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A note on rotations in separable Banach spaces

by

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Abstract. We show that every separable Banach space X is a complemented subspace of a separable Banach space Y which has the following rotation property:

There is a dense subset S of the unit sphere of Y so that for every $x, y \in S$ there is an isometric automorphism $T: Y \to Y$ with $T(x) = y$. As a consequence, there is a separable Banach space satisfying this rotation property which, on the other side, fails to have the approximation property.

This paper is concerned with the following Banach space property:

(M) *Let X be a Banach space (real or complex). There is a dense subset S of the unit sphere of X such that for every $x, y \in S$ there is an isometric automorphism $T: X \to X$ with $T(x) = y$.*

An isometric automorphism of a Banach space X is sometimes called *rotation*. We obtain immediately:

Let X be a Banach space having (M). *For every $\varepsilon > 0$ and $x, y \in X$ with $\|x\| = \|y\| = 1$ there is an automorphism $T: X \to X$ with $(1-\varepsilon)\|z\| \leqslant \|T(z)\| \leqslant (1+\varepsilon)\|z\|$ for all $z \in X$ and $T(x) = y$.*

Clearly, the separable Hilbert space satisfies (M). In [7] it was shown that there is a separable Banach space G with property (M) whose dual space G^* is isometrically isomorphic to an abstract L-space (cf. [5]). It turns out that the rotation property of (M) holds on the set of all smooth points of G (i.e. on the points x with $\|x\| = 1$ and there is only one linear functional x^* with $x^*(x) = \|x^*\| = 1$). The set of smooth points is a dense G_δ-subset of the unit sphere of any separable Banach space (Mazur [9]).

On the other hand, the unit sphere of G contains points x, y which do not admit a rotation T of G carrying x onto y. Thus G is an example of a Banach space having (M) which is different from a Hilbert space. Exploiting Banach's characterization of the rotations in $L_p(0,1)$; $1 \leqslant p < \infty$; $p \neq 2$; ([1] Chap. XI), we obtain:

Let $f, g \in L_p(0,1)$; $1 \leqslant p < \infty$; $p \neq 2$; (with respect to the Lebesgue measure λ) so that $\|f\| = \|g\| = 1$.

(i) *If either* $\lambda(\operatorname{supp} f) = \lambda(\operatorname{supp} g) = 1$ *or* $\lambda(\operatorname{supp} f), \lambda(\operatorname{supp} g) < 1$, *then there is a rotation* $T\colon L_p(0,1) \to L_p(0,1)$ *with* $T(f) = g$.

(ii) *If* $\lambda(\operatorname{supp} f) = 1$ *but* $\lambda(\operatorname{supp} g) < 1$, *then* f *and* g *do not admit a rotation* T *with* $T(f) = g$.

The latter assertion follows from the facts that

$$\{f \in L_1(0,1) \mid \|f\| = 1,\ \lambda(\operatorname{supp} f) = 1\}$$

is the set of smooth points in $L_1(0,1)$ and

$$\|f-g\|^p + \|f+g\|^p = 2\|f\|^p + 2\|g\|^p$$

iff $\lambda(\operatorname{supp} f \cap \operatorname{supp} g) = 0$; $1 \leqslant p < \infty$; $p \neq 2$ ([1]). Hence $L_p(0,1)$ has property (M), too, and the above space G (the *Gurarij space*) seems to fit into this concept of $L_p(0,1)$-spaces taking over the position of $L_\infty(0,1)$.

In view of these examples it seems natural to ask in how far property (M) depends on the intrinsic structure of a given Banach space. The purpose of our note is to show that property (M) is a universal property which is related in some sense (theorem below) to every separable Banach space. This provides us even with a separable Banach space satisfying (M) which fails to have the approximation property. So, in particular, we obtain an example which is extremely different from a Hilbert space. Notice that all Banach spaces previously discussed do have the approximation property.

THEOREM. *Let* X *be an arbitrary separable Banach space. Then there is a separable Banach space* $Z \supset X$ *satisfying* (M), *and a contractive projection* $P\colon Z \to X$.

