow from the next three lemmas of which the first two are quite simple.

LEMMA 2.5. *A bounded linear operator $A\colon X\to Y$ is norm attaining $\Leftrightarrow$ there is a $y^* \in Y^*$ and $x \in X$ with $\|y^*\| = \|x\| = 1$ such that $\|A^*\| = \|A^*y^*\| = A^*y^*(x)$, i.e. A^* is norm attaining and achieves its norm at a norm attaining functional in X^*.*

LEMMA 2.6. *A bounded linear operator $A\colon X\to C(S)$ is norm attaining $\Leftrightarrow$ there is an $s \in S$ such that $\mu(s) = A^*(\delta_s) = \|A\|$ and $\mu(s)$ is norm attaining where μ is as in Lemma 2.1.*

LEMMA 2.7. *Let $A\colon X\to C(S)$ be a bounded linear operator with representation $\mu\colon S\to X^*$ given by Lemma 2.1. Let $\varepsilon > 0$ be given. Suppose there is a norm attaining functional $x^* \in X^*$ with $\|x^*\| = \|A\|$ and a weak* open set $U \subset X^*$ such that $U\cap\mu(S) \neq \emptyset$ and, for every $y^* \in U\cap\mu(S)$, $\|x^*-y^*\| \leqslant \varepsilon$. Then there is a norm attaining bounded linear operator $A'\colon X\to C(S)$ such that $\|A'-A\| \leqslant \varepsilon$.*

Proof. Suppose x^* and U are as given. Choose $s_0 \in S$ with $\mu(s_0) \in U$. Then $W = \{s \in S\colon \mu(s) \in U\}$ is an open set containing s_0. Urysohn's Lemma provides a function $\varphi \in C(S)$ such that $\varphi(s_0) = 1$, $0 \leqslant \varphi \leqslant 1$ and $\varphi = 0$ on $S\setminus W$. Define $\mu'\colon S\to X^*$ by $\mu'(s) = \varphi(s)x^* + \big(1-\varphi(s)\big)\mu(s)$. Then

$$\|\mu(s)-\mu'(s)\| = \varphi(s)\|x^*-\mu(s)\| \leqslant \varepsilon \quad \text{for all } s \in S,$$

$$\|\mu'(s)\| \leqslant \varphi(s)\|x^*\| + \big(1-\varphi(s)\big)\|\mu(s)\| \leqslant \|\mu\| \quad \text{for all } s \in S,$$

and

$$\|\mu'(s_0)\| = \|x^*\| = \|A\| = \|\mu'\|.$$

Thus μ' represents a norm attaining operator by Lemma 2.6.

Proof of Theorem 2. Suppose $A: X \to C(S)$ is a bounded linear operator and let $\varepsilon > 0$ be given. Let $\mu: S \to X^*$ represent A. Since $\|A\| = \sup\{\|\mu(s)\|: s \in S\}$, we can choose $s_0 \in S$ and $x_0 \in X$ such that $\langle x_0, \mu(s_0)\rangle > \|A\| - \varepsilon/2$. Let $U_1 = \{x^* \in X^*: \langle x_0, x^*\rangle > \|A\| - \varepsilon/2\}$. Then $U_1 \cap \mu(S)$ is nonempty and bounded. Since X is an Asplund space, we can choose a weak* open set $U_2 \subset X^*$ such that $\operatorname{diam}(U_2 \cap U_1 \cap \mu(S)) \leqslant \varepsilon/2$ and $U_2 \cap U_1 \cap \mu(S)$ is nonempty. Let $U = U_1 \cap U_2$ and let $x_0^* \in U \cap \mu(S)$. Then $\|A\| \geqslant \|x_0^*\| \geqslant \langle x_0, x_0^*\rangle > \|A\| - \varepsilon/2$. So by the Bishop–Phelps Theorem [1] there is a norm attaining functional x^* such that $\|x^*\| = \|A\|$ and $\|x^* - x_0^*\| \leqslant \varepsilon/2$. Then x^* and U satisfy the last lemma since, if $y^* \in U \cap \mu(S)$, then

$$\|x^* - y^*\| \leqslant \|x^* - x_0^*\| + \|x_0^* - y^*\| \leqslant \varepsilon/2 + \operatorname{diam}(U \cap \mu(S)) \leqslant \varepsilon$$

and the theorem is proved.

