STUDIA MATHEMATICA, T. LXV. (1979)

The L^p mapping problem for well-behaved convolutions

by

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Abstract. In this paper we discuss convolutions

$$(K*f)(x) = \int K(x-y)f(y)\,dy$$

over $\boldsymbol{R}^n$. For kernels K which are "weakly oscillating" we give necessary and sufficient conditions (in terms of K) in order that K maps L^p into L^q.

§ 0. Introduction. In this paper we discuss convolutions of complex-valued functions of several real variables,

$$(K*f)(x) = \int K(x-y)f(y)\,dy = \int K(y)f(x-y)\,dy, \tag{0}$$

where the integration is over $\boldsymbol{R}^n$ ($n \geqslant 1$ integral) and $f \in L^p(\boldsymbol{R}^n)$, $1 \leqslant p \leqslant \infty$. For precise definitions, see § 1. The mapping problem consists of finding necessary and sufficient conditions for K to map L^p into L^q. The conditions should be in terms of K and sufficiently simple to become applicable. This is a very important problem, but in general quite hopeless at the present time except for the cases $p = 1$ or $q = \infty$ (Toeplitz, Banach–Steinhaus), $p > q$ (Hörmander), and $p = q = 2$ which is known at least in terms of the (distributional) Fourier transform $\hat{K}$. A reasonable approach, however, is to ask this question for various classes of "well-behaved" kernels. Here we will be concerned with weak forms of the condition

$$|\nabla K(y)| < B\,|y|^{-1}\,|K(y)|, \quad y \neq 0 \tag{1}$$

with $B = B(K) > 0$. Condition (1) is satisfied rather generally by nice kernels whose growth is similar in every direction and restricted by $|x|^{\pm a}$. We describe this as "weak oscillation" which is related to but somewhat less restrictive than "slow oscillation".

In § 1 we introduce several averaged forms of (1) which resemble Lipschitz conditions. For the corresponding classes we prove various necessary conditions in order to map L^p into L^q strongly or even weakly

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* The research of the first author was supported in part by the National Science Foundation.

(Theorems I, II, IV). In case $p = q$, several of these conditions have been considered before by Hörmander [3], Stein [5], Benedek, Calderón, Panzone [1] and Muckenhoupt [4] and are known to be sufficient. So they represent a solution of the mapping problem (Theorem III). In case $p < q$ our necessary conditions are also sufficient (Theorem V), but this fact seems to have been overlooked before. In dimension 1 the necessary conditions can be simplified and agree again with known sufficient conditions of Hardy–Littlewood.

§ 1. Definitions and related properties. For standard notations the reader is referred to Stein [5], pp. 28–53. We always assume K to be locally integrable except possibly at 0, $K \in L_{\mathrm{loc}}(\mathbf{R}^n - 0)$. We introduce *generalized Cauchy–Lebesgue integrals* by

(2) $$\oint'_{|x|\leqslant 1} f(x)\,dx = \lim_{\varepsilon\to+0} \int_{|x|\leqslant 1} |x|^{\varepsilon} f(x)\,dx,$$

where we also require $|x|^{\varepsilon} f(x)$ to be integrable for $|x| \leqslant 1$ and all $\varepsilon > 0$, and by

(3) $$\oint'_{\mathbf{R}^n} f(x)\,dx = \oint'_{|x|\leqslant 1} f(x)\,dx + \int_{|x|\geqslant 1} f(x)\,dx,$$

where both integrals on the right should exist, the second in the Lebesgue sense. The *Cauchy–Lebesgue class* $\mathrm{CL}(\mathbf{R}^n)$ consists of those $K \in L_{\mathrm{loc}}(\mathbf{R}^n - 0)$ for which $\oint'_{|x|\leqslant 1} K(x)\,dx$ exists.

