An application of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers by

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Dedicated to Ralph Boas on the occasion of his sixty-fifth birthday

Abstract. Localized Bessel potential spaces \( S_\gamma(q, \gamma), \gamma > 0 \), were recently introduced by Connett and Schwartz in connection with ultraspherical multipliers and characterized for integer \( \gamma \) in terms of sequence spaces. Analogous results are obtained in this paper for all real \( \gamma > 1/q \), where \( 1 < q < \infty \). These results are then used to derive best possible multiplier criteria of Marcinkiewicz type for Jacobi expansions by interpolating between end-point results due to Askey and to the authors and to derive analogous multiplier criteria for Hankel transforms.

1. Introduction. In [11] Connett and Schwartz showed that localized Bessel potential spaces \( S_\gamma(q, \gamma) \) are useful in the theory of ultraspherical multipliers. However, one disadvantage of these spaces is that it is hard to verify when a sequence is the restriction (to the positive integers) of an element in \( S_\gamma(q, \gamma) \). In case of \( \gamma \) being a positive integer, Connett and Schwartz characterized \( S_\gamma(q, \gamma) \) by means of (finite) difference conditions upon the sequence. The main result of this paper, Theorem 1, extends this characterization to all \( \gamma > 1/q \) for \( 1 < q < \infty \). We also give a neat description (Theorems 4 and 5) of the embedding behavior of the wbv and WBV-spaces (defined below), which are important in multiplier theory. These results are then used to derive various multiplier criteria for Jacobi expansions (Theorem 6) and Hankel transforms (Theorem 7).

To define the localized Bessel potential spaces we first recall that the standard space of Bessel potentials \( L^2_\gamma(R) \), \( \gamma > 0, 1 < q < \infty \), is defined by (see [30], p. 134)

\[ L^2_\gamma = \{ g \in L^q(0, \infty) : g = G_\gamma * h, \|g\|_{L^q} = \|h\|_{L^q} < \infty \}, \]

where the Bessel kernel \( G_\gamma(x) \) is a function whose Fourier transform is given by

\[ \hat{G}_\gamma(\sigma) = \frac{1}{\sigma} \int_{-\infty}^{\infty} G_\gamma(y)e^{-iy\sigma}dy = (1 + |y|^\gamma)^{-\gamma/2}. \]

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References


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$I_2^c$ is localized as follows: Let $\Phi(t) \in C^0(-\infty, \infty)$ be monotone decreasing for $t > 0$ with

$$
\Phi(t) = \begin{cases} 1, & 0 \leq t \leq 1/3, \\ 1 - \frac{1}{t}, & -1/3 < t \leq 0, \\ 0, & t \geq 2/3,
\end{cases}
$$

so that, obviously, $\sum_{k=0}^\infty \Phi(t-k) = 1$ for all $t \in \mathbb{R}$. Let $\phi_k(t) = (\Phi(t-k))^{1/2}$; then $\phi_k \in C^0(-\infty, \infty)$, $\phi_k(t) > 0$, $\phi_k(t)$ is monotone decreasing for $t > k$ and $\sum_{k=0}^\infty \phi_k(t) = 1$. For any measurable function $g$ on $(0, \infty)$, let $g^*(x) = g(\phi(x))$. Then, for $\gamma > 0$ and $1 < q < \infty$, the space of localized Bessel potentials is defined by

$$
S(\gamma, q) = \{ g : \|g\|_{S(\gamma, q)} = \sup_{k \in \mathbb{Z}} \|\phi_k g\|_{L^q} < \infty \},
$$

where $\mathbb{Z}$ is the set of all integers. We also let $N = \{1, 2, \ldots\}$, $L^\infty$ denote the space of bounded sequences $\eta = (\eta_k)_{k \in \mathbb{N}}$, with the supremum norm $\|\eta\|_\infty$, $L^\infty = L^\infty(0, \infty)$, $L^\infty_q$ and $L^\infty_{\infty}$ denote the corresponding spaces of sequences (and bounded functions) with compact support in $(0, \infty)$, and let $C(0, \infty)$ denote the space of continuous functions on $(0, \infty)$. Generic positive constants will be denoted by $C$.

As is customary we shall identify a function which coincides a.e. with a continuous function with that continuous function. When $\gamma > 1/q$, $1 < q < \infty$, each function in $S(\gamma, q)$ can be identified with a continuous function [11] and so we can consider the restriction of the continuous function to the normal numbers and state the Cotton and Schwartz characterization of $S(\gamma, q)$ for $\gamma \in N$ as follows:

**Theorem A.** Let $1 < q < \infty$ and $\gamma \in \mathbb{N}$.

(a) If $g \in S(\gamma, q)$, $\|g\|_\infty \leq C \|\phi_k g\|_{L^q}$ and $\eta_k = g(k)$, $k \in \mathbb{N}$, then

$$
\sum_{k=0}^\infty \left( \sum_{m=0}^{k-1} k^{-1} A^m \|\phi_k g\|_{L^q} \right)^{1/q} \leq C \|g\|_{S(\gamma, q)},
$$

where, as usual, $C$ is a constant independent of $g$.

(b) Conversely, for each sequence $\eta$ for which the left side of (1.2) is finite there exists a $g \in S(\gamma, q)$ with $g(k) = \eta_k$, $k \in \mathbb{N}$, so that (1.2) also holds with the inequality reversed (and a different positive constant $C$).

Here the (fractional) difference operator $A^k$ is defined by [5]

$$
A^k \eta_k = \sum_{j=0}^k A^{k-j} \eta_j, \quad A^k \eta_k = \left( \begin{array}{c} k+1 \\
\end{array} \right) \frac{\Gamma(k+1)}{\Gamma(k+1)},
$$

whenever the series converges. To keep the notation compact we also define the following (weak bounded variation) sequence spaces: for $\gamma > 0$, $1 < \infty$,

$$
\mathbb{W}v_{\gamma, q} = \{ \eta \in L^\infty : \|\eta\|_{\mathbb{W}v_{\gamma, q}} = \sup_{n \in \mathbb{N}} \left( \sum_{k=0}^{n-1} k^{-1} A^k \|\phi_k \eta\|_{L^q} \right)^{1/q} \}< \infty,
$$

and for $\gamma > 0$, $q = \infty$,

$$
\mathbb{W}v_{\gamma, \infty} = \{ \eta \in L^\infty : \|\phi_k \eta\|_{L^\infty} = \sup_{n \in \mathbb{N}} \|A^k \|\phi_k \eta\|_{L^\infty} \}< \infty.
$$

Corresponding to $\mathbb{W}v_{\gamma}$ we also consider the space $\mathbb{S}(\gamma, q)$, $\gamma > 1/q$, $1 < q < \infty$, of all sequences $\eta$ for which there is a (continuous) $g \in S(\gamma, q)$ such that $\eta_k = g(k)$, $k \in \mathbb{N}$. The norm is defined by

$$
\|\eta\|_{\mathbb{S}(\gamma, q)} = \inf \{ \|g\|_{S(\gamma, q)} : \phi_k g \text{ and } g(k) = \eta_k, \quad k \in \mathbb{N} \}.$$

Then Theorem A is equivalent to the statement that

$$
\mathbb{W}v_{\gamma} = \mathbb{S}(\gamma, q), \quad 1 < q < \infty, \quad \gamma \in \mathbb{N},
$$

with equivalent norms, and our extension of Theorem A to each $\gamma > 1/q$ can be stated in the form

**Theorem 1.** If $1 < q < \infty$ and $\gamma > 1/q$, then

$$
\mathbb{W}v_{\gamma} = \mathbb{S}(\gamma, q),
$$

with equivalent norms.

An intermediate step in the proof of Theorem A uses the fact that (on account of Theorem 3 in [20], p. 130) when $\gamma \in \mathbb{N}$

$$
\|\phi_k \|_{L^\infty} + \sup_{t > 0} \left( \int_{\gamma - 1}^{\gamma} t^{-1} g(t)^2 \, dt \right)^{1/2} \leq C
$$

is an equivalent norm for $S(\gamma, q)$. To generalize this to fractional $\gamma$ we first define for $0 < \delta < 1$ and a locally integrable function $g$ the fractional integral

$$
I^\delta_+ (g)(t) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^t (s-t)^{\delta-1} g(s) \, ds, & 0 < t < \infty, \\
0, & t = \infty
\end{cases}
$$

and the fractional derivatives

$$
g^{(\delta)}(t) = \lim_{n \to \infty} \frac{d}{dt} I^{(\delta)}_+ (g)(t), \quad 1, \gamma > 0.
$$
whenever the right sides exist ([γ] denotes the integer part of γ). See COSMAR [12] and [26], § 3.3.1. Then we form the function analog of the \( \text{wbv}_{d,p} \) spaces defined for \( \gamma > 0, 1 < q < \infty \), by

\[
\text{WBV}_{d,p} = \{ g \in L^p \cap C(0, \infty); \quad I^{\gamma}(g) \in AC_{M0}, \quad \omega > 0, \quad \text{if} \quad \delta > 0; \quad g^{(\delta)}, \ldots, g^{(\gamma-1)} \in AC_{M0}, \quad \text{if} \quad \gamma > 1, \quad \text{and} \quad \|g\|_{M(\gamma,p)} < \infty \},
\]

where \( \delta = \gamma - [\gamma] \).

\[
\|g\|_{M(\gamma,p)} = \|g\|_{M0} + \sup_{t \neq 0} \left( \int_{t-1}^{t+1} \left| g^{(\gamma)}(t) \right| d\ell \right)^{1/q}, \quad 1 < q < \infty,
\]

and

\[
\|g\|_{M(\gamma,p)} = \|g\|_{M0} + \|g^{(\gamma)}(t)\|_{M0}.
\]

In the above and what follows \( AC_{M0} (L^p_{M}) \) is the space of functions which are absolutely continuous (integrable) on every compact subinterval of \((0, \infty)\).

Analogously to the definition of \( s(g, \gamma) \) we define \( \text{wbv}(g, \gamma), \gamma > 0, 1 < q < \infty \), to be the space of all sequences \( \eta \) for which there is a \( g \in \text{WBV}_{d,p} \) such that \( \eta_n = g(k), k \in N \). The norm is defined by

\[
\|\eta\|_{\text{wbv}(\gamma,p)} = \inf \{ \|g\|_{M(\gamma,p)}; \quad g \in \text{WBV}_{d,p}, \quad \text{and} \quad g(k) = \eta_k, k \in N \}.
\]

We shall prove in Section 5 that \( \text{wbv}_{d,p} \) and \( \text{wbv}(g, \gamma) \) are related as follows:

**Theorem 2.** If \( \gamma > 1/2, 1 < q < \infty \) or \( \gamma \geq 1, q = 1, \) then

\[
\text{wbv}_{d,p} = \text{wbv}(g, \gamma)
\]

with equivalent norms.

Then, to complete the proof of Theorem 1 it only remains to prove (in Section 6)

**Theorem 3.** If \( 1 < q < \infty \) and \( \gamma > 1/2 \), then

\[
\text{WBV}_{d,p} = S(g, \gamma)
\]

and hence

\[
\text{wbv}(g, \gamma) = s(g, \gamma)
\]

with equivalent norms.

Before proving these theorems we shall first derive some required fundamental properties of the \( \text{wbv}_{d,p}, \quad \text{WBV}_{d,p}, \) and \( L^p_{M} \) spaces in Sections 2-4. The imbedding behavior of the \( \text{wbv}_{d,p} \) and \( \text{WBV}_{d,p} \) spaces is presented in Section 7 and then these results are used in Section 8 to derive our multiplier criteria for Jacobi expansions. Our multiplier criteria for Hankel transforms are presented in Section 9 along with several remarks concerning related results.

2. \( \text{wbv}_{d,p} \) spaces. In this section we present some basic properties of the \( \text{wbv}_{d,p} \) spaces which are needed for the proof of Theorem 2 and for the applications.

**Lemma 1.** If \( 0 < \mu < \gamma, 1 < q < \infty \), then

\[
\text{wbv}_{d,p} \subset \text{wbv}_{d,\gamma}
\]

where, as elsewhere, the inclusion sign means that the identity map is continuous.

**Proof.** The case \( q = 1 \) is essentially proved in [13], § 2. A slight modification of that proof shows that the Lemma also holds for \( 1 < q < \infty \).