3. Approximating compact operators by finite rank norm attaining operators. Theorems 3 and 4 are proved in this section by obtaining general conditions on either the domain or the range of a compact operator in order to insure that it can be approximated by a norm attaining finite rank operator: Namely if either the domain has lots of norm one complemented finite dimensional subspaces or the range has a large uniformly complemented family of finite dimensional subspaces whose unit balls are polyhedral. The proof of the next lemma is omitted.

LEMMA 3.1. *The following two conditions on a Banach space X are equivalent:*

(I) *For every finite dimensional Banach space F, every bounded linear operator $A: X \to F$ and every $\varepsilon > 0$, there is a projection $P: X \to X$ such that P has finite dimensional range, $\|P\| = 1$ and $\|A - AP\| \leqslant \varepsilon$.*

(II) *Given $\varepsilon > 0$ and $\{x_1^*, \ldots, x_n^*\} \subset X^*$, there is a projection $P: X \to X$ such that P has finite rank, $\|P\| = 1$ and for each $i = 1, \ldots, n$ there is a $y_i^* \in X^*$ such that $\|x_i^* - P^* y_i^*\| \leqslant \varepsilon$.*

Since the operator AP of condition (I) is norm attaining, these two equivalent conditions on X imply that finite rank operators on X can be approximated by norm attaining finite rank operators.

We note that not every Banach space X satisfies the conditions of this lemma since they imply that X^* has the 1-metric approximation property (see Johnson, Rosenthal, Zippin [8]). Thus the predual of a reflexive Banach space failing the approximation property does not satisfy the conditions of the lemma.

PROPOSITION 3.2. *If S is a compact Hausdorff space, then $C(S)$ satisfies the conditions of the last lemma.*

Proof. We verify condition (II). Let $\varepsilon > 0$ and

$$\{\mu_1, \ldots, \mu_n\} \subset C(S)^* = M(S)$$

be given. Let $\mu = \sum_{i=1}^{n} |\mu_i|$. Since $\mu_i \ll \mu$ for each i, there is a $g_i \in L^1(\mu)$ such that $\mu_i = g_i \mu$. Choose simple functions $s_i \in L^1(\mu)$ such that $\int |g_i - s_i| d\mu \leqslant \varepsilon/2$. Let $\{A_1, \ldots, A_m\}$ be a disjoint collection of Borel sets with $\mu(A_j) \neq 0$ such that for each i there are real numbers a_j^i with $s_i = \sum_{j=1}^{m} a_j^i \chi_{A_j}$. For each $j = 1, \ldots, m$ choose a compact set $K_j \subset S$ with $K_j \subset A_j$ and $\mu(A_j \setminus K_j) < \varepsilon/2mM$ where $M = \max|a_j^i|$. Let $\{\varphi_1, \ldots, \varphi_m\}$ be disjointly supported functions in $C(S)$ such that $\varphi_j(K_j) = 1$ for each j. Define $P: C(S) \to C(S)$ by

$$Pf = \sum_{j=1}^{m} \left(\frac{1}{\mu(K_j)} \int_{K_j} f\,d\mu\right) \varphi_j.$$

Then P is a norm 1 projection onto $\operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$. For $k = 1, \ldots, m$, $P^* \chi_{K_k}\mu = \chi_{K_k}\mu$ since, for $f \in C(S)$,

$$\int_{K_k} Pf\,d\mu = \sum_{j=1}^{m} \left(\frac{1}{\mu(K_j)} \int_{K_j} f\,d\mu\right) \int_{K_k} \varphi_j\,d\mu = \int_{K_k} f\,d\mu.$$