COROLLARY. *There is a separable Banach space* X *which satisfies* (M) *but does not have the approximation property.*

Proof of the Corollary: There is a separable Banach space $\tilde{X}$ which fails to have the approximation property ([2], [3]). Our above theorem yields a separable Banach space $X \supset \tilde{X}$ with property (M) and a contractive projection $P\colon X \to \tilde{X}$. Hence X cannot have the approximation property. Indeed, otherwise let $K \subset \tilde{X}$ be compact and $\varepsilon > 0$, so that $T\colon X \to X$ is a linear operator with finite rank and $\|T(x) - x\| \leqslant \varepsilon$ for all $x \in K$. But then $P \cdot T|\tilde{X}$ is a linear, finite rank operator from $\tilde{X}$ into $\tilde{X}$ and

$$\|(P\cdot T)(x) - x\| = \big\|P\big(T(x)\big) - P(x)\big\| \leqslant \|T(x) - x\| \leqslant \varepsilon \quad \text{for all } x \in K.$$

Hence $\tilde{X}$ would have the approximation property if this were true for X. ■

In order to prove the above theorem we need a lemma first.

LEMMA. *Let* Y *be a separable Banach space. Let* $E_n \subset Y$ *be a sequence of subspaces of* Y *and let* $T_n\colon E_n \to Y$ *be isometric linear. Furthermore, assume that there are contractive projections* $P_n\colon Y \to E_n$ *and* $Q_n\colon Y \to T_n(E_n)$ *for all* n. *Then there are a separable Banach space* $\tilde{Y} \supset Y$, *isometric extensions* $\tilde{T}_n\colon Y \to \tilde{Y}$ *of* T_n *and contractive projections* $P\colon \tilde{Y} \to Y$, $\tilde{Q}_n\colon \tilde{Y} \to \tilde{T}_n(Y)$ *for all* n.

Proof. Consider $(\bigoplus_{i=1}^{\infty} Y)_{(1)}$ (endowed with the norm $\|(y_i)\| = \sum_{i=1}^{\infty} \|y_i\|$ for all $y_i \in Y$) and let V be the closed linear span of all vectors $(-T_n(e), 0, \ldots, 0, \underset{n+1}{e}, 0, \ldots)$ where $e \in E_n$, $n \in \mathbf{N}$. Set $\tilde{Y} = (\bigoplus_{i=1}^{\infty} Y)_{(1)}/V$. Since

$$\|y\| \leqslant \inf\Big\{\Big\|y - \sum_{n=1}^{\infty} T_n(e_n)\Big\| + \sum_{n=1}^{\infty} \|e_n\| \;\Big|\; e_n \in E_n;\ n \in \mathbf{N}\Big\} \leqslant \|y\|$$

for all $y \in Y$, we can Y identify with the subspace spanned by the elements $(y, 0, 0, \ldots) + V \in \tilde{Y}$, $y \in Y$.

We observe that $(T_n(e), 0, 0, \ldots) + V = (0, \ldots, 0, \underset{n+1}{e}, 0, \ldots) + V$ for all $e \in E_n$; for all $n \in \mathbf{N}$. Furthermore, we obtain

$$\|y\| \geqslant \inf\Big\{\|y - e_n\| + \sum_{\substack{i=1\\ i\neq n}}^{\infty} \|e_i\| + \Big\|\sum_{i=1}^{\infty} T_i(e_i)\Big\| \;\Big|\; e_i \in E_i;\ i \in \mathbf{N}\Big\}$$

$$\geqslant \inf\Big\{\|y - e_n\| + \|T_n(e_n)\| + \sum_{\substack{i=1\\ i\neq n}}^{\infty} \big(\|e_i\| - \|T_i(e_i)\|\big) \;\Big|\; e_i \in E_i;\ n \in \mathbf{N}\Big\}$$

$$\geqslant \|y\|$$

so that we can T_n extend isometrically by setting $\tilde{T}_n(y) = (0, \ldots, 0, \underset{n+1}{y}, 0, \ldots) + V$ for all $y \in Y$ and all $n \in \mathbf{N}$. Set

$$P\big((y_i) + V\big) = \big(y_1, P_1(y_2), P_2(y_3), \ldots\big) + V$$

and

$$\tilde{Q}_n\big((y_i) + V\big) = \Big(Q_n\Big(y_1 + \sum_{\substack{i=1\\ i\neq n}}^{\infty} (T_i P_i)(y_{i+1})\Big), 0, 0, \ldots, 0, \underset{n+1}{y}, 0, \ldots\Big) + V$$

for all $y_i \in Y$ and $n \in \mathbf{N}$.