To define the convolution with K we restrict ourselves to test functions $f \in C_0^\infty(\mathbf{R}^n)$, i.e. infinitely differentiable functions with compact support. If we only know that $K \in L_{\mathrm{loc}}(\mathbf{R}^n - 0)$ we consider

(4) $$K_\varepsilon(x) = \begin{cases} K(x) & \text{for } |x| \geqslant \varepsilon, \\ 0 & \text{elsewhere} \end{cases}$$

for $\varepsilon > 0$ or $K_{\varepsilon,\eta} = K_\varepsilon - K_\eta$ for $0 < \varepsilon < \eta < \infty$. Then $K_\varepsilon * f$ is defined by (0) for $f \in C_0^\infty$. If, however, $K \in \mathrm{CL}(\mathbf{R}^n)$, then also $K\varphi \in \mathrm{CL}$ for any $\varphi \in C_0^\infty$ in view of

(5) $$\oint'_{|x|\leqslant 1} K(x)\varphi(x)\,dx = \varphi(0) \oint'_{|x|\leqslant 1} K(x)\,dx + \int_{|x|\leqslant 1} K(x)\big(\varphi(x) - \varphi(0)\big)\,dx.$$

Therefore we give the definition

(6) $$(K*f)(x) = \oint'_{\mathbf{R}^n} K(y) f(x-y)\,dy, \quad f \in C_0^\infty\ (K \in \mathrm{CL}),$$

which reduces to (0) in the case that K is also integrable at 0.

We make use of the "weak norm" for measurable g

(7) $$\|g\|_q^{\#} = \sup_{\lambda>0} \lambda \big(m\{x\colon |g(x)| \geqslant \lambda\}\big)^{1/q}, \quad 1 \leqslant q < \infty$$

and the "operator norms" (if $K \in \mathrm{CL}$)

(8) $$\|K\|_{p,q} = \sup_f \frac{\|K*f\|_q}{\|f\|_p}, \quad \|K\|_{p,q}^{\#} = \sup_f \frac{\|K*f\|_q^{\#}}{\|f\|_p},$$

when $1 \leqslant p \leqslant q < \infty$, $f \in C_0^\infty$, $\|f\|_p \neq 0$. In case $K \notin \mathrm{CL}$ we work with K_ε. We say that $K \in L_{p,q}^{\#}$ (resp. $K \in L_{p,q}$) if $\|K\|_{p,q}^{\#} < \infty$ (resp. $\|K\|_{p,q} < \infty$) and speak of *weak type* (p, q) (resp. *strong type*).

We say that "$K_\varepsilon \in L_{p,q}^{\#}$ uniformly" if $\|K_\varepsilon\|_{p,q}^{\#} \leqslant B(K, p, q) < \infty$ and similarly for uniformly strong type. It is convenient to observe that (7) may be replaced for $1 < q < \infty$ $((1/q) + (1/q') = 1)$ by an equivalent (actual) norm

(7′) $$\sup_{a,f} a^{-1/q'} \Big| \int_{\mathbf{R}^n} g(x) f(x)\,dx \Big|, \quad g \in L_{\mathrm{loc}}(\mathbf{R}^n),$$

where $a > 0$ and $f \in C_0^\infty$, $|f| \leqslant 1$, $m\{\mathrm{supp} f\} \leqslant a$. (This has the same effect as varying f over L^∞ with the same restrictions.)

Suppose that $1 \leqslant s \leqslant \infty$. Hörmander's condition $K \in H_s$ requires

(8) $$\Big(\int_{|y|\geqslant\varrho} |K(y+x) - K(y)|^s\,dy\Big)^{1/s} \leqslant B(K, s) < \infty \quad \text{for} \quad \varrho > 0,\ |x| \leqslant \varrho/2$$

or equivalently

(8′) $$\Big(\int_{|y|\geqslant 2|x|} |K(y+x) - K(y)|^s\,dy\Big)^{1/s} \leqslant B(K, s) < \infty \quad \text{for} \quad x \neq 0.$$

We introduce the following two conditions concerning *weak oscillation*:

$K \in V_s$ requires for $\varrho > 0$, $|x| \leqslant \delta\varrho$

(9) $$\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y+x) - K(y)|^s\,dy\Big)^{1/s} \leqslant \tfrac{1}{2}\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|^s\,dy\Big)^{1/s} < \infty$$

with suitable $\delta = \delta(K, s) \in (0, 1)$.