So for each i, if $U_i = \left(\sum_{j=1}^{m} a_j^i \chi_{K_j}\right)\mu$, $P^* U_i = U_i$ and thus

$$\begin{aligned}\|\mu_i - P^* U_i\| = \|g_i\mu - U_i\| &\leqslant \|g_i\mu - s_i\mu\| + \|s_i\mu - U_i\| \\ &\leqslant \int |g_i - s_i|\,d\mu + \int \Big|s_i - \sum_{j=1}^{m} a_j^i \chi_{K_j}\Big|\,d\mu \\ &\leqslant \frac{\varepsilon}{2} + \sum_{j=1}^{m} |a_j^i|\,\mu(A_j \setminus K_j) \leqslant \varepsilon.\end{aligned}$$

This proves the proposition.

An argument similar to the above shows that if μ is a finite measure, then $L^p(\mu)$ satisfies the conditions of Lemma 3.1 for $1 \leqslant p \leqslant \infty$.

Since any compact operator on a $C(S)$ or $L^1(\mu)$ space can be approximated by a finite rank operator, this proves the "from" half of Theorems 3 and 4. The "into" half of these two results follow from the next proposition.

PROPOSITION 3.3. *If either Y or Y^* is isometric to an $L^1(\mu)$ space, then for any Banach space X, $\overline{\mathrm{FRNA}(X, Y)} = K(X, Y)$.*

We denote by l_1^n (l_∞^n) the n-dimensional Banach space of real n-tuples $x = (x_1, \ldots, x_n)$ with $\|x\| = \sum_{i=1}^{n} |x_i|$ ($= \sup |x_i|$). A simple argument shows that, for any n, if $Y = l_1^n$ or $Y = l_\infty^n$ then for any X, $\overline{\mathrm{NA}(X, Y)} = L(X, Y)$ (cf. [11], pp. 146–147).

If Y^* is isometric to $L^1(\mu)$, then for any $\varepsilon > 0$ and any finite set $\{y_1, \ldots, y_n\} \subset Y$ there is a norm 1 complemented subspace E of Y which is isometric to l_∞^m for some m and such that for each $i = 1, \ldots, n$ there is an $e \in E$ with $\|y_i - e\| \leqslant \varepsilon$ (cf. [10], Theorem 3.1). Also, if $Y = L^1(\mu)$ then the same property holds with l_1^m replacing l_∞^m (the subspaces E are obtained by taking the spans of finite sets of simple functions). Thus Proposition 3.3 follows from the next lemma.

LEMMA 3.4. *Let Y be a Banach space. Suppose there is a $\lambda \geqslant 1$ such that for every $\delta > 0$ and finite set $\{y_1, \ldots, y_n\} \subset Y$ there is a subspace E of Y and a projection P of Y onto E such that $\|P\| \leqslant \lambda$, for each y_i there is an $e \in E$ with $\|y_i - e\| \leqslant \delta$, and, for every X, $\overline{\mathrm{NA}(X, E)} = L(X, E)$. Then for every Banach space X, $\overline{\mathrm{FRNA}(X, Y)} = K(X, Y)$.*

Proof. Let a compact $A: X \to Y$ and $\varepsilon > 0$ be given. Choose $\{y_1, \ldots, y_n\} \subset Y$ such that for each $x \in X$ with $\|x\| \leqslant 1$ there is a y_i with $\|Ax - y_i\| \leqslant \varepsilon/8\lambda$. Choose E and P as provided with $\delta = \varepsilon/8\lambda$. Then for $x \in X$ with $\|x\| \leqslant 1$ there is an $e \in E$ with $\|Ax - e\| \leqslant \varepsilon/4\lambda$ so

$$\|Ax - PAx\| \leqslant \|Ax - e\| + \|Pe - PAx\| \leqslant \varepsilon/2.$$

Thus $\|A - PA\| \leqslant \varepsilon/2$. Now choose a norm attaining operator $A': X \to E$ with $\|PA - A'\| \leqslant \varepsilon/2$ and the lemma is proved.