First note that for \( \eta \in L^p \) we have [5], Lemma 1

\[
\left( \sum_{k=0}^{\infty} k^{|\gamma|} |A_k^d \eta_k|^p \right)^{1/p} 
\]

Thus, for \( 1 < q < \infty \),

\[
\left( \sum_{k=0}^{\infty} k^{|\gamma|} |A_k^d \eta_k|^p \right)^{1/p} 
\]

Hölder’s inequality gives

\[
\sum_{k=0}^{\infty} A_k^{d-\gamma} \eta_k \eta_k \leq \sum_{k=0}^{\infty} A_k^{d-\gamma} \eta_k \eta_k \left( \sum_{k=0}^{\infty} A_k^{d-\gamma} \eta_k \eta_k \right)^{p/2}
\]

with \( p = q/(q-1) \). Observing that the last factor on the right side is majorized by \( O(A_j^{-\gamma} \eta_j^{p/2}), \quad 3^{-\gamma} \leq j < 2^\gamma \), we can estimate \( I_4 \), after an interchange of summation, by

\[
I_4 \leq O \left( \sum_{j=3^{-\gamma}}^{2^\gamma} j^{p/2} \eta_j \eta_j \sum_{k=0}^{\infty} A_k^{d-\gamma} \eta_k \eta_k \left( \sum_{k=0}^{\infty} A_k^{d-\gamma} \eta_k \eta_k \right)^{p/2} \right)
\]

uniformly in \( m \). The sums \( I_4 \) and \( I_4 \) are estimated analogously (use also the method in [13]), which gives the case \( 1 < q < \infty \). Analogously is even simpler to treat, it is omitted.


**Lemma 2.** Let \( \gamma > 0, 1 \leq q < \infty \), \( G \in C^\infty([0, \infty)) \) be monotone decreasing with
\[
G(\xi) = \begin{cases} 
1, & \xi \leq 1, \\
0, & \xi > d,
\end{cases}
\]
where \( d > 1 \) is fixed, and let \( G_n(\xi) = G(\xi/n) \), \( n > 0 \). Then there exist constants \( C_1, C_2 \) independent of \( \eta \) and \( u > 0 \) such that
\[
C_1 \| \eta \|_{u, \infty} \leq \sup_{n > 0} \| [G_n(\xi) \eta] \|_{u, \infty} \leq C_2 \| \eta \|_{u, \infty}.
\]
The left inequality also holds with \( \sup \) replaced by \( \max \), where \( a > 0 \).

**Proof.** We first prove the right-side inequality. Let \( \eta \in \mathcal{W}^p_{b, x} \). If \( \gamma \) is a natural number, Leibniz' formula gives
\[
\begin{align*}
\Delta^\gamma [G_n(\xi) \eta] &= \sum_{j=0}^\gamma \binom{\gamma}{j} \Delta^j \eta \Delta^{\gamma-j} G_n(\xi + k),
\end{align*}
\]
and the inequality follows as in the proof of (13), Lemma 2.

So now let \( \gamma \) be strictly fractional. Note that by a formula due to Peyerimhoff [18], p. 3 we have
\[
(2.2) \quad \Delta^\gamma [G_n(\xi) \eta] = \eta \Delta^\gamma G_n(\xi) + \sum_{k=0}^{\gamma} \binom{\gamma}{k} \Delta^k \eta \Delta^{\gamma-k} G_n(\xi + k),
\]
where the remainder term is given by
\[
R_k = (-1)^k [\sum_{n=0}^\infty A_n^{-1} \eta_n - \eta] \sum_{n=0}^\infty \sum_{j=1}^{n-1} A_j^{-1} \eta_j \Delta^j G_n(\xi) - \Delta^\gamma G_n(\xi).
\]
Also note that, by [26], p. 37, we have for any \( \delta > 0 \)
\[
(2.3) \quad \left\| \int_0^\infty \left\| \frac{d^2}{ds^2} G_n(s) \right\| ds \right\| \leq C
\]
We shall show that
\[
(2.4) \quad \sup_k |A_k^* R_k| \leq C \| \eta \|_{u, \infty}.
\]
For the case \( 0 < \gamma < 1, 2^{m-1} - 1 < k < 2^m \), we have from (2.3) (observe that \( A_n^{-1} \), \( n > k + 1 \), and \( \Delta^\gamma G_n(\xi) \) do not change sign)
\[
|A_k^* R_k| \leq 2 \| \eta \|_{u, \infty} \sum_{n=0}^m A_n^{-1} \sum_{j=1}^{n-1} \Delta^j G_n(\xi) +
\]
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\[
+ 2 \| \eta \|_{u, \infty} \sum_{n=0}^m A_n^{-1} \sum_{n=1}^{n-1} A_j^{-1} \eta_j \Delta^j G_n(\xi) +
\]
\[
\leq C \| \eta \|_{u, \infty} \sum_{n=0}^m A_n^{-1} \sum_{n=1}^{n-1} A_j^{-1} \eta_j \Delta^j G_n(\xi) +
\]
\[
\leq C \| \eta \|_{u, \infty}.
\]
Similarly, for \( \gamma > 1, \gamma \notin N, 2^{m-1} - 1 < k < 2^m \), we have
\[
|A_k^* R_k| \leq 2 \| \eta \|_{u, \infty} \sum_{n=0}^m A_n^{-1} \sum_{j=1}^{n-1} \Delta^j G_n(\xi) +
\]
\[
\leq C \| \eta \|_{u, \infty} \sum_{n=0}^m A_n^{-1} \sum_{j=1}^{n-1} \Delta^j G_n(\xi) +
\]
\[
\leq C \| \eta \|_{u, \infty}.
\]
Analogously, for \( \gamma > 1, \gamma \notin N, 2^{m-1} - 1 < k < 2^m \), we have
\[
|A_k^* R_k| \leq 2 \| \eta \|_{u, \infty} \sum_{n=0}^m A_n^{-1} \sum_{j=1}^{n-1} \Delta^j G_n(\xi) +
\]
\[
\leq C \| \eta \|_{u, \infty} \sum_{n=0}^m A_n^{-1} \sum_{j=1}^{n-1} \Delta^j G_n(\xi) +
\]
\[
\leq C \| \eta \|_{u, \infty}.
\]
uniformly in \( k \). Thus (2.4) holds. The right-side inequality of Lemma 2 now follows easily. By (2.2)–(2.4), it is immediately obvious in the case \( \gamma = \infty \). If \( 1 \leq q < \infty \), then we essentially observe that from (2.2)–(2.4) and Lemma 1 we have
\[
(2.5) \quad \left\| [A_k^* \Delta^\gamma G_n(\xi) \eta] \right\|_{u, \infty} \leq C \| \eta \|_{u, \infty} + \sum_{k=0}^\infty \left\| \frac{d^2}{ds^2} G_n(s) \right\|_{u, \infty} + \| \eta \|_{u, \infty}.
\]
Conversely, let \( \| [G_n(\xi) \eta] \|_{u, \infty} \) be uniformly bounded in \( u > 0 \).

Fix \( m \) and choose \( u > 2^{m-1}; 2^m - 1 < k < 2^m \). Then
\[
|A_k^* \Delta^\gamma G_n(\xi) \eta| = \left| A_k \sum_{j=1}^{n-1} \Delta^j G_n(\xi) \eta_j \right|
\]
\[
\leq C \| \eta \|_{u, \infty}.
\]
for \( u \) sufficiently large. By the triangle inequality,
\[
\left( \sum_{k=0}^\infty \left| [A_k^* \Delta^\gamma G_n(\xi) \eta] \right|^q \right)^{1/q} \leq \sup_{m \geq 0} \sum_{k=0}^\infty \left| [A_k^* \Delta^\gamma G_n(\xi) \eta] \right|^q + C \| \eta \|_{u, \infty}
\]
\[
\leq \| [G_n(\xi) \eta] \|_{u, \infty}.
\]
uniformly in $m$, which gives the left inequality in the lemma. The case $q = \infty$ follows analogously.

3. WBV$_{\gamma}$ spaces. Suppose that $\gamma > 0$, $1 \leq q \leq \infty$ and $g \in$ WBV$_{\gamma}$.
If $\gamma > 1$ and $g$ has compact support in $(0, \infty)$, it follows from [26], Lemma 3.14 and [27] that

$$g(t) = \pm \frac{1}{\Gamma(\gamma)} \int_0^t (s-t)^{\gamma-1} g(s) ds \quad \text{a.e.,}$$

which can easily be used to show that

$$g^{(\mu)}(t) = \pm \frac{1}{\Gamma(\gamma-\mu)} \int_0^t (s-t)^{\gamma-\mu-1} g^{(\mu)}(s) ds \quad \text{a.e.,} \quad 0 < \mu < \gamma.$$  

Since formula (3.2) will be needed for all $g \in$ WBV$_{\gamma}$ to derive the imbedding properties of these spaces, it is natural to try to prove (3.2) by first proving (3.1) for these spaces. Unfortunately, if $g$ does not have compact support, then the integral in (3.1) might not converge a.e. even though the integral in (3.2) clearly converges a.e. since, for $2^\delta > t$,

$$\int_0^t (s-t)^{\gamma-\mu-1} g^{(\mu)}(s) ds \leq C \sum_{k=0}^\infty 2^{-k} \int_0^\infty (s-t)^{\gamma-\mu-1} g^{(\mu)}(s) ds \leq C \sum_{k=0}^\infty 2^{-k} \int_0^\infty (s^{\gamma-\mu} g^{(\mu)}(s))^{\frac{1}{\gamma-\mu}} ds < \infty$$

and $\int g^{(\mu)}(s) ds$ converges a.e. Therefore for functions without compact support we shall have to use a more delicate approach which requires the following preliminary results:

**Lemma 3.** Suppose $0 < \mu < \delta < 1$, $g \in L^\infty \cap C(0, \infty)$, and $I^{\alpha}_{-\mu} \in C_0$ for each $\omega > 0$. Then

$$\int_0^\infty (s-t)^{\gamma-\mu-1} I^{\alpha}_{-\mu}(g)(s) ds \leq C \|g\|_{L^\infty}, \quad \omega > 0,$$

where $C$ depends only on $\delta$, and $I^{\alpha}_{-\mu}(g) \in C_0$, $\omega > 0$.

Proof. From the hypotheses and the definition of $g^{(\mu)}$ it follows for a.e. $t$ in $(0, \omega)$ that

$$g^{(\mu)}(t) = \frac{1}{\Gamma(\delta-\mu)} \int_0^t (s-t)^{\delta-\mu-1} g(s) ds - \frac{\delta}{\Gamma(\delta-\mu)} \int_0^t (s-t)^{\delta-\mu-1} g(s) ds = I_1(t) + I_2(t),$$

where

$$I_1(t) = \frac{1}{\Gamma(\delta-\mu)} \int_0^t (s-t)^{\delta-\mu-1} g(s) ds$$

and

$$I_2(t) = \frac{\delta}{\Gamma(\delta-\mu)} \int_0^t (s-t)^{\delta-\mu-1} g(s) ds.$$

Clearly,

$$|| I^{\alpha}_{-\mu}(I_1)(s) || \leq C \|g\|_{L^\infty} \int_0^\infty (t-s)^{\delta-1} (s-t)^{\delta-1} ds dt \leq C \|g\|_{L^\infty}$$

and from

$$\int_0^\infty I^{\alpha}_{-\mu}(I_2)(s) ds = \left[ \frac{1}{\Gamma(\delta)} \int_0^t \left( \left[ \int_0^s (s-t)^{\delta-1} g(t) dt \right] ds \right) \right]$$

we have that $I^{\alpha}_{-\mu}(I_1)(s) = g(s)$ for a.e. $s$ in $(0, \omega)$, which gives (3.4).

Since $I^{\alpha}_{-\mu}(g)(s) = 0$ for $s \geq \omega$ and is continuous at $s = \infty$ it only remains to show that if $0 < a < \omega$, then this function is absolutely continuous on $[a, \omega]$. By hypothesis there is an $f \in L^1[a, \omega]$ such that

$$I^{\alpha}_{-\mu}(g)(s) = \int_a^s f(t) dt, \quad a \leq x \leq \omega.$$

Then, for $x \in [a, \omega]$,

$$I^{\alpha}_{-\mu}(g)(x) = \frac{1}{\Gamma(\delta-\mu)} \int_0^x (x-s)^{\delta-\mu-1} g(s) ds$$

$$= \frac{1}{\Gamma(\delta-\mu)} \int_0^x (t-x)^{\delta-\mu-1} I^{\alpha}_{-\mu}(g)(t) dt$$

$$= \frac{1}{\Gamma(\delta-\mu)} \int_0^x (t-x)^{\delta-\mu-1} f(t) dt$$

which shows that $I^{\alpha}_{-\mu}(g) \in AC[a, \omega]$, since the inner integral is in $L^1[a, \omega]$. 

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**Lemma 4.** Suppose $1 < \gamma < 2$ and $g \in W^{1,1}_p$. Then $g^{(\gamma-1)} \in AC_{loc}$, $1 < \lambda < \gamma$, and formula (3.2) holds. Also $\int_1^\infty (y)^{-1-\lambda}a(y)dy$ for $0 < \mu < 1$, $\omega > 0$. 