$K \in W_s$ requires for $\varrho > 0$, $|x| \leqslant \varrho/2$

(10) $$\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y+x) - K(y)|^s\,dy\Big)^{1/s} \leqslant B|x|\varrho^{-1}\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|^s\,dy\Big)^{1/s} < \infty$$

with suitable $B = B(K, s) > 0$.

Clearly, (9) follows from (10) and this follows from $K \in W$ requiring

(11) $$|K(y+x) - K(y)| < B|x|\,|y|^{-1}|K(y)| \quad \text{for} \quad |y| > 0,\ |x| \leqslant |y|/2$$

with suitable $B = B(K) > 0$. Condition (11) replaces condition (1) and does not use V. If $K \in C^1(\mathbf{R}^n - 0)$, condition (1) is equivalent to (11) since (1) implies $|K(y_1)/K(y_2)| \leqslant B$ for $1/2 \leqslant |y_1|/|y_2| \leqslant 2$, $y_2 \neq 0$ by estimating the variation of $\log|K|$. The constant $1/2$ in (9) could be replaced by $1-\varepsilon$, but we use $1/2$ for simplicity.

For finite positive absolute constants depending at most on n, p, q, s we write generically A; if also dependency on K is permitted, we use B generically. Since the conditions on weak oscillation are central to this paper, we make additional comments on W_s for $1 \leqslant s < \infty$. First we consider the case of dimension $n = 1$. Then one can show that $K \in W_1$ is equivalent to

$$\underset{\varrho\leqslant|y|\leqslant 2\varrho}{\operatorname{var}} K(y) \leqslant B\varrho^{-1}\int_{\varrho}^{2\varrho} |K(y)|\,dy < \infty,$$

$$\int_{\varrho/2}^{4\varrho} |K(y)|\,dy \leqslant B\int_{\varrho}^{2\varrho} |K(y)|\,dy, \qquad \varrho > 0, \tag{12}$$

and $K \in W_s$, $1 < s < \infty$, is equivalent to $K(y)$ being absolutely continuous for $y \neq 0$ (after adjusting the definition on a set of measure 0) and

$$\int_{\varrho}^{2\varrho} |K'(y)|^s\,dy \leqslant B\varrho^{-s}\int_{\varrho}^{2\varrho} |K(y)|^s\,dy < \infty,$$

$$\int_{\varrho/2}^{4\varrho} |K(y)|^s\,dy \leqslant B\int_{\varrho}^{2\varrho} |K(y)|^s\,dy, \qquad \varrho > 0. \tag{13}$$

Also observe that (12) implies

$$\sup_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)| \leqslant B\varrho^{-1}\int_{\varrho}^{2\varrho} |K(y)|\,dy, \qquad \varrho > 0, \tag{12'}$$

and that (13) implies $|K|^s \in W_1$.

Similar considerations apply to dimension $n > 1$ in which case we assume $K \in C^1(\mathbf{R}^n - 0)$ for simplicity. Then $K \in W_s$ is equivalent to

$$\int_{\varrho\leqslant|y|\leqslant 2\varrho} |\nabla K(y)|^s\,dy \leqslant B\,\varrho^{-s}\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|^s\,dy < \infty,$$

$$\int_{\varrho/2\leqslant|y|\leqslant 4\varrho} |K(y)|^s\,dy \leqslant B\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|^s\,dy, \qquad \varrho > 0. \tag{14}$$