4. Open problems and remarks. The following questions about norm attaining operators between classical Banach spaces seem to be open.

QUESTION 1. Are the norm attaining operators dense in $L(C[0,1], L^1)$, $L(L^1, C[0,1])$, $L(C[0,1], l^2)$?

The following special case of the next proposition has a possible bearing on the $C(S)$ to L^1 problem: If S is a dispersed compact Hausdorff space (i.e., it contains no perfect subsets), then

$$\overline{\mathrm{NA}(C(S), L^1)} = L(C(S), L^1).$$

PROPOSITION 4.1. *Suppose S is a dispersed compact Hausdorff space and suppose Y is either a $C(K)$ space or an $L^p(\mu)$ space for $1 \leqslant p \leqslant \infty$. Then*

$$\overline{\mathrm{NA}(C(S), Y)} = L(C(S), Y).$$

Proof. Since $L^\infty(\mu)$ is a $C(K)$ space, Theorem 1 takes care of the cases $Y = C(K)$ or $L^\infty(\mu)$. For p finite all bounded linear operators from $C(S)$ to Y are compact ([9], Lemma 9) and so the proposition follows from Theorem 3.

The results and techniques of this paper and the literature show that if X is either a $C(S)$ space or an $L^p(\mu)$ space or if Y is either a $C(S)$ space or an $L^1(\mu)$ space then $\overline{\mathrm{FRNA}(X, Y)} = K(X, Y)$. In general, a compact operator need not be approximable by any finite rank operator, however, the following question raised in [4] seems to be unanswered:

QUESTION 2. Does $\overline{\mathrm{KNA}(X, Y)} = K(X, Y)$ for any pair of Banach spaces X and Y?

The simplest case of this question which is unresolved is Question 6, below.

Lindenstrauss [11] introduced the following two properties of a Banach space. The Banach space X has *property* A if, for any Banach space Y, $\overline{\mathrm{NA}(X, Y)} = L(X, Y)$. The Banach space Y has *property* B if, for any Banach space X, $\overline{\mathrm{NA}(X, Y)} = L(X, Y)$.

The remarkable paper of Bourgain [2] answers most questions about property A. He shows that if X has the Radon–Nikodym property then X has property A. The converse seems to be open.

QUESTION 3. Does property A imply the Radon–Nikodym property?

This question is equivalent in the separable case (via [2]) to:

QUESTION 4. Is property A an isomorphism invariant?

Bob Huff in [15] has recently shown that Questions 3 and 4 are equivalent. (This fact was also pointed out to us by the referee.)

Problems concerning property B even for classical spaces are still numerous, however. Lindenstrauss proved in [11] that $\boldsymbol{R}^n$ equipped with a polyhedral norm has property B and asked if every reflexive space has property B. It is easy to show that l^∞ and c have property B. Surprisingly, the following question is still open.

QUESTION 5. Do any of the following classes of Banach spaces have property B: (a) finite dimensional spaces, (b) reflexive Banach spaces, (c) $C(S)$ spaces, (d) $L^1(\mu)$ spaces, including l_1?

There are Banach spaces which fail property B (cf. [11]) but it is not known if any classical Banach space fails B.

The most irritating problem about norm attaining operators which is linked to property B is the following:

QUESTION 6. Let X be an arbitrary Banach space and let $\boldsymbol{R}^2$ be a 2-dimensional Euclidean space. Can every operator from X to $\boldsymbol{R}^2$ be approximated by a norm attaining operator?

QUESTION 7. Can every Banach space be equivalently renormed to have property B?

We close with the following proposition which characterizes those pairs of classical Banach spaces (X, Y) satisfying $\mathrm{NA}(X, Y) = L(X, Y)$.