Proof. Let $\delta = \gamma - 1$ and define 

$$
(t-x)^{\gamma-1} = \begin{cases} 
(t-x)^\delta, & t > x, \\
0, & t \leq x.
\end{cases}
$$

Use the decomposition in (3.5) to write 

$$
\mathcal{I}_s\varphi g^{(\gamma-1)}(x) - \mathcal{I}_s\varphi g^{(\gamma-1)}(y) = g(x) - g(y) + R_s(x, y)
$$

for a.e. $x$, $y \in (0, \omega)$. Then from 

$$
|R_s(x, y)| = C \left| \int_1^\infty \left[ (t-x)^{\gamma-1} - (t-y)^{\gamma-1} \right] \int_0^1 (s-t)^{-1-\lambda}g(s)ds dt \right|
$$

$$
\leq C\|g\|_s \int_1^\infty \left[ (t-x)^{\gamma-1} - (t-y)^{\gamma-1} \right] (t-x)^{-\lambda} dt + 
$$

$$
+ C\|g\|_s \int_1^\infty \left[ (t-x)^{\gamma-1} - (t-y)^{\gamma-1} \right] (t-y)^{-\lambda} dt
$$

it is easily seen that $R_s(x, y) \to 0$ as $\omega \to \infty$. For, by the dominated convergence theorem, the penultimate term tends to zero as $\omega \to \infty$, and the last integral also tends to zero since, for fixed $x, y$ with $0 < x, y < 3\omega/4$, $|\cdot| \leq 3\omega/4 \to 0$ as $\omega \to \infty$. Therefore we have the representation 

$$
0 < \delta < 1
$$

(3.6)

$$
g(x) - g(y) = \frac{1}{\Gamma(\gamma-1)} \int_1^\infty \left[ (t-x)^{\gamma-1} - (t-y)^{\gamma-1} \right] g^{(\gamma-1)}(t) dt
$$

for a.e. $x, y \in (0, \omega)$ and hence for all $x, y > 0$ since both sides are continuous functions and the integral is absolutely convergent for $x, y > 0$. 

Since $g^{(\mu)} \in AC_{loc}$ and, by (3.3),

$$
g^{(\mu)}(y) - g^{(\mu)}(x) = \int_x^y g^{(\mu)}(t) dt
$$

tends to zero as $x, y \to \infty$, $\lim_{y \to \infty} g^{(\mu)}(y) = 1$ exists and

$$
g^{(\mu)}(x) = 1 - \int_x^\infty g^{(\mu)}(t) dt.
$$

(3.7)

In fact, $l = 0$. For if $l > 0$ and $g^{(\mu)}(x) = l > 1$, where $|b(x)| < l/2$ when $x > X$, then

$$
R^\mu_{2X}(g^{(\mu)})(X) = C \left\{ \int_X^X \frac{1}{\delta} X^{\gamma-1} + \int_X^{X^{1-\lambda}} (t-X)^{-\gamma-1}h(t) dt \right\}
$$

$$
\geq C \left\{ \int_X^X \frac{1}{\delta} X^{\gamma-1} - \frac{1}{2} \int_X^{X^{1-\lambda}} (t-X)^{-\gamma-1} dt \right\}
$$

$$
= C\lambda X^{\gamma-2\delta} \to \infty
$$

as $X \to \infty$, in contradiction to Lemma 3. Similarly, $l$ cannot be negative. Substitution of (3.7) (with $l = 0$) into (3.6) and some obvious manipulations yield

$$
g(x) - g(y) = \frac{1}{\Gamma(\gamma-1)} \int_x^\infty (t-x)^{-\gamma} g^{(\mu)}(t) dt
$$

which shows that $g \in AC_{loc}$ since the inner integral is in $\mathcal{F}_s$. Hence the right side of (3.8) also equals $\int_y g^{(\mu)}(t) dt$, so

$$
g^{(\mu)}(t) = \frac{1}{\Gamma(\gamma-1)} \int_x^\infty (t-x)^{-\gamma} g^{(\mu)}(t) dt
$$

(3.9)

If $0 < \mu < 1$, then $\lim_{y \to \infty} g^{(\mu)}(y)^{-\gamma} = 0$ since $g$ is bounded, and by an integration by parts and (3.9) we have

$$
g^{(\mu)}(x) = \lim_{y \to \infty} \int_x^y g^{(\mu)}(t) dt
$$

(3.10)

$$
= \frac{1}{\Gamma(2-\mu)} \lim_{y \to \infty} \frac{d}{dx} \int_x^y (t-x)^{-\gamma} g^{(\mu)}(t) dt
$$

$$
= \frac{1}{\Gamma(1-\mu)} \lim_{y \to \infty} \frac{d}{dx} \int_x^y g^{(\mu)}(t) (t-x)^{-\gamma} dt
$$

$$
= \frac{1}{\Gamma(1-\mu)} \int_x^y (t-x)^{-\gamma} g^{(\mu)}(t) dt
$$

$$
= \frac{1}{\Gamma(1-\mu)} \int_x^y (t-x)^{-\gamma} \int_x^t (t-s)^{-\gamma} g^{(\mu)}(s) ds dt
$$

$$
= \frac{1}{\Gamma(1-\mu)} \Gamma(\gamma-1) \int_x^y (t-x)^{-\gamma} \int_x^t (t-s)^{-\gamma} g^{(\mu)}(s) ds dt
$$

$$
= \frac{1}{\Gamma(1-\mu)} \Gamma(\gamma-1) \int_x^y (t-x)^{-\gamma} \int_x^t (t-s)^{-\gamma} g^{(\mu)}(s) ds dt
$$

$$
= \frac{1}{\Gamma(1-\mu)} \Gamma(\gamma-1) \int_x^y \frac{d}{dx} (t-x)^{-\gamma} g^{(\mu)}(t) dt
$$

$$
= \frac{1}{\Gamma(1-\mu)} \Gamma(\gamma-1) \int_x^y \frac{d}{dx} (t-x)^{-\gamma} g^{(\mu)}(t) dt
$$

a.e.
Comparison of the first and fourth lines in (3.10) also shows that \( I_{\infty}^{\alpha-\gamma}(\omega) \in AC_{\infty} \) \( \omega > 0 \).

If \( 1 < \mu < \gamma \), then it follows from (3.10) that

\[
g^{\alpha-\gamma}(\omega) = \frac{1}{\Gamma(\gamma - \mu)} \int_0^\infty (s - t)^{\gamma - \mu - 1} g(t) dt.\]

Since the inner integral in \( I_{\infty}^{\alpha-\gamma}(\omega) \in AC_{\infty} \) and (3.2) holds, which completes the proof.

**Lemma 5.** If \( \gamma = 1 \) or \( 2 \) and \( g \in WBV_{1,0} \), then \( I_{\infty}^{\alpha-\gamma}(g) \in AC_{\infty} \) for \( 0 < \mu < 1 \), \( \omega > 0 \), and (3.2) holds.

The proof is analogous to that of Lemma 4, except that when \( \gamma = 2 \) it is shown that if \( I > 0 \) in the formula

\[
g'(x) = 1 - \int_0^x g''(t) dt
\]

and \( g'(x) - 1 = h(x) \), where \( h(x) < t/2 \) for \( x \geq X \), then

\[
g(2X) - g(X) = \int_x^{2X} g''(t) dt = \frac{1}{\xi} \int_x^X h(t) dt
\]

\[
\geq LX/\xi \to \infty \quad \text{as} \quad X \to \infty
\]

which contradicts the boundedness of \( g \).

**Lemma 6.** If \( 0 < \gamma < 1 \) and \( g \in WBV_{1,0} \), then (3.2) holds.

Proof. Since the hypotheses do not imply that \( g \) is locally absolutely continuous, we have to proceed in a way which is quite different from the proof of Lemma 4. If \( 0 < \sigma < \omega \), then

\[
\int_0^\omega \int_0^{\omega} (s - t)^{\gamma - 
\mu - 1} \frac{d}{ds} I_{\infty}^{\alpha-\gamma}(g)(s) ds dt
\]

\[
= \frac{1}{\gamma - \mu} \int_0^\omega (s - t)^{\gamma - \mu - 1} \frac{d}{ds} I_{\infty}^{\alpha-\gamma}(g)(s) ds
\]

\[
= \frac{1}{\Gamma(1 - \gamma)} \int_0^\omega (s - t)^{\gamma - \mu - 1} \int_t^\omega (y - t)^{-\gamma} g(y) dy ds
\]

\[
= \frac{\Gamma(\gamma - \mu)}{\Gamma(1 - \mu)} \int_0^\omega \int_0^\omega (s - t)^{\gamma - \mu - 1} \frac{d}{ds} I_{\infty}^{\alpha-\gamma}(g)(s) ds dt
\]

Hence

\[
\frac{d}{dt} I_{\infty}^{\alpha-\gamma}(g)(t) = \frac{1}{\Gamma(1 - \gamma)} \int_t^\omega (s - t)^{\gamma - \mu - 1} \frac{d}{ds} I_{\infty}^{\alpha-\gamma}(g)(s) ds dt
\]

for a.e. \( t \) in \( (0, \omega) \) and the argument on p. 36 of [26] shows that if \( a > 0 \) and \( \Omega \) is a countable set dense in \( (a, \omega) \), then this formula also holds for all \( \omega \in \Omega \) for a.e. \( t \) in \( (0, \omega) \). In addition, for \( \omega \in \Omega \) and a.e. \( t \in (0, \omega) \)

\[
(3.11) \int_0^\omega \int_0^{\omega} (s - t)^{\gamma - \mu - 1} \frac{d}{ds} I_{\infty}^{\alpha-\gamma}(g)(s) ds dt
\]

\[
= C \int_0^\omega (s - t)^{\gamma - \mu - 1} \int_0^\omega (y - s)^{-\gamma} g(y) dy ds
\]

\[
\leq C ||g||_{\infty} (s - t)^{\gamma - \mu - 1} \to 0 \quad \text{as} \quad \omega \to \infty \quad (\omega \in \Omega).
\]

Since the first integral in (3.11) is a continuous function of \( \omega \) \( (0 < t < \omega < a) \) and for a.e. \( t \) in \( (0, \omega) \) it converges as \( \omega \to \infty \), it follows that \( g^{\alpha}(t) \) exists for a.e. \( t \) in \( (0, \omega) \) and that (3.2) holds a.e. in \( (0, \omega) \). But a is arbitrary, so (3.2) holds a.e. in \( (0, \infty) \).

**Lemma 7.** If \( \gamma = 2 \), then

\[
WBV_{1,0} \subset WBV_{1,0}, \quad k \leq \gamma - 1, \quad k \in N.
\]

**Lemma 8.** If \( \gamma > 0 \), \( 1 < \sigma < \infty \) and \( g \in WBV_{1,0} \), then (3.2) holds.

Lemmas 7 and 8 follow by the standard arguments in [26], pp. 36, 37, and the trivial observation that \( WBV_{1,0} \subset WBV_{p,0} \) when \( 1 < p < q \), \( \gamma > 0 \).

**Lemma 9.** If \( \gamma > 0 \) and \( 1 < \sigma < \infty \), then

\[
WBV_{1,0} \subset WBV_{1,0}, \quad 0 < \mu < \gamma.
\]

The proof is analogous to that of Lemma 1, with (3.2) being used instead of (2.1) and, naturally, the sums being replaced by integrals; therefore we omit it. The next result is an analog of Lemma 2.