We also introduce for $1 \leqslant s < \infty$

$$k_s(t) = \int_{|z|=1} |K(tz)|^s\,\omega(dz), \qquad t > 0, \tag{15}$$

where ω is the measure (surface area) on the unit sphere. If $K \in W_s$, it follows that k_s is absolutely continuous $(t > 0)$ and

$$|k_s'(t)| \leqslant s\int_{|z|=1} |K(tz)|^{s-1}|\nabla K(tz)|\,\omega(dz), \qquad t > 0.$$

Hence, by Hölder

$$\int_{\varrho}^{2\varrho} |k_s'(t)|\,dt \leqslant A\varrho^{-n+1}\int_{\varrho\leqslant|y|\leqslant 2\varrho} |\nabla K(y)|\,|K(y)|^{s-1}\,dy \leqslant B\,\varrho^{-1}\int_{\varrho}^{2\varrho} k_s(t)\,dt, \tag{16}$$

so that $k_s \in W_1$ and can be estimated by its average.

In conclusion, we mention the two-dimensional example $K(x, y) = (x^4+y^2)^\alpha$ with $\alpha \neq 0$ $(x, y, \alpha \in \mathbf{R})$. It satisfies $K \in W_s$ for $1 \leqslant s < \infty$, but $K \notin W$. Observe that its growth is quite different in various parts of the plane, however, if we approach 0 or ∞ along fixed rays, we see the same behavior in the end with only two exceptions.

§ 2. Necessary conditions for type (p, q). Here we give necessary conditions for weak type (p, q) based on the assumption $K \in V_s$ which is our least restrictive condition. We use the notations $(1/p)+(1/p') = 1$, $(1/p)-(1/q) = 1-(1/r)$.

THEOREM I. *Assume $1 \leqslant p \leqslant q < \infty$, $1 \leqslant s \leqslant p'$, $K \in L_{\mathrm{loc}}(\mathbf{R}^n - 0)$ and $K \in V_s$. Now, if $K_\varepsilon \in L^{\#}_{p,q}$ uniformly, then*

$$\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|^s\,dy\Big)^{1/s} \leqslant B\,\varrho^{n((1/s)-(1/r))}, \qquad \varrho > 0. \tag{17}$$

If also $K \in \mathrm{CL}(\mathbf{R}^n)$, the same conclusion can be drawn from $K \in L^{\#}_{p,q}$.

Before we enter the proof, we make various comments. Condition (17) becomes stronger for larger s. We have $1 \leqslant r < p'$, so $s = r$ is a possible choice. In case $s < r$ the integrability of $|K|^s$ at ∞ is implied, in case $s < r$ the integrability of $|K|^s$ at 0 is implied. If $s = \infty$ is possible, we have

$$|K(x)| \leqslant B\,|x|^{-n/r}, \qquad x \neq 0. \tag{18}$$

In general, our conclusions can be somewhat improved if we use the stronger condition $K \in W_s$.

COROLLARY I. *Assume the situation of Theorem I with V_s replaced by W_s and $s < \infty$. Now, in case $n = 1$, condition (18) is always necessary. In case $n > 1$ we obtain the stronger necessary condition*

$$k_s(t) \leqslant Bt^{-n(s/r)}, \qquad t > 0, \tag{19}$$

provided that $K \in C^1(\mathbf{R}^n - 0)$ also.

The corollary follows in case $n = 1$ since $|K|^s \in W_1$ and can be estimated by its mean value, cf. (12'). In case $n > 1$ we use $k_s \in W_1$, cf. (16).

The condition $K \in W_s$ can also be used to make a connection with Hörmander's condition H_r.

COROLLARY II. *Assume the situation of Theorem I with V_s replaced by W_s and let $1 \leqslant \sigma \leqslant s$, $0 < 1/\sigma < (1/r)+(1/n)$. Then it is necessary that for $\varrho > 0$, $|x| \leqslant \varrho/2$*

$$\Big(\int_{|y|\geqslant\varrho} |K(y+x)-K(y)|^\sigma dy\Big)^{1/\sigma} \leqslant B(\sigma)\,\varrho^{n((1/\sigma)-(1/r))}. \tag{20}$$

In particular, if $s \geqslant r$, then $\sigma = r$ is possible and $K \in H_r$ is necessary.