PROPOSITION 4.2. *Among the infinite dimensional classical Banach spaces, i.e. L^p spaces and $C(S)$ spaces, $\mathrm{NA}(X, Y) = L(X, Y)$ if and only if $X = L^p(\mu)$, $Y = L^r(\nu)$, $1 \leqslant r < p < \infty$ and one of the following holds:*

(a) $1 < r$ *and μ and ν are atomic.*

(b) $1 < r < 2$ *and ν is atomic.*

(c) $p > 2$, $r > 1$ *and μ is atomic.*

(d) $r = 1$ *and ν is atomic.*

(e) $r = 1$, $p > 2$ *and μ is atomic.*

Proof. First, if $\mathrm{NA}(X, Y) = L(X, Y)$ then X is reflexive since each element of X^* must attain its norm. Thus among the classical spaces, X can only be $L^p(\mu)$ for $1 < p < \infty$. Next, let us show that Y cannot be a space of type $C(S)$: Suppose $\mathrm{NA}(X, c_0) = L(X, c_0)$. Then X is reflexive so there is a basic sequence in the unit sphere of X^*. Let $\{x_n^*\}$ be a weakly convergent subsequence, which must converge weakly to zero. (See I. Singer, *Bases in Banach spaces. I*, p. 297.) Define $T\colon X \to c_0$ by letting the nth coordinate of Tx be $\left(1-\frac{1}{n}\right)x_n^*(x)$. Then $\|T\| = 1$ but $\|Tx\| < \|x\|$ for each $x \notin X$ and we obtain a contradiction. Since c_0 embeds isometrically in any infinite dimensional $C(S)$ space, Y cannot be of this type. Hence we are led to consider only L^p spaces.

Now, in [6] (also cf. [3], page 16) it is proved that if X and Y are reflexive and one of them has the approximation property then $\mathrm{NA}(X, Y) = L(X, Y)$ iff $L(X, Y) = K(X, Y)$ iff $L(X, Y)$ is reflexive. In [13], Theorem A2, those L^p spaces X and Y satisfying $L(X, Y) = K(X, Y)$ were characterized. In case $r > 1$, this yields the proposition in cases (a), (b), and (c). The easy observation that a compact operator from a reflexive space to $L^1(\nu)$ with ν atomic attains its norm, along with [13], Theorem A2, yields the sufficiency of (d) and (e). It remains to show that, given $r = 1$, (d) and (e) are necessary. The following examples are due to J. Bourgain:

EXAMPLE 1. For $1 < p < 2$, define $T\colon l^p \to L^1[0, 1]$ by

$$Tx = \sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)x_n p_n$$

where $x = (x_n)$ and (p_n) is the sequence of Rademacher functions on $[0, 1]$. Then

$$\|Tx\|_1 \leqslant \|Tx\|_2 = \left[\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^2 x_n^2\right]^{1/2} < \left[\sum_{n=1}^{\infty} x_n^2\right]^{1/2} \leqslant \|x\|_p.$$

Thus $\|T\| \leqslant 1$. Since $Te_n = (1-1/n)p_n$, $\|T\| = 1$ but the above inequality shows it is not attained.

EXAMPLE 2. For $2 \leqslant p < \infty$ define $T\colon L^p[0, 1] \to L^1[0, 1]$ by

$$Tf = \sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)\left[\int_0^1 f p_n\,dx\right]p_n$$

where (p_n) are again the Rademacher functions. Then $T = I_2 SPI_1$ where $I_1\colon L^p \to L^2$ and $I_2\colon L^2 \to L^1$ are inclusions, P is the projection of L^2 on the span of (p_n) and S maps the span of the Rademacher functions onto itself by $S(\sum a_n p_n) = \sum (1-1/n)a_n p_n$. Since each of these operators has norm 1, so does T. But T does not attain its norm since S does not.

It is easy to show that if H is the range of a norm one projection on E then $\mathrm{NA}(E, F) = L(E, F)$ implies $\mathrm{NA}(H, F) = L(H, F)$. Hence Example 1 shows that $\mathrm{NA}(L^p, L^1) \neq L(L^p, L^1)$ for $1 < p < 2$. This completes the proof.

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