**Lemma 10.** Let \( \gamma > 0 \), \( 1 < \sigma < \infty \), \( g \in L^\sigma \) and \( g_n = g(G_n - G), u_n > 0 \), where \( G_n \) is the function defined in Lemma 2. Then \( g \in WBV_{1,0} \) if and only if \( g_n \in WBV_{1,0} \) for each \( u_n > 0 \) and \( \sup_{u_n} ||g_n||_{1,0} < \infty \). Moreover, there exists constants \( C_1, C_2 \) independent of \( g \) and \( u_n > 0 \) such that

\[
(3.12) \sup_{u > 0} ||g_n||_{1,0} < C_1 \sup_{u > 0} ||g||_{1,0} < C_2 \sup_{u > 0} ||g||_{1,0}
\]

and the left inequality also holds with \( \sup \) replaced by \( \sup \), where \( a > 0 \).
Proof. Suppose \(0 < \gamma < 1\), \(1 \leq q \leq \infty\) and \(g \in \text{WBV}_{\omega, r}\). Then formula (3.6) holds with \(\delta = \gamma\) and setting

\[
h(t) = \frac{1}{\Gamma(\gamma)} \int_0^t s^{1-\gamma} (g(s+t) - g(t)) \, ds,
\]

we have for \(0 < a < b\) that

\[
\int_a^b h(t) \, dt = \frac{1}{\Gamma(\gamma)} \int_a^b \int_0^\infty \left[ (w-w-t)_+^{\gamma-1} - (w-t)_+^{\gamma-1} \right] t^{-\gamma} g^{(2)}(w) \, dw \, ds \, dt
\]

\[
= \frac{1}{\Gamma(\gamma) \Gamma'(-\gamma)} \int_0^\infty \left[ \int_0^t (\cdots) \, dt \, ds \right] g^{(2)}(w) \, dw.
\]

Clearly, \(\{\cdots\} = 0\) if \(w < a\), so \(\int_0^\infty \cdots \, dw = \int_0^a \cdots \, dw\). By considering the intervals \((0, w-b), (w-b, w-a), (w-a, \infty)\), we find that the above double integral in brackets equals zero for \(w > b\) and equals \(\Gamma(-\gamma) \Gamma'(-\gamma)\) for \(a < w < b\). Hence

\[
\int_a^b h(t) \, dt = \frac{1}{\Gamma(\gamma)} \int_a^b g^{(2)}(w) \, dw
\]

and then \(g^{(2)}(t) = h(t)\), a.e., since \(a\) and \(b\) are arbitrary.

Now let \(w > 0\), \(A_\gamma g(t) = g(s+t) - g(t)\), and

\[
h_a(t) = \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{1-\gamma} A_\gamma g_a(t) \, ds.
\]

Then, setting \(H_a(t) = G_a(t) - G_a(t)\) and

\[
\Gamma(-\gamma) h_a(t) = \int_0^\infty s^{1-\gamma} A_\gamma g_a(t) \, ds +
\]

\[
- H_a(t) = \int_0^\infty s^{1-\gamma} A_\gamma g(t) \, ds - g(t) \int_0^\infty s^{1-\gamma} A_\gamma H_a(t) \, ds,
\]

we have

\[
\left( \int_{|t| \leq 1} |\mathcal{F} H_a(t)|^2 \frac{dt}{T} \right)^{1/2} \leq 2 \|g\|_{\omega, \gamma} \left( \int_{|t| \leq 1} \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{1-\gamma} |A_\gamma H_a(t) ds|^2 \frac{dt}{T} \right)^{1/2}.
\]

uniformly in \(m\), where we used the fact that \(|A_\gamma H_a(t)| \leq \|A_\gamma G_a(t)\| +
\]

\[
+ \|A_\gamma G_a(t)\| = \Gamma(-\gamma) \Gamma'(-\gamma) = C(g, \omega, \gamma)
\]

where \((-i\omega)^\gamma\) is defined by

\[
(-i\omega)^\gamma = |\omega|^{\gamma} \left( \cos \frac{\pi \gamma}{2} - i (\text{sign} \omega) \sin \frac{\pi \gamma}{2} \right).
\]

Since the right side of (3.14) equals the Fourier distributional transform of \((g_a)^{(0)} \ast S'\), where \(S'\) is the set of tempered distributions, it follows that \((g_a)^{(0)} \ast S' = h_a(t)\), a.e. and that the right-side inequality in (3.12) holds.

Since, for \(0 < a \leq \omega < b \leq \omega < a'\),

\[
I_{a'}^{-\gamma}(g_a(x)) = I_{a'}^{-\gamma}(g_a(x)) + \frac{1}{\Gamma(-\gamma)} \int_a^{a'} (t-a')^{-\gamma} g_a(t) \, dt
\]

and, by means of [23], p. 371, Ex. 6, the integral on the right side is in \(AC[a, b]\), to show that \(I_{a'}^{-\gamma}(g_a) \in AC_{\infty}, \omega > 0\), it suffices to show this for \(\omega\) so large that \(g_a(t) = 0\) for \(t \geq \omega - 1\). Then

\[
I_{a'}^{-\gamma}(g_a(x)) = \frac{1}{\Gamma(-\gamma)} \int_a^{a'} (t-a')^{-\gamma} g_a(t) \, dt
\]

and extending this function to \(x \leq 0\) by letting \(g_a(t) = 0\) for \(t \leq 0\), we see that the function

\[
f(x) = I_{a'}^{-\gamma}(g_a(x)) - I_{a'}^{-\gamma}(g_a(x) + \omega)
\]

belongs to \(L^1(-\infty, \infty)\) and

\[
f'(x) = (-i\omega)^{\gamma} - (-i\omega)^{\gamma} \circ (g_a)\circ (x) + \omega)
\]

where \((-i\omega)^{\gamma}\) is defined by (3.15). Thus, from (3.14),

\[
(-i\omega)^{\gamma} - (-i\omega)^{\gamma} \circ (g_a)\circ (x) + \omega)
\]

is

\[
(h_a(x) - (h_a(x) + \omega))
\]
which, by [5], Prop. 5.1.15, implies that $f \in A_{\text{loc}}(-\infty, \infty)$, and hence $I_{\nu}^{-\gamma}(g_{\nu}) \in A_{\text{loc}}(0, \infty)$ since $f(x) = I_{\nu}^{-\gamma}(g_{\nu})(x)$ for $x > 0$.

To show that $I_{\nu}^{-\gamma}(g) \in A_{\text{loc}}$ if $g_{\nu} \in \text{WBV}_{\nu}$ for each $\nu > 0$, one need but observe that if $0 < a < x < b < \infty$ and if $u$ is chosen so large that $g_{\nu} = g$ on $[a, u]$, then $I_{\nu}^{-\gamma}(g_{\nu})(x) = I_{\nu}^{-\gamma}(g)(x)$ for $x > 0$. For $\gamma > 1$ it easily follows by applying (3.6) to $g_{\nu}$ that $g_{\nu}^{(\nu)} \in A_{\text{loc}}$ if $g_{\nu} \in \text{WBV}_{\nu}$, $\nu > 0$. The left-side inequality in (3.12) is proved for $\gamma > 0$ just as in the corresponding case of Lemma 2, with sums replaced by integrals.

The remaining part of the proof for the case $\gamma \geq 1$ is a slight modification of the proof of relation (1.2) in [27] and so it is omitted. Furthermore, it should be pointed out that, as in [26], p. 37, if $g \in \text{WBV}_{\nu}$, $\nu > 0$, then

$$g^{(-\Gamma)}(t) = \{g^{(\nu)}(t)\}^{\nu-\Gamma}, \text{ a.e.}$$

4. Bessel potential spaces. Before turning to the proofs of the theorems we shall also need the following characterization of the Bessel potential space $L_{\nu}^{p}$ by an appropriate space of hypersingular integrals.

**Lemma 11.** Let $1 < q < \infty$.

(a) If $\gamma \in \mathbb{N}$, then

$$||g||_{L_{\nu}^{p}} \simeq \sum_{\nu=0}^{\gamma} ||g^{(\nu)}||_{L_{a}^{p}},$$

(b) If $\gamma > 0$ and $\delta = \gamma - \lfloor \gamma \rfloor > 0$, then

$$||g||_{L_{\nu}^{p}} \simeq \sum_{\nu=0}^{\lfloor \gamma \rfloor} ||g^{(\nu)}||_{L_{a}^{p}} + \left[ \frac{1}{f(\nu)} \int_{0}^{\infty} s^{1-\gamma} A_{\nu} g^{(\nu)}(t) \, dt \right]_{L_{a}^{p}} = ||g||_{L_{\nu}^{p}},$$

where $A_{\nu}$ is as defined in Section 3.

**Proof.** Part (a) is due to Calderón; for a proof see Stein [20], p. 135. Consider (b). From (a) and (20), p. 136,

$$||g||_{L_{\nu}^{p}} \simeq \sum_{\nu=0}^{\lfloor \gamma \rfloor} ||g^{(\nu)}||_{L_{a}^{p}} + ||g^{(\lfloor \gamma \rfloor)}||_{L_{a}^{p}},$$

which implies that we may restrict ourselves to the case $0 < \gamma < 1$. Let $g \in E_{2}$, $1 < q < \infty$. Then from Wheeden [31] we have

$$2 \left\| \int_{0}^{\infty} s^{1-\gamma} A_{\nu} g(t) \, dt + \int_{(a-b)} \frac{g(a)}{t} \, dt \right\|_{L_{d}^{q}} \leq C \left\| \int_{0}^{\infty} |s|^{1-\gamma} A_{\nu} g(t) \, dt \right\|_{L_{d}^{q}}.$$

Conversely, let $\varphi \in \overline{E}$, $\overline{E}$ being the set of all infinitely differentiable functions on $(-\infty, \infty)$ that are rapidly decreasing at infinity. By [20], pp. 133, 134 there is a bounded measure $\mu \in (-\infty, \infty)$ with

$$(1 + |s|)^{\gamma-1} = (d\mu)^{\gamma-1}(s)(1 + |s|).$$

From (3.15),

$$|\varphi|^{\gamma} = (-i\varphi)^{\gamma} \left( \cos \frac{\pi}{2} \gamma + i(\sin \varphi) \sin \frac{\pi}{2} \gamma \right)$$

and so letting $g^{-}$ denote the Hilbert transform of $g$ defined by $g^{-}(\varphi) = -i(\sin \varphi) g^{+}(\varphi)$ and letting $F^{-1}$ denote the inverse Fourier transform, we have

$$||\varphi||_{L_{d}^{\gamma}} \leq \left\| \left[ F^{-1}(1 + |s|)^{\gamma-1} \varphi \right] \right\|_{L_{d}^{\gamma}} \leq C \left[ ||\varphi||_{L_{d}^{\gamma}} + \left\| F^{-1}((-i\varphi)^{\gamma} \varphi) \right\|_{L_{d}^{\gamma}} \right] \leq C \left[ ||\varphi||_{L_{d}^{\gamma}} + \int_{0}^{\infty} s^{1-\gamma} A_{\nu} \varphi(t) \, dt \right]_{L_{d}^{\gamma}}$$

with $C$ independent of $\varphi \in \overline{E}$. The desired result then follows by applying the density argument in Wheeden [31], I, p. 432.

5. **Proof of Theorem 2.** Since, by Lemma 9, the case $\gamma \in \mathbb{N}$ is already proved in [11], we shall only consider the case $\gamma > 0$, $\gamma \notin \mathbb{N}$. Without loss of generality we may assume that $n_{\gamma} = 0$ for any sequence $n_{\gamma}$ under consideration.

We shall first show that if $\gamma > 1/q$, $1 < q \leq \infty$ or $\gamma > 1$, $q = 1$, and if $n_{k} = g(k)$, $k \in \mathbb{N}$, where $g \in \text{WBV}_{\overline{E}} \cap E_{1}^{\infty}$, then

$$(1) \quad ||n_{k}||_{L_{\nu}^{p}} \leq C \left[ \sum_{k} \left| A_{k}^{n_{k}} \sum_{k=1}^{m} \left( f^{(k)} \right)^{-1} \varphi^{(k)}(s) \, ds \right| \right],$$

which implies, by Lemmas 2 and 10, that $\text{wbv}(g, p) \subset \text{wbv}_{\nu}$. Consider the case $g = \infty$ and let $2^{m+1} \leq k < 3^{m+1}$. Looking at the proof of Lemma 3 it is clear that (3.1) even holds for $\gamma > 0$ provided $g$ has compact support. Thus

$$||A_{k} g^{(k)}||_{L_{d}^{q}} = C \left[ \int_{k}^{k+1} \left| A_{k}^{n_{k}} \sum_{k=1}^{m} \left( f^{(k)} \right)^{-1} \varphi^{(k)}(s) \, ds \right| \right],$$

For $\gamma > 0$, $1 < s < 1+1$,

$$(5.2) \quad \sum_{k=1}^{m} \left( f^{(k)} \right)^{-1} A_{k}^{n_{k}+1} \leq C (1 + 1 - k)^{-1} \left( |s-1|^{-1} + 1 \right).$$
Then \( g(t) \) is a bounded function which vanishes identically for large \( t \) and satisfies \( g(k) = \eta_k \) for \( k \in \mathbb{N} \). Also, \( g^{(0)}, \ldots, g^{(r-1)} \in AC([0, \infty)) \) and

\[
q(t) = \frac{-j}{(s-t)^\nu} \left[ - \psi(0)(s) - \psi(s) \right] A'\eta_{k, L-1},
\]

for \( k = t \leq k+1/2 \), \( k+1/2 \leq t \leq k+1 \), \( k = 0, 1, \ldots \). Since it is clear that \( g \in WBV_{L+1} = WBV_{L+1} \), we now have to show that \( |g|_{L+1, \infty} \leq C \) independent of \( \eta \). By (3.2)

\[
g^{(0)}(t) = \int (s-t)^\nu g'^{(0)}(s) ds, \quad \mu = [\gamma]+1-\gamma,
\]

and hence, for \( k \leq t \leq k+1 \),

\[
g^{(0)}(t) = - \sum_{\ell=k+1}^{\infty} \int (s-t)^\nu \left[ - \psi(0)(s) - \psi(\ell)(s) \right] A'\eta_{\ell-1} ds + \int (s-t)^\nu \psi(\ell)(s) A'\eta_{\ell-1} ds + \int (s-t)^\nu g'^{(0)}(s) ds = I_1 + I_2 + I_3.
\]