To see this apply $K \in W_s$ and (17) to obtain for $\varrho > 0$, $|x| < \varrho/2$

$$\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y+x)-K(y)|^s dy\Big)^{1/s} \leqslant B\frac{x}{\varrho}\,\varrho^{n((1/s)-(1/r))}.$$

In this condition we can replace s by σ and ϱ by $2^j\varrho$. Adding over $j = 0, 1, \ldots$ yields

$$\int_{|y|\geqslant\varrho} |K(y+x)-K(y)|^\sigma dy \leqslant B^\sigma \frac{|x|^\sigma}{\varrho^\sigma}\,\varrho^{n(1-(\sigma/r))}, \qquad |x| < \varrho/2. \tag{21}$$

As a special case of Corollary II the condition $K \in W_r$ will ensure that $K \in H_r$ is necessary. Note, however, that for well-behaved kernels, $K \in W_r$ is satisfied rather generally while $K \in H_r$ means an additional restriction similar to (18). For well-behaved kernels $K \in H_r$ is an important part of the mapping property as seen from the necessary side and from the results of Hörmander.

Proof of Theorem I. We may assume, given $\varrho > 0$, that

$$\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|^s dy\Big)^{1/s} = 4\lambda > 0 \qquad (\lambda < \infty).$$

Now select $f \in C_0^\infty$ with support in $\{\varrho \leqslant |y| \leqslant 2\varrho\}$ such that

$$\Big|\int_{\varrho\leqslant|y|\leqslant 2\varrho} K(y)f(-y)\,dy\Big| \geqslant 3\lambda, \qquad \|f\|_{s'} = 1.$$

This is also possible for $s = 1$ or $s = \infty$ (if permitted). Then, by $K \in V_s$ for $|x| \leqslant \delta\varrho$

$$\Big|\int_{\varrho\leqslant|y|\leqslant 2\varrho} (K(y+x)-K(y))f(-y)\,dy\Big| \leqslant 2\lambda, \tag{22}$$

hence, with $\varepsilon = (1-\delta)\varrho$,

$$|(K_\varepsilon * f)(x)| = \Big|\int_{\varrho\leqslant|y|\leqslant 2\varrho} K(y+x)f(-y)\,dy\Big| \geqslant \lambda,$$

$$m\{x\colon |(K_\varepsilon * f)(x)| \geqslant \lambda\} \geqslant A\delta^n\varrho^n = B\varrho^n.$$

The condition $K_\varepsilon \in L^\#_{p,q}$ uniformly therefore implies

$$\lambda(B\varrho^n)^{1/q} \leqslant \|K_\varepsilon * f\|^\#_q \leqslant B\|f\|_p.$$

In view of $p \leqslant s' \leqslant \infty$,

$$\|f\|_p^p \leqslant \|f\|_{s'}^p (A\varrho^n)^{1-(p/s')} \leqslant A\varrho^{n(1-(p/s'))}.$$

Hence

$$\lambda \leqslant B\varrho^{n((1/p)-(1/s')-(1/q))} = B\varrho^{n((1/s)-(1/r))}.$$

The case $K \in L^\#_{p,q}$ does not require the introduction of ε, and the proof is complete.