We have defined projections with $\|P\| \leqslant \sup_{i\in\mathbf{N}} (\|P_i\|) \leqslant 1$ and $\|\tilde{Q}_n\| \leqslant \sup_{i\in\mathbf{N}} (\|Q_n\|\|P_i\|) \leqslant 1$. ■

Proof of the Theorem. Define $Y_0 = X$. Assume that we have introduced already Y_{2n} and

$$\Omega_{2n} = \{T_m\colon E_m \to Y_{2n} \mid T_m \text{ isometric, linear}\},$$

where E_m are certain subspaces of Y_{2n}. Assume furthermore, that there are contractive projections $P_m^{2n}\colon Y_{2n} \to E_m$; $Q_m^{2n}\colon Y_{2n} \to T_m(E_m)$ and $R_{2n}\colon Y_{2n} \to X$. We use the above Lemma to define $Y_{2n+1} \supset Y_{2n}$, isometric extensions $\tilde{T}_m\colon Y_{2n} \to Y_{2n+1}$ of T_m and contractive projections $P\colon Y_{2n+1}$

$\to Y_{2n}$; $Q_m^{2n+1}\colon Y_{2n+1}\to \tilde{T}_m(Y_{2n})$ for all $m\in N$. Another application of the above Lemma by replacing T_m by $\tilde{T}_m^{-1}$, E_m by $\tilde{T}_m(Y_{2n})$, P_m^{2n} by Q_m^{2n+1}, $T_m(E_m)$ by Y_{2n} and Q_m^{2n} by P for all m yields $Y_{2n+2}\supset Y_{2n+1}$, isometric extensions S_m of $\tilde{T}_m$, $S_m\colon F_m\to Y_{2n+1}$, where F_m are subspaces of Y_{2n+2} with $Y_{2n}\subset F_m$ and $S_m(F_m)=Y_{2n+1}$ for all $m\in N$. We obtain in addition contractive projections $P_m^{2n+2}\colon Y_{2n+2}\to F_m$; $Q\colon Y_{2n+2}\to Y_{2n+1}=S_m(F_m)$. We set $R_{2n+2}=R_{2n}\cdot P\cdot Q$.

Consider now a countable dense subset Γ of the unit sphere of Y_{2n+2} and define

$$\Omega=\{T\colon \langle x\rangle\to\langle y\rangle \mid T \text{ linear isometric; } x,y\in\Gamma;\ T(x)=y\},$$

where $\langle x\rangle$ denotes the linear span of x.

Then certainly Ω is countable. The theorem of Hahn–Banach provides us with contractive projections from Y_{2n+2} onto $\langle x\rangle$ for all $x\in\Gamma$. Set $\Omega_{2n+2}=\Omega_{2n}\cup\Omega$ and continue the induction by defining Y_{2n+3}.

Finally, set $Z=\overline{\bigcup_{n\in N} Y_n}$ and define a contractive projection from Z onto X by the R_{2n}, $n\in N$. ■

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A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers

by

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Dedicated to Ralph Boas on the occasion of his sixty-fifth birthday

Abstract. Localized Bessel potential spaces $S(q,\gamma)$, $\gamma>0$, were recently introduced by Connett and Schwartz in connection with ultraspherical multipliers and characterized for integer γ in terms of sequence spaces. Analogous results are obtained in this paper for all real $\gamma>1/q$, where $1<q<\infty$. These results are then used to derive best possible multiplier criteria of Marcinkiewicz type for Jacobi expansions by interpolating between end-point results due to Askey and to the authors and to derive analogous multiplier criteria for Hankel transforms.

1. Introduction. In [11] Connett and Schwartz showed that localized Bessel potential spaces $S(q,\gamma)$ are useful in the theory of ultraspherical multipliers. However, one disadvantage of these spaces is that it is hard to verify when a sequence is the restriction (to the positive integers) of an element in $S(q,\gamma)$. Jn case of γ being a positive integer, Connett and Schwartz characterized $S(q,\gamma)$ by means of (finite) difference conditions upon the sequence. The main result of this paper, Theorem 1, extends this characterization to all $\gamma>1/q$ for $1<q<\infty$. We also give a neat description (Theorems 4 and 5) of the imbedding behavior of the wbv and WBV-spaces (defined below), which are important in multiplier theory. These results are then used to derive various multiplier criteria for Jacobi expansions (Theorem 6) and Hankel transforms (Theorem 7).

To define the localized Bessel potential spaces we first recall that the standard space of Bessel potentials $L_\gamma^q(\mathbf{R})$, $\gamma>0$, $1\leqslant q\leqslant\infty$, is defined by (see [20], p. 134)

$$L_\gamma^q=\{g\in L^q(0,\infty)\colon g=G_\gamma * h,\ \|g\|_{q,\gamma}=\|h\|_q<\infty\},$$

where the *Bessel kernel* $G_\gamma(x)$ is a function whose Fourier transform is given by

$$G_\gamma^\wedge(v)=\int_{-\infty}^{\infty}G_\gamma(x)e^{-ivx}dx=(1+|v|^2)^{-\gamma/2}.$$

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