Since \( \psi^{(0)}(s) = \psi(0)(s) \) and, by (2.1),

\[
A'\eta_{k, L-1} = A'^{(k+1)}A'\eta_{k, L-1} = \sum_{\ell=0}^{\infty} A_\ell A'^{-(k+1)}A'\eta_{\ell+1},
\]

it follows that

\[
|I_i| \leq C \left| \sum_{\ell=k+1}^{\infty} \int (s-t)^\nu \left[ - \psi(0)(s) - \psi(\ell)(s) \right] A'\eta_{\ell-1} ds \right| \psi^{(0)}(s) ds.
\]

Now set \( i = n-L \) and use (5.2) to obtain

\[
|I_i| \leq C \left| \sum_{\ell=k+1}^{\infty} \int (s-t)^\nu \left[ - \psi(0)(s) - \psi(\ell)(s) \right] A'\eta_{\ell-1} ds \right| \psi^{(0)}(s) ds.
\]

Since \( 0 \leq k+1+s-t \leq 2/3 \) and \( \psi^{(0)}(s) \leq C \). Similarly,

\[
|I_i| \leq C \left| \sum_{\ell=k+1}^{\infty} \int (s-t)^\nu \left[ - \psi(0)(s) - \psi(\ell)(s) \right] A'\eta_{\ell-1} ds \right| \psi^{(0)}(s) ds.
\]

Since, by (5.4),

\[
|g^{(0)}(s)\psi(\ell)\rangle \leq C \left| \langle A'\eta_{\ell-1} | A'\eta_{\ell} \rangle \right|, \quad 0 \leq s \leq 1,
\]
we have
\begin{align}
|I_k| & \leq C \left( |(d^s \eta_{k-1})| + |d^s \eta_k| \right) \\
& \leq C \sum_{n=0}^m \left( |(d^s \eta_{k+n-1})| + |d^s \eta_{k+n}| \right)
\end{align}
and hence for \( k < t < k + 1, \gamma > 0 \),
\begin{align}
|g(t)| & \leq C \sum_{n=0}^m \left( |(d^s \eta_{k+n-1})| + |d^s \eta_{k+n}| + |d^s \eta_{k+n+1}| \right)
\end{align}
by (5.5). Therefore \( |g(t)| \leq C |\eta|_{\infty} \) and if \( m \) is a non-positive integer, then
\begin{align}
\left( \int_{s^n-1}^{s^n} \left| |g(t)| \right|^2 \frac{dt}{t} \right)^{1/2} & \leq C |\eta|_{\infty} \leq C |\eta|_{L_{\infty, s^n}}
\end{align}
If \( m \) is a positive integer and \( g = 1, \gamma \gg 1 \), then (5.7) gives
\begin{align}
\int_{s^n-1}^{s^n} \left( |g(t)| \right)^2 \frac{dt}{t} & \leq C \sum_{n=0}^m \int_{s^n-1}^{s^n} \left( |(d^s \eta_{k+n-1})| + |d^s \eta_{k+n}| + |d^s \eta_{k+n+1}| \right) dt \\
& \leq C \sum_{n=0}^m \sum_{k} \left( |(d^s \eta_{k+n-1})| + |d^s \eta_{k+n}| \right) A_{k+1} \leq C |\eta|_{L_{\infty, s^n}}
\end{align}
since
\begin{align}
\sum_{k} A_{k+1} |d^s \eta_{k+n}| & \leq 2 |\eta|_{L_{\infty, s^n}}
\end{align}
uniformly in \( t \). If \( g = \infty \), the desired estimate immediately follows from (5.7). If \( 1 < q < \infty \), \( \gamma \gg 1 / q \), then
\begin{align}
\int_{s^n-1}^{s^n} \left| \frac{|g(t)|}{t^{1/q}} \right|^q \frac{dt}{t} & \leq C \sum_{n=0}^m \sum_{k} \left( |(d^s \eta_{k+n-1})| + |d^s \eta_{k+n}| \right) A_{k+1} \leq 2 |\eta|_{L_{\infty, s^n}}
\end{align}
which completes the proof for the case where \( \eta \) has compact support.

For arbitrary \( \eta \in \text{wbv}_{\infty, \gamma} \), with \( \eta_0 = 0 \), let \( g \) be the function defined by (6.3). Also let \( u \gg 1 \) and \( f_u \) be the function constructed in the same way as \( g \), but with the sequence \( \eta_k \) replaced by the sequence \( \{G(k) \eta_k\} \), where \( G \) is as defined in Lemma 2. Then \( g(k) = \eta_k, f_u(k) = G(k) \eta_k \) for \( k \in \mathbb{N} \) and \( f_u \in L^\infty \). Since \( d^s \eta_k \) depends only on \( \eta_k, \ldots, \eta_0 \) and \( G(k) \eta_k \) = 1, \( k \leq u \), it follows for \( 0 \leq t \leq |[k] - j| + 1 \) that both \( g(t) \) and \( f_u(t) \) depend only on \( \eta_0, \ldots, \eta_{[k]} \); hence \( g(t) = f_u(t), 0 \leq t \leq |[k] - j| + 1 \). With \( 1 < v \leq |(k+1) - j| + 1 \) it then follows that
\begin{align}
g_u(t) = (G_u(t) - G_{u-1}(t)) g(t) = (G_u(t) - G_{u-1}(t)) f_u(t), \quad t > 0.
\end{align}
Thus, by Lemma 2, the above compact support case, and Lemma 10,
\begin{align}
|g_u(t)|_{L^\infty} & \leq C |G(k) \eta_k|_{L^\infty} \leq C |f_u|_{L^\infty} \leq C |g|_{L_{\infty, s^n}}
\end{align}
uniformly in \( v \gg 1 \), which completes the proof by an application of Lemma 10.

6. Proof of Theorem 3. Since the case \( \gamma \in \mathbb{N} \) is already proved in Connett and Schwartz [11], we shall assume that \( \gamma > 1/2, 1 < q < \infty \), \( \gamma \notin \mathbb{N} \).

First observe that if \( g \in S(q, \gamma) \) and we have proved that \( g_{\nu}^{(k)} \in AC_{\infty, \infty} \) for \( u > 0, k = 1, \ldots, [q] \), where \( g_u \) is the function defined in Lemma 10, then it easily follows that \( g_{\nu}^{(k)} \in AC_{\infty, \infty} \). For \( 0 < a \leq t \leq b \) and \( u \) is chosen so large that \( g_u = 0 \) on \([a, b+1] \), then
\begin{align}
(g_u - g_{u-1})(t) = - \frac{\delta}{(1 - \delta)} \int_{t \in [a, b]} \frac{(s - t)^{-1/2}}{s^{1/2}} (g_u - g_{u-1})(s) ds
\end{align}
for \( \delta = \gamma - \gamma' \) and this function can be repeatedly differentiated for \( a \leq t \leq b \). Thus \( g_{\nu}^{(k)} = (g_u - g_{u-1}) \in AC_{\infty, \infty} \). Similarly, if \( L_{\nu}^{(k)}(g_u) \) \( \in AC_{\infty, \infty} \) for \( a > 0 \), then \( L_{\nu}^{(k)}(g) \in AC_{\infty, \infty} \). Therefore, by Lemma 10 and the fact [11] that
\begin{align}
|g|_{L^q(s^n)} \leq \sup \{ |g_u|_{L^q(s^n)} \}
\end{align}
in proving Theorem 3 it suffices to consider only functions with compact support in \((0, \infty)\).

Also observe that in the definition of \( WBV_{\infty, \gamma} \) spaces the fact that we decompose \((0, \infty)\) by intervals of the form \([2^{n-1}, 2^n] \) is unessential; see Remark 6.2.2 in [20], p. 109. In what follows it is more convenient to replace \([2^{n-1}, 2^n] \) by \([2^m, 2^{m+1}] \). Furthermore, if \( g \in S(q, \gamma), \gamma > 1/q \), then \( |g|_{L^\infty} \leq C |g|_{L^{1/2, s}} \), while if \( g \in WBV_{\infty, \gamma} \) then \( |g|_{L^q} \leq C |g|_{L^\infty} \), where \( \varphi_\nu \) and \( \varphi \) are as defined in Section 1. These observations reduce the proof to showing that if \( g \in S(q, \gamma), g \in L^\infty, then g \) satisfies the \( AC_{\infty, \infty} \) conditions in the definition of \( WBV_{\infty, \gamma} \) and
\begin{align}
\left( \int_{s^n} \left| |g^{(\nu)}(t)| \right|^2 \frac{dt}{t} \right)^{1/2} \leq C |g|_{L^{1/2, s}}
\end{align}
uniformly in \( m \in \mathbb{Z} \), and, conversely, by Lemma 11, that if \( g \in \text{WBV}_{a,r} \cap L^\infty \), then
\[
\left\| \frac{1}{r(t)} \int_0^t s^{1-a} A_s'(\rho^s g^*(t)^{[1]} g(t)^{[0]}) ds \right\|_{L^1} \leq C \|g\|_{\text{WBV}_{a,r}}
\]
uniformly for \( h \in \mathbb{Z} \).

We shall first show for \( g \in S(\mathbb{R}, \gamma) \) that
\[
(6.3) \quad \sum_{h=-m}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\rho^s}{r(t)} \int_0^t s^{1-a} A_s'(\rho^s g^*(t)^{[1]} g(t)^{[0]}) ds \right| dt \frac{dI}{t} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
uniformly in \( m \). After (6.3) is established we will then show that
\[
(6.4) \quad |\rho^s g^*(t)^{[1]} g(t)^{[0]}| \leq \frac{1}{r(t)} \int_0^t s^{1-a} A_s'(\rho^s g^*(t)^{[1]} g(t)^{[0]}) ds, \quad \text{a.e.}
\]
which then yields (6.1) because
\[
\left( \sum_{m} \int_{\mathbb{R}} \left| \rho^s g^*(t)^{[1]} g(t)^{[0]} \right|^2 dt \right)^{1/2} \leq C \sum_{m} \int_{\mathbb{R}} \left| \rho^s g^*(t)^{[1]} g(t)^{[0]} \right| dt \frac{dI}{t} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
In the following estimates we will omit the factor \( 1/r(t) \), keeping in mind that it keeps the constant multiples of \( 1/\delta \) and \( 1/(1 - \delta) \) occurring in the integrations bounded. With the aid of
\[
(6.5) \quad e^{s} g^{[0]}(t) = \sum_{h=-m}^{m} \left( \int_{\mathbb{R}} \left( 1 - e^{-\delta s}\right)^{1-\delta} e^{-\delta t} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(u) du \right) \frac{dI}{t} g(t)
\]
and the successive substitutions \( s = r - t, \quad r = e^s, \quad t = e^r \), the left side of (6.3) can be estimated by
\[
C \sum_{h=-m}^{m} \sum_{l=0}^{l} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( 1 - e^{-\delta s}\right)^{1-\delta} e^{-\delta t} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(u) du \frac{dI}{t} \leq C \sum_{l=0}^{l} \sum_{h=-m}^{m} I_{h,l},
\]
Consider the single terms of the sum. The most critical ones occurring are those with \( h = m \) or \( h = m + 1 \). Let us first discuss
\[
I_{h,m} = \int_{\mathbb{R}} \int_{\mathbb{R}} (1 - e^{-\delta s})^{1-\delta} e^{-\delta t} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(t-u) du \frac{dI}{t} \frac{dI}{u} \|
\]
Since \( \rho^s(u) = 0 \) for \( u \geq m + 2\beta \),
\[
I_{h,m} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} (1 - e^{-\delta s})^{1-\delta} e^{-\delta t} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(t-u) du \frac{dI}{t} \|
\]
uniformly for \( h \in \mathbb{Z} \).