Remark I. Essentially the same proof works if $K \in V_s$ is replaced by

$$\Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y+x)-K(y)|^s dy\Big)^{1/s} \leqslant B\varrho^{n((1/s)-(1/r))} \tag{23}$$

$$\text{for} \quad \varrho > 0,\ |x| \leqslant \varrho/2.$$

Observe that in (22) the right side becomes $B\varrho^{n((1/s)-(1/r))}$ which can be assumed $\leqslant 2\lambda$ since we are finished otherwise (provided that the objective was to prove $\lambda \leqslant B\varrho^{n((1/s)-(1/r))}$ with B independent of ϱ and the choice of f). Also observe that the proof can be modified even if $\lambda = \infty$. So we obtain $K \in L^s_{\mathrm{loc}}(\mathbf{R}^n - 0)$ as a necessary condition (without having to assume it). We note that condition (23) becomes stronger for larger s. Finally, we remark that condition (23) in case $s = r$ is somewhat weaker than $K \in H_r$. So the condition $K \in V_s$ of Theorem I ($s = r$) may also be replaced by $K \in H_r$.

§ 3. Further necessary conditions for type (p, p). Besides the growth conditions (17), (18) there are further necessary conditions in case $p = q$.

THEOREM II. *Assume $1 \leqslant p < \infty$, $K \in L_{\mathrm{loc}}(\mathbf{R}^n - 0)$, and*

$$\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|\,dy \leqslant B, \qquad \varrho > 0. \tag{24}$$

Now, if $K_\varepsilon \in L^\#_{p,p}$ uniformly, then

$$\Big|\int_{\varrho_1\leqslant|y|\leqslant\varrho_2} K(y)\,dy\Big| \leqslant B, \qquad 0 < \varrho_1 < \varrho_2 < \infty. \tag{25}$$

If also $K \in \mathrm{CL}(\mathbf{R}^n)$, the same conclusion can be drawn from $K \in L^\#_{p,p}$.

It should be pointed out that the necessary condition (17) with $p = q$ implies (24), so Theorem II is applicable in the situation of Theorem I ($p = q$).

Proof. We select $f \in C_0^\infty$ such that $0 \leqslant f(y) \leqslant 1$ always, $f(y) = 1$ for $|y| \leqslant 2\varrho$, $f(y) = 0$ for $|y| \geqslant 3\varrho$ ($\varrho > 0$). Take $0 < \varepsilon < \varrho$ and let

$$2\lambda = \Big|\int_{|y|\leqslant\varrho} K_\varepsilon(y)\,dy\Big|, \qquad \int_{\varrho\leqslant|y|\leqslant 4\varrho} |K(y)|\,dy \leqslant B'.$$

We may assume $\lambda \geqslant B'$ since we are finished otherwise. Hence, for $|x| \leqslant \varrho$,

$$(K_s * f)(x) = \int_{|y|\leqslant\varrho} K_s(y)\,dy + \int_{\varrho\leqslant|y|\leqslant 4\varrho} K(y) f(x-y)\,dy,$$

$$|(K_s * f)(x)| \geqslant 2\lambda - B' \geqslant \lambda,$$

$$m\{x\colon\ |(K_s * f)(x)| \geqslant \lambda\} \geqslant A\varrho^n.$$

The condition $K_s \in L^{\#}_{p,p}$ uniformly therefore implies

$$\lambda (A\varrho^n)^{1/p} \leqslant \|K_s * f\|_p^{\#} \leqslant B\,\|f\|_p \leqslant B\varrho^{n/p}$$

giving the conclusion $\lambda \leqslant B$.

In case $K \in L^{\#}_{p,p}$ we drop s and write generalized Cauchy integrals. Thus we get

$$\Big| \int'_{|y|\leqslant\varrho} K(y)\,dy \Big| \leqslant B,$$

which implies (25) by taking differences.

§ 4. Necessary and sufficient conditions for type (p, p). Here we use results of Hörmander [3], Stein [5], Benedek–Calderón–Panzone [1] to show that our necessary conditions are sufficient as well, at least in the case of weak oscillation. Since several ideas are familiar, the exposition will be brief.