By the integral Minkowski inequality and a result of Strichartz [22], \( \| \| \), it follows for \( j \leq \gamma \) that
\[
J_{h,m} \leq C \|\rho^s g^*[l_{h,m}] \leq C \|\rho^s g^*[l_{h,m}]*_{\mathbb{R}} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
where we used the fact that if \( h \in I_{h, j} \), then
\[
(6.6) \quad |\rho^s|_{l_{h,m}} \leq \|\rho^s|_{l_{h,m}}|_{l_{h,m} \leq \rho^s, \quad 0 \leq \rho^s < \lambda}.
\]
In view of Lemma 11 we want to replace \( \int_{\mathbb{R}} \) in \( J_{h,m} \) by
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} (1 - e^{-\delta s})^{1-\delta} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(u) dt \frac{dI}{t} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
As in the estimation of \( J_{h,m} \), we have
\[
(6.7) \quad \int_{\mathbb{R}} \left( 1 - e^{-\delta s}\right)^{1-\delta} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(u) du \frac{dI}{t} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
so it remains to estimate
\[
(6.8) \quad \int_{\mathbb{R}} \left( 1 - e^{-\delta s}\right)^{1-\delta} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(u) du \frac{dI}{t} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
and
\[
(6.9) \quad \int_{\mathbb{R}} \left( 1 - e^{-\delta s}\right)^{1-\delta} (\rho^s g^*(t)^{[1]} g(t)^{[0]})(u) du \frac{dI}{t} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
Since
\[
1 - e^{-\delta s} \geq e^{-\delta s} > e^{-\delta s} > 0, \quad (1 - e^{-\delta s}) \leq 0
\]
for \( 0 < t < 1 \), the above formula can be estimated after routine reformulations by
\[
C \|\rho^s g^*[l_{h,m}] \leq C \|\rho^s g^*[l_{h,m}]*_{\mathbb{R}} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]
So a combination of these estimates yields
\[
I_{h,m} \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})} + C \|g\|_{L^2(\mathbb{R}, \mathbb{R})} \leq C \|\rho^s g^*[l_{h,m}] \leq C \|g\|_{L^2(\mathbb{R}, \mathbb{R})},
\]

by (6.6) and Lemma 11. Similarly,

\[ I_{f,m+1} \leq C \|g\|_{L^p(0,1)}^* \]

so it remains to estimate \( \sum_{k=m+2}^{\infty} \sum_{j \leq k} I_{f,k} \). Since \( \varphi_k = [k-2/3, k+2/3] \),

\[ \sum_{k=m+2}^{\infty} \sum_{j \leq k} I_{f,k} \]

\[ \leq \sum_{k=m+2}^{\infty} \sum_{j \leq k} (1 - e^{-C})^{-1} \left( \int_{k-2/3}^{k+2/3} \int_{k-2/3}^{k+2/3} e^{-|t-u|} \| \varphi_j g^* \|^2_{L^2} \right) \frac{dt}{du} \left|\frac{du}{dt}\right|^{1/2} \]

\[ \leq C \sum_{k=m+2}^{\infty} \sum_{j \leq k} \int_{k-2/3}^{k+2/3} e^{-|t|} \| \varphi_j g^* \|^2_{L^2} dt \]

\[ \leq C \|g\|_{L^p(0,1)}^* \sum_{k=m+2}^{\infty} \sum_{j \leq k} \left( \int_{k-2/3}^{k+2/3} e^{-|t|} \| \varphi_j g^* \|^2_{L^2} dt \right) \leq C \|g\|_{L^p(0,1)}^* \]

and hence (6.3) is established.

To consider (6.4), we set

\[ g_k(t) = \varphi_k(\log t) g(t), \quad t > 0, \]

\[ g_k(t) = 0, \quad t \leq 0. \]

Since \( S(g, \gamma) \subseteq S(g, [\gamma]) \) and \( g_k \) has compact support, it is clear that \( g_k, g_k^{(0)} \in L^1(-\infty, \infty) \). By (6.3)

\[ h_k(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t} s^{-\alpha-1} A_s g_k^{(0)}(s) ds \in L^1(\alpha^{-\alpha}, \infty), \quad \alpha \geq m \]

for each \( m \in Z \), and, by a direct elementary calculation, it also belongs to \( L^1(-\infty, \alpha^{-\alpha}) \). Hence we may take the (classical) Fourier transform

\[ (\hat{g_k^{(0)}})(v) = (-1)^{b+1} \int_{-\infty}^{\infty} \frac{1}{\Gamma(1-\alpha)} (s - t)^{-\alpha} \hat{g_k}(v) ds \]

\[ = (-1)^{b+1} (-i\alpha)^{-\alpha} \hat{g_k}(v). \]

Hence, by the uniqueness theorem for Fourier transforms, (6.4) holds and to complete the proof of \( S(g, \gamma) \subseteq \overline{W_{2p}^s} \), it only remains to show that \( I_{f,1}^{-\alpha}(g), g^{(0)}, \ldots, g^{(m-1)} \in \overline{AC_{\text{loc}}} \).

Since \( g \) has compact support, it follows as in the proof of Lemma 10 that \( I_{f,1}^{-\alpha}(g) \in \overline{AC_{\text{loc}}} \), \( \alpha > 0 \). Also, from the above, if \( \gamma > 1 \), then

\[ (-i\alpha)^{-\alpha}\hat{g_k}(v) = (-i\alpha)^{\alpha} \hat{g_k}(v) = (1)^{\alpha} h_k(v) \]

and hence, by (6). Prop. 5.1.15, \( g^{(0)}, \ldots, g^{(m-1)} \in \overline{AC_{\text{loc}}} \). But \( g \) is a finite sum of the \( g_k \) and fractional differentiation is linear, so \( g^{(0)}, \ldots, g^{(m-1)} \in \overline{AC_{\text{loc}}} \).

As mentioned before, to prove the converse we have to show that if \( g \in \overline{W_{2p}^s} \), then \( \gamma > 1/g \), \( 1 < q < \infty \), then (6.2) holds. By the integer case of the theorem,

\[ \left\| \int_{\infty}^{\infty} e^{-t^\gamma} A_s (\varphi_k g^*)^{(0)}(t) ds \right\|_q \leq C \left\| I_1 + \int_{0}^{\infty} e^{-t^\gamma} A_s (\varphi_k g^*)^{(0)}(t) ds \right\|_q \]

\[ \leq I_1 + C \|g\|_{L^p(0,1)}^* \]

We shall handle the term \( I_1 \) essentially reading the proof of (6.3) backwards. So again by the integer case,

\[ I_1 \leq \left\| \int_{0}^{1} (1 - e^{-t})^{-1} A_s (\varphi_k g^*)^{(0)}(t) ds \right\|_q \]

\[ + \left\| \int_{1}^{\infty} A_s (\varphi_k g^*)^{(0)}(t) (e^{-t})^{-1} (1 - e^{-t})^{-1} ds \right\|_q \]

\[ = I_1 + C \|g\|_{L^p(0,1)}^* \]

where

\[ I_1 \leq \int_{0}^{1} (1 - e^{-t})^{-1} (1 - e^{-t}) (\varphi_k g^*)^{(0)}(s+t) ds \]

\[ + \int_{1}^{\infty} (1 - e^{-t})^{-1} (1 - e^{-t}) (\varphi_k g^*)^{(0)}(t) ds \]

\[ + \int_{0}^{\infty} (1 - e^{-t})^{-1} e^{-t} (\varphi_k g^*)^{(0)}(s) - e^{-t} (\varphi_k g^*)^{(0)}(t) \]

\[ + \int_{0}^{\infty} (1 - e^{-t})^{-1} (1 - e^{-t}) (\varphi_k g^*)^{(0)}(t) \]

\[ = I_1 + I_4 + I_5. \]

Clearly, by the integral Minkowski inequality,

\[ I_1 + I_4 + I_5 \leq C \|g\|_{L^p(0,1)}^* \]

so the essential term is

\[ I_5 = \left\| \int_{0}^{\infty} (1 - e^{-t})^{-1} e^{-t} (\varphi_k g^*)^{(0)}(s) - e^{-t} (\varphi_k g^*)^{(0)}(t) ds \right\|_q \]

\[ = e^{-t} (\varphi_k g^*)^{(0)}(t) \frac{dt}{ds}. \]
By first substituting \( e^t = u \) and then \( e^t = s \), observing that \( \text{supp } \Delta_p \phi_k(\log u)g(u) \cap \{ e^{k+1}, \infty \} \) is empty, and then letting \( s = t + u \), one arrives at

\[
I = C \int_0^{e^k} \left| \int_0^t (t+u)^{-\beta} \left( g(t+u) \frac{d}{du} \right) \left( \frac{\phi_k(\log(t+u))}{u} \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} du \leq C \sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left| \int_0^t (t+u)^{-\beta} \left( g(t+u) \frac{d}{du} \right) \left( \frac{\phi_k(\log(t+u))}{u} \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} dt \right)
\]

Setting \( f_\beta(u) = \phi_k(\log u)g(u), u > 0 \), and using the fact that

\[
(6.7) \quad \left( \frac{t}{\text{d}t} \right)^n g(t) = \sum_{n=0}^{\infty} a_n e^{n\beta} g_\beta^n(t), \quad a_{n,0} = 0 \quad \text{for } n \geq 1,
\]

we have

\[
J_k \leq C \sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left| \int_0^t (t+u)^{-\beta} \Delta_p \left( (u+1)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} dt \right)
\]

\[
\leq C \sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left| \int_0^t (t+u)^{-\beta} \Delta_p \left( (u+1)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} dt \right) + C \sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left| \int_0^t (t+u)^{-\beta} \left( 1 - (1+\beta) f_\beta^0(u) g_\beta^0(u) \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} dt \right).
\]

By (3.13), (3.16) and Lemma 9 the first sum on the right side is majorized by \( C \| g \|_{L^p(V)} \). In the second sum the substitution \( u = (u+1) \) leads to

\[
\sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left| \int_0^t (t+u)^{-\beta} \left( (u+1)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} dt \right)
\]

\[
\leq C \sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left| \int_0^t (t+u)^{-\beta} \left( (u+1)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} dt \right)
\]

by Lemmas 9 and 10. Thus it remains to estimate \( \tilde{J}_k \). Since \( \text{supp} f_\beta \subseteq \{ u : e^{k+1} \leq u \leq e^{k+3j} \} \), setting \( m = k - 2j \), and using (6.7) we have for \( j \geq 1 \)

\[
J_k \leq C \sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left| \int_0^t (t+u)^{-\beta} \left( (u+1)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right| \left( \frac{u}{u+1} \right)^{\frac{1}{2}} dt \right)
\]

\[
\leq C e^{k+2j} \| g \|_{L^p(V)} \| g \|_{L^p(V)}
\]

which concludes the proof since \( (e^{k})^{-\beta} \int_0^{e^{k+2j}} \left( (u+1)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \leq O(1).

7. Imbeddings. Our results lead to some important information concerning the imbedding behavior of the \( W_{B} \) and \( W_{B}^0 \) spaces.

Theorem 4. Let \( \gamma, \mu > 0 \).

(a) If \( 1 < p < q < \infty \), then

\( W_{B} \subseteq W_{B} \) \( p \leq W_{B}^0 \).

(b) If \( 1 < p < q < \infty \), then

\( W_{B} \subseteq W_{B} \) \( p \leq W_{B}^0 \).

(c) If \( 1 < q < \infty \), then

\( W_{B} \subseteq W_{B} \).

(d) \( W_{B} \subseteq W_{B} \) \( \gamma > 1 \).

(e) \( 1 < q < \gamma - \mu \), then

\( W_{B} \subseteq W_{B} \).

Proof. Part (a) follows directly from Hölder's inequality and (b) follows from the well-known imbedding property of Bessel potential spaces [8]. For (c) first note that by (3.3)

\[
\left( \int_0^{e^k} \left( (t+u)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right)^{\frac{1}{\beta}} \leq C \left( \int_0^{e^k} \left( (t+u)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right)^{\frac{1}{\beta}}
\]

\[
\leq C \left( \int_0^{e^k} \left( (t+u)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right)^{\frac{1}{\beta}} + C \sum_{j=0}^{\infty} \left( \int_0^{e^{k+2j}} \left( (t+u)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \right)^{\frac{1}{\beta}} = \Sigma_1 + \Sigma_2.
\]

Since \( \gamma - \mu + 1 < 1 \), the integral Minkowski inequality gives

\[
\Sigma_1 \leq C \int_0^{e^k} \left( (t+u)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \leq C \| g \|_{L^p(V)}
\]

\[
\Sigma_2 \leq C \int_0^{e^{k+2j}} \left( (t+u)^{-\beta} f_\beta^0(u) g_\beta^0(u) \right) du \leq C \| g \|_{L^p(V)}
\]

The sum \( \Sigma_1 \) is estimated analogously. Finally parts (d) and (e) follow similarly by straightforward computations and so we omit the proof.

Note that in (d), \( W_{B} \subseteq W_{B} \) \( \gamma > 1 \).

Theorem 5. Let \( \gamma, \mu > 0 \).

(a) If \( 1 \leq p \leq q \leq \infty \), then

\( W_{B} \subseteq W_{B} \).

(b) If \( 1 < p < q < \infty \), then

\( W_{B} \subseteq W_{B} \).

(c) If \( 1 < q < \infty \), then

\( W_{B} \subseteq W_{B} \).

(d) \( W_{B} \subseteq W_{B} \) \( \gamma > 1 \).

(e) \( 1 < q < \gamma - \mu \), then

\( W_{B} \subseteq W_{B} \).

Proof. Part (a) follows directly from Hölder's inequality and (b) follows from the well-known imbedding property of Bessel potential spaces [8]. For (c) first note that by (3.3)
(b) If \(1 < p < q < \infty, 1/p - 1/q \leq \gamma - \mu, \gamma > 1/p\), then
\[
\text{wbv}_{\alpha, \gamma} \subset \text{wbv}_{\alpha, \mu};
\]
(c) If \(1 \leq q < \infty, 1 - 1/q \leq \gamma - \mu, \gamma \geq 1\), then
\[
\text{wbv}_{\alpha, \gamma} \subset \text{wbv}_{\alpha, \mu};
\]
(d) We have
\[
\text{wbv}_{\lambda, \gamma} \subset \text{wbv}_{\lambda, \gamma - 1} \quad \text{for} \quad \gamma \geq 1;
\]
(e) If \(1/q < \gamma - \mu, 1 < q \leq \infty, \gamma \geq 1\), then
\[
\text{wbv}_{\alpha, \gamma} \subset \text{wbv}_{\alpha, \mu}.
\]

8. Multipliers for Jacobi expansions. Fix \(\alpha \geq \beta \geq -1/2\) and let
\(L^p = L^p_{\alpha, \beta}, 1 \leq p < \infty\), denote the space of measurable functions \(f(x)\) on \([-1, 1]\) for which
\[
\|f\|_p = \left( \int_{-1}^{1} |f(x)|^p (1-x)^\alpha (1+x)^\beta \, dx \right)^{1/p} < \infty.
\]

To each \(f \in L^p\) there can be associated the formal expansion [24], Chap. 9
\[
f(x) \sim \sum_{n=0}^{\infty} a_n h_n P_n^{(\alpha, \beta)}(x),
\]
where \(P_n^{(\alpha, \beta)}(x)\) is the Jacobi polynomial of order \((\alpha, \beta)\),
\[
h_n = h_n^{(\alpha, \beta)} = (\|P_n^{(\alpha, \beta)}(x)\|_2)^{-1},
\]
and
\[
a_n = \int_{-1}^{1} f(x) P_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta \, dx.
\]

A sequence \(\eta \in \ell^p\) is called a multiplier of type \((p, q)\), notation \(\eta \in M^p_q\), if for each \(f \in L^p\) there exists a function \(f' \in L^q\) with
\[
(f')(x) \sim \sum_{n=0}^{\infty} \eta_n h_n P_n^{(\alpha, \beta)}(x), \quad \|f'\|_q \leq C\|f\|_p,
\]
where \(C\) is independent of \(f\). The smallest constant \(C\) independent of \(f\) for which (8.1) holds is called the multiplier norm of \(\eta\), and it is denoted by \(\|\eta\|_{M^p_q}\). We note that in deriving multiplier criteria we may assume that \(\eta_n = 0\).

The following end-point results concerning Jacobi multipliers are known:

\[\text{Theorem B. (a) } M^\infty = M^1;\]
\[\text{(b) } \text{wbv}_{\lambda, \gamma} \subset M^p_{\alpha, \beta} \text{ if } 1 \leq \frac{2a+2}{a+1/2} < p < \frac{2a+2}{a+1/2} \leq \infty;\]
\[\text{(c) } \text{wbv}_{\lambda, \gamma} \subset M^p_{\alpha, \beta} \text{ if } 1 < p < \infty, \gamma > a+1/2;\]
\[\text{(d) } \text{wbv}_{\lambda, \gamma} \subset M^p_{\alpha, \beta} \text{ if } 1 < p < \infty, \gamma > [a+2].\]

Part (a) is well-known and a consequence of the Parseval formula for complete orthogonal expansions. Part (b) in the case \(\alpha = \beta = -1/2\) is the classical Marcinkiewicz multiplier criterion for Fourier series (see, e.g. [32], p. 232); in the ultraheretical case \(\alpha = \beta > -1/2\) it is due to Muckenhoupt and Stein [17], and the general case is due to Askey [1]. Part (c) is proved in Gasper and Trebels [13]. Both (b) and (c) are best possible in the sense that in (b) the \(p\)-range cannot be enlarged, and (c) is not true for \(\gamma < a+1/2\); Theorem 3(a) below shows that (c) also holds for \(\gamma = a+1/2\). For (d) see Connell and Schwartz [9] and Gasper and Trebels [13].

Our aim here is to derive new multiplier criteria for Jacobi expansions by interpolating between the various end-point results collected in Theorem B. This will be accomplished by using Theorem 1 and the following theorem, established by Connell and Schwartz [11] in connection with ultraheretical multipliers but which immediately carries over to Jacobi polynomials.

\[\text{Theorem C. Define the operator } T: S(q, \gamma) \to M^1_q \text{ by}\]
\[Tq(\omega) = \sum_{\gamma=1}^{\infty} \eta_\gamma h_\gamma P_n^{(\alpha, \beta)}(\omega), \quad q \in S(q, \gamma), \quad g(\omega) = \eta_\gamma, \quad k \in N,\]
\[\text{and suppose that } T: S(q, \gamma) \to M^p_{\alpha, \beta}, 1 < p < \infty, q \leq \infty, \gamma > 0, \text{ is continuous for } \gamma = 0, 1. \text{ If } 0 < s < 1 \text{ and }\]
\[(\gamma_1, 1/p; 1/q) = (1-s)(\gamma_1, 1/p_1, 1/q_1) + s(\gamma_1, 1/p_1, 1/q_1),\]
\[\text{then } T: S(q, \gamma) \to M^p_{\alpha, \beta} \text{ is also continuous.}\]

Our multiplier results are contained in

\[\text{Theorem 6. Let } \alpha \geq \beta \geq -1/2, 1 < p < \infty, \text{ and } \gamma > 1, q \leq \infty.\]
\[\text{(a) If } \gamma > (2a+2)(1/p - 1/2) + 1/2 \text{ and } \gamma > 1, \text{ then } \text{wbv}_{\alpha, \gamma} \subset M^p_{\alpha, \beta}.\]
\[\text{This result is best possible in the sense that } \text{wbv}_{\alpha, \gamma} \subset M^p_{\alpha, \beta} \text{ when } \mu \leq (2a+2)(1/p - 1/2) + 1/2.\]
\[\text{(b) If } \gamma > (2a+2)(1/p - 1/2) > 1/q, \quad \frac{2a+2}{a+1/2} < p < \frac{2a+2}{a+1/2}, \text{ then } \text{wbv}_{\alpha, \gamma} \subset M^p_{\alpha, \beta}.\]
\[\text{(c) If } \gamma > (2a+3)(1/p - 1/2), q > p/|2-p|, \text{ then } \text{wbv}_{\alpha, \gamma} \subset M^p_{\alpha, \beta}.\]
(We shall omit the statements of the various criteria arising when Theorem B(d) is one end-point of the interpolation since this end-point is not sharp.)

Proof. Part(a) follows by interpolating between parts (b) and (c) of Theorem B. First let \( \varepsilon, \delta > 0 \) be small. By Theorem B(a) and Theorems 1-3 the hypotheses of Theorem C are satisfied if we choose

\[
\gamma_0 = 3/2 + \varepsilon, \quad \gamma_1 = 1, \quad \gamma_2 = 1 + \delta, \\
p_0 = 1 + \delta, \quad p_2 = (2a+2)/(a+3/2 + \delta).
\]

Hence \( T : S(q, \gamma') \to M^p_p \) is continuous when

\[
0 < s < 1, \quad q = 1 + \delta, \quad \gamma' = (1-s)(a+3/2 + \varepsilon) + s, \\
1/p = (1-s)/(1+\delta) + s/\left(\frac{2a+2}{a+3/2 + \delta}\right).
\]

Note that \( \gamma' - (2a+2)/1/p - 1/2 > 0 \) and tends to zero as \( \varepsilon, \delta \to 0 \).

Similarly, with \( \gamma_0, \gamma_1, \gamma_2 \) defined as above and \( p_0 \) defined as above and \( p_2 = (2a+2)/(a+1/2) - \delta \) we find that \( T : S(q, \gamma') \to M^p_p \) is continuous when

\[
0 < s < 1, \quad q = 1 + \delta, \quad \gamma' = (1-s)(a+3/2 + \varepsilon) + s, \\
1/p = (1-s)/(1+\delta) + s/\left(\frac{2a+2}{a+3/2 + \delta}\right),
\]

where \( \gamma' - (2a+2)/1/p - 1/2 > 0 \) and tends to zero as \( \varepsilon, \delta \to 0 \).

By Theorem 5(a),

\[
\text{wBV}_{q,p} = \text{wBV}_{q,p^*} = M^p_p,
\]

where \( \gamma > \gamma' \) may be chosen arbitrarily near \( \gamma' \) since \( q = 1 + \delta \) may be chosen arbitrarily near 1; hence \( \text{wBV}_{q,p} = M^p_p \) for \( \gamma > (2a+2)/1/p - 1/2 \).

The fact that the \((p, \gamma)\)-range is best possible can be seen from the counterexample of the Oskarlo kernel. Askey and Hirschman [2] have shown that the sequence \( \eta_{a,n} \),

\[
\eta_{a,n}(k) = \begin{cases} 
A^\gamma_{a,n}/A^{a,n}, & 0 \leq k < a, \\
0, & k > n,
\end{cases}
\]

is not a uniformly (in \( n \)) bounded \( M^p_p \)-multiplier family for expansions in ultraspherical polynomials \((a = \beta > -1/2)\) provided \( 0 \leq \mu < (2a+2)/1/p - 1/2 \). Their proof carries over to the general Jacobi polynomial case (use [24], § 9.41 and Ex. 91) and so \( \eta_{a,n} \) is not a uniformly bounded family of Jacobi \( M^p_p \)-multipliers in the \((p, \mu)\)-range. But \( \eta_{a,n} \) satisfies

\[
\sum_{k=0}^{n} A^\gamma_{a,n}/A^{a,n}(k) \leq C
\]

uniformly in \( n > 0 \) (see the Remark in [28], p. 44), so \( \eta_{a,n} w \text{BV}_{q,p} \leq \text{wBV}_{q,p^*} \) with \( \mu + 1 \leq (2a+2)/1/p - 1/2 + 1/2 \); which completes the proof of (a). Also see Bomani and Clerc [4].

Parts (b) and (c) of Theorem 6 follow easily by interpolating between (a), (b) and (c), of Theorem B, respectively.

Remarks. The special case \( a = \beta > -1/2, \gamma \in \mathbb{N}_0 \) of Theorem 6(a) is contained in Benami [3], which also gives an extension to weighted \( L^p \)-spaces. Also Theorem 6 supplements some results in Connell and Schwartz. What seems necessary now is to extend Theorem 6 by replacing the Marcinkiewicz type conditions by Hörmander ones; \( \text{wBV}_{q,p} \), where the parameter \( \kappa \) is governed by the imbedding relations in Theorem 5 applied to Theorem 6(a), i.e. \( \kappa > (2a+2)/1/p - 1/2 \).

Looking at Theorem 6 there naturally arises the question if one can give similar conditions for \( M^p_p \)-multipliers, \( 1 < p < q < \infty \). This can indeed be accomplished by the following idea. Write \( \{\eta_k\} = \{\xi_k\}^\gamma \); then for appropriate \( \sigma > 0 \) the multiplier sequence \( \{r_{k}^{(n)}\} \) transforms \( f \in L^p \) into a corresponding \( f^{*} \in L^q \). Now one has only to check if \( \{\xi_k\} \) satisfies the sufficient \( M^p_p \)-condition of Theorem 6. This again leads to "sharp" multiplier conditions in terms of modified weak bounded variation spaces. Details will be given in a subsequent paper.

9. Multipliers for Hankel transforms. Fix \( \alpha > -1 \) and let \( L^p = L^p_{\alpha,1}, 1 < p < \infty \), denote the space of measurable functions \( f(x) \) on \((0, \infty)\) for which

\[
\|f\|_p = \left( \int_0^\infty |f(x)|^p x^{2n+1} x^p \right)^{1/p}
\]

is finite. Following Hirschman [15], we define for \( f \in L^p \) the (modified) Hankel transform of order \( \alpha \) by

\[
H_{\alpha f}(y) = \int_0^\infty f(x) (x/y)^{\alpha} J_{\alpha}(x/y) x^{2n+1} x^p dx,
\]

where \( J_{\alpha}(x) \) is the Bessel function of the first kind; thus, the Hankel transform of order \( \alpha = (n-2)/2 \) coincides with the Fourier transform of a radial function integrable on \( \mathbb{R}^n \) (see [21], p. 155). The multiplier transformation associated with a function \( \psi(y) \) is defined formally by

\[
U_{\psi} f(x) = \int_0^\infty \psi(y) f(y) (y/y)\alpha J_{\alpha}(y) y^{2n+1} y^p dy.
\]

\( \psi \in L^p((0, \infty)) \) is called a multiplier of type \((p, p)\), notation \( \psi \in M^p_p(R^p) \) if to each \( f \in L^p \cap L^p \) there exists a function \( U_{\psi} f = f^{*} \in L^p \) such that

\[
\|U_{\psi} f\|_p \leq C \|f\|_p \quad (f \in L^p \cap L^p);
\]
the least constant \( C \) for which (9.1) holds defines the operator norm of the multiplier \( \psi \), notation \( \| \psi \|_{M_p^b} \). Igari [16] has proved the following interesting connection between Hankel and Jacobi multipliers of strong type \((p, p)\); an analogous result for weak type multipliers can be found in Connell and Schwartz [10].

**Theorem D.** Let \( 1 \leq p < \infty \) and \( \alpha, \beta > -1 \). Let \( \psi \) be a continuous function on \((0, \infty)\), set \( \psi^r(y) = \psi(\alpha y) \), and denote by \( (\psi_k^r)_{k=1}^{\infty} \) the sequence for which \( \psi_k^r = \psi(\alpha k) \). Assume \( \psi^r \) to be a uniformly bounded family of Jacobi multipliers (for small \( \epsilon > 0 \)). Then \( \| \psi \|_{M_p^b} \) is finite and

\[
\| \psi \|_{M_p^b} \leq \liminf_{s \to 0^+} \| (\psi_k^r)_{k=1}^{\infty} \|_{M_p^b}^{1/r}
\]

where \( \| \cdot \|_{M_p^b} \) denotes the Jacobi multiplier norm.

This enables us to use Theorem 6 to obtain analogous results for Hankel multipliers. We restrict ourselves to the case \( q = 1 \) (part (a) of Theorem 6).

**Theorem 7.** Let \( 1 < p < \infty, \alpha > -1/2 \). Then

\[
W \in L^2_{m, \nu} \subset M^2_{\nu}(\mathbb{R}) \subset W \in L^2_{m, \nu},
\]

where \( 0 < \mu < (2a + 2)/|1/p - 1/2| > 1/2 < y < 1 \).

**Proof.** The right-hand-side inclusion is proved in [33] for \( \alpha = (n - 2)/2 \); the extension to arbitrary \( \alpha > -1/2 \) is completely analogous.

The left-hand inclusion is proved in Theorem D, Lemma 2, and Theorem 2 to an arbitrary \( \psi \in W \in L^2_{m, \nu} \):

\[
\| \psi \|_{M_p^b} \leq \liminf_{s \to 0^+} \| (\psi_k^r)_{k=1}^{\infty} \|_{M_p^b}^{1/r} \leq \liminf_{s \to 0^+} \| (\psi_k^r)_{k=1}^{\infty} \|_{L_p^b}^{1/r} \leq \liminf_{s \to 0^+} \| (\psi_k^r)_{k=1}^{\infty} \|_{L_p^b}^{1/r},
\]

where \( \psi_k^r = (G_k^r - G_{k+1}^r) \psi^r \). But by direct computation (cf. [33], p. 41)

\[
\frac{d}{dt} \psi_k(t) = e^\nu \psi_k(t)
\]

so that for arbitrary integer \( m \), by Lemma 10,

\[
\int_{-m}^{+m} \left| \psi_k(t) \right|^2 \frac{dt}{t} \leq \int_{-m}^{+m} \left| \psi_k(t) \right|^2 \frac{dt}{t} \leq C \| \psi \|_{L_p^b}^{1/r}
\]

which completes the proof.

**Remarks.** (i) On account of Theorem 3 and Theorem 4(c), Theorem 7 is equivalent to a result (in terms of localized Besel potential spaces) in Connell and Schwartz [11]. In the special case \( \alpha = (n - 2)/2 \), \( \psi(t) = (1 - t^{-2}) \), Theorem 7 yields Weiland's result [29] on the boundedness of the Bochner–Riesz means for radial functions on \( L^p(R^n) \).

To show that the \((p, \gamma)\)-range in Theorem 7 is best possible it suffices to consider the Bochner–Riesz kernel

\[
m(y) = m(x) = (1 - y^2)^{1/2}, \quad \gamma > 1,
\]

and use a modification of Weiland's proof of the special case \( \alpha = (n - 2)/2 \) (which utilized formula (3.1) in [29]; personal communication). In place of (29), (3.1) we shall use the fact that if \( f \) is the characteristic function of an interval \([0, \varepsilon]\) and \( \mathcal{J}_\alpha(x) = e^{-i\alpha} \mathcal{J}_\alpha(x) \), then

\[
\mathcal{J}_\alpha(x) = \int_0^1 (1 - y^2)^{-\gamma} \int_0^1 f(y) \mathcal{J}_\alpha(y) \gamma^{-1} dy
\]

\[
= C_\alpha \int_0^1 f(x) \gamma^{-1} \left( \int_0^1 \mathcal{J}_\alpha(y) \left( \left( x^2 + \gamma^2 - 2xy \cos \varphi \right)^{1/2} \sin \varphi d\varphi \right) dx \right).
\]

This formula follows easily by using the product formula \((a > 1/2)\)

\[
\mathcal{J}_\alpha(x) \mathcal{J}_\alpha(y) = \int_0^1 \mathcal{J}_\alpha(x) \mathcal{J}_\alpha(y) \gamma^{-1} \alpha x^{2n+1} dx
\]

\[
= C_\alpha \int_0^1 \mathcal{J}_\alpha \left( \left( x^2 + \gamma^2 - 2x \gamma \cos \varphi \right)^{1/2} \sin \varphi d\varphi \right) dx.
\]

where the kernel \( \mathcal{K}_\alpha(x, y, z) \) is symmetric in \( x, y, z \), and Lemma 4.13 in [31]. In view of the asymptotic formula [24], (1.7.17)

\[
\mathcal{J}_\alpha(x) = \alpha x^{2n+1} \cos (x + c_2) + O(\alpha^{2n+3} x^{2n+3}), \quad x \to +\infty,
\]

if \( \alpha, b \), \( 0 < a < b \), is an interval such that

\[
\cos (x + c_2) \leq 1/2, \quad a \leq x \leq b,
\]

and if \( a = (b/2)/4, \) then, by (9.3),

\[
V_n \mathcal{J}_\alpha(x) \geq C_\alpha \left( x^{2n+1} \right), \quad a + \varepsilon + 2\varepsilon \leq x \leq b + \varepsilon + 2\varepsilon,
\]

for \( k = k_0 \), where \( k_0 \) is sufficiently large. Hence

\[
\| V_n f \|_{L_p^b} \geq C_\alpha \sum_{k=0}^{k_0} \left( x^{2n+1} \right) \geq \infty
\]

when \( p \leq (a + 1)/(2a + 2b + 1) \).

(ii) By Theorem 2, Theorem D, and Theorem 7 we also obtain a necessary condition for a family of sequences \( \psi^r \) with \( \psi_k^r = \psi(\alpha k) \), \( \epsilon > 0 \), to be a uniformly bounded Jacobi multiplier family.

(iii) The case \( \gamma = 1 \) follows analogously by Theorem B(b) and Theorem D, and reads as follows.
If \( \varphi \in \mathcal{D}'(0, \infty) \) is locally of bounded variation with
\[
|\varphi|_{L^1} + \sup_{m} \int \left| \frac{d\varphi(t)}{dt} \right| < \infty,
\]
then \( \varphi \in M_\varphi^p(H) \) provided
\[
1 \leq \frac{2a+2}{\alpha+3/2} < p < \frac{2a+2}{\alpha+1/2} \leq \infty.
\]
This method is due to Igari [16]; for another proof giving multiplier criteria even in weighted \( L^p \)-spaces see Guy [14], p. 187; a heuristic proof using Littlewood–Paley functions can be found in Sonnichsen [23].

(iv) It is interesting to note what happens for \( p = 1 \) (or \( p = \infty \)). To this end introduce the space of functions of bounded variation of order \( \gamma \) (cf. [26], p. 36)
\[
BV_\gamma = \{ g \in C_0[0, \infty) : \int_0^\infty (t+\delta t)^{-\delta} g(t) dt \leq C(t) \text{ for all } t, \delta > 0; \ g(t_1), \ldots, g(t_r) \in AC_\omega \text{ and } \|g\|_{BV_\gamma} < \infty, \}
\]
where \( \delta = \gamma - 1 \).
\[
\|g\|_{BV_\gamma} = \left| g(0) \right| + \int_0^\infty (t+\delta t)^{-\delta} \|g(t)\| \, dt
\]
and \( C_0[0, \infty) \) is the set of all continuous functions on \( [0, \infty) \) which vanish at infinity. Denote by \( \mathcal{L}^\gamma \) the set of all functions which are Hankel transforms of some \( \mathcal{L}^1 \)-function. Then there holds
\[
BV_\gamma \subset \mathcal{L}^\gamma \subset BV_\gamma, \quad 0 < \mu < \alpha+1/2 < \gamma - 1.
\]
The left-side inclusion is proved in Butzer, Nessel, and Trebels [7], the right-side inclusion is only a simple modification of [28], noting that the Riemann–Lebesgue Lemma holds for Hankel transforms (see [19]).

By the same technique as in [28] one can also show
\[
\mathcal{L}^\gamma \subset M_\varphi^p(H) \subset WBV_\omega, \quad 0 < \mu < \alpha+1/2.
\]
(\( \gamma \) There is a differentiation gap of about \( 1/\alpha \) (\( \alpha+1 < \gamma \)) between necessary (\( \mu \)) and sufficient (\( \gamma \)) Hankel multiplier criteria. Theorem 4(d) and the example (1–\( \alpha \)) show that this gap cannot essentially be diminished as long as we stick to \( WBV_\omega \) and \( WBV_\omega, \gamma \) classes. As in the Jacobi multiplier case the conjecture for an improvement is
\[
WBV_\omega, \gamma \subset M_\varphi^p(H), \quad \gamma > (2a+2)(1-p-1/2), \quad 1 < p < \infty.
\]
One result in this direction is due to Connett and Schwartz [10] who verified the embedding for \( \gamma = [\alpha+1]+1, \alpha > -1/2 \).
Factorization in Banach algebras

by

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Abstract. Let \( A \) be a Banach algebra with bounded left approximate identity and let \( \mathcal{P} \) be a left Banach \( A \)-module. For each sequence \( y(n) \in \mathcal{A} \mathcal{F}^* \) satisfying a certain condition there exists an \( a \in A \) and another sequence \( z(n) \in \mathcal{A} \mathcal{F}^* \) such that \( y(n) = a \cdot z(n) \).

Introduction. We present a simple proof of a generalization of the Rudin-Cohen factorization theorem using the method of nondiscrete mathematical induction. This method is based on a simple abstract theorem about families of sets, the so-called induction theorem. The induction theorem is closely related to the closed graph theorem and is nothing more than the abstract description of a class of iterative constructions in analysis. One of the advantages of this method consists in the fact that the construction of the sequence of iterations is dealt with by the abstract theorem; this reduces the amount of work required to an investigation of the improvement of the degree of approximation which can be achieved within a given distance from a given point. In this manner, by separating the hard analysis part from the construction this approach not only yields considerable simplifications of proofs but also evidences more clearly the substance of the problem.

1. Preliminaries. Given a positive number \( r \) and a set \( M \) in a metric space \( (E, d) \), we define \( U(M, r) = \{ y \in E; d(y, M) < r \} \). Let \( T \) be an interval of the form \( [t_0; 0 < t < t_0^+ \} \), where \( t_0 \) is positive or \( \infty \). If \( A(t) \), \( t \in T \) is a family of subsets of \( E \), we define its limit \( A(0) \) as follows

\[
A(0) = \bigcap_{0 < t < t_0} \left( \bigcup_{s < r} A(s) \right).
\]

A mapping \( \omega \) transforming \( T \) into itself is called a small function or a rate of convergence on \( T \) if \( \omega(t) = t + \omega(t) + \omega(\omega(t)) + \ldots \) is finite for each \( t \in T \). The method of nondiscrete mathematical induction is based on the following simple result.