THEOREM III. *Assume* $1 \leqslant p < \infty$, $1 \leqslant s \leqslant p'$, $K \in L_{\mathrm{loc}}(\boldsymbol{R}^n - 0)$, *and* $K \in W_s$. *Now,* $K_s \in L^{\#}_{p,p}$ *uniformly if and only if for* $\varrho > 0$, $0 < \varrho_1 < \varrho_2 < \infty$

$$(26)\qquad \Big(\int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|^s\,dy \Big)^{1/s} \leqslant B\varrho^{-n/s'}, \qquad \Big| \int_{\varrho_1\leqslant|y|\leqslant\varrho_2} K(y)\,dy \Big| \leqslant B.$$

If also $K \in \mathrm{CL}(\boldsymbol{R}^n)$, *the same conclusion holds with respect to* $K \in L^{\#}_{p,p}$.

The simplest case is $s = 1$ if one fixes s in the assumption. It follows that the weak mapping properties are equivalent for $1 \leqslant p \leqslant s'$ $(p < \infty)$ and therefore also equivalent for the strong mapping properties for $1 < p < s'$ $(s < \infty)$ by the Marcinkiewicz convexity theorem. Some of the restrictions for p can be removed by duality and the use of the Stein–Weiss convexity theorem [6].

Proof. The necessity has been shown already in Theorems I and II. For the sufficiency observe that $K \in W_s$ and (26) imply $K \in H_1$ as shown in (20), (21) with $r = 1 = \sigma$. In view of the simple inequality

$$\int_{|y|\geqslant 2|x|} |K_s(y+x) - K_s(y)|\,dy \leqslant \int_{|y|\geqslant 2|x|} |K(y+x) - K(y)|\,dy + 2 \int_{s\leqslant|y|\leqslant 3s} |K(y)|\,dy,$$

it follows that $K_s \in H_1$ uniformly and $K_{\varepsilon,\eta} \in H_1$ uniformly. By Benedek–Calderón–Panzone it also follows in view of (26) that $K_{\varepsilon,\eta} \in L_{2,2}$ uniformly. Next, by Hörmander it follows that $K_{\varepsilon,\eta} \in L^{\#}_{1,1}$ uniformly and (by convexity and duality) that $K_{\varepsilon,\eta} \in L^{\#}_{p,p}$ uniformly. Now, a Fatou-type argument as $\eta \to \infty$ shows that $K_s \in L^{\#}_{p,p}$ uniformly. This completes the proof unless we are in the case $K \in \mathrm{CL}$. Here we introduce for $0 < \varepsilon < 1 < \eta < \infty$

$$\tilde K_{\varepsilon,\eta}(x) = |x|^{\varepsilon} K(x) \quad \text{for} \quad |x| \leqslant 1, \qquad \tilde K_{\varepsilon,\eta}(x) = K_{1,\eta}(x) \quad \text{for} \quad |x| > 1$$

and observe that for $x \neq 0$

$$\tilde K_{\varepsilon,\eta}(x) = \varepsilon \int_0^1 t^{\varepsilon-1} K_{t,\eta}(x)\,dt.$$

So the kernel $\tilde K_{\varepsilon,\eta}$ is a convex combination of the kernels $K_{t,\eta}$. Therefore $\tilde K_{\varepsilon,\eta} \in H_1$ uniformly and $\tilde K_{\varepsilon,\eta} \in L_{2,2}$ uniformly, hence as before $\tilde K_{\varepsilon,\eta} \in L^{\#}_{p,p}$ uniformly. Letting first $\eta \to +\infty$ and then $\varepsilon \to +0$ it follows by a Fatou-type argument again that $K \in L^{\#}_{p,p}$. This completes the proof in the second case.

Remark II. Essentially the same proof works if in Theorem III $(s = 1)$ the condition $K \in W_1$ is replaced by $K \in H_1$ (use Remark I for the necessity). We have explained why this may be less satisfactory though. However, a slight extension of Hörmander's proof (in combination with convexity theorems) gives that under the assumption $K \in H_1 \cap \mathrm{CL}$ the condition $K \in L^{\#}_{p,p}$ is equivalent to $\hat K \in L^{\infty}$ for any $p \in (1, \infty)$. So we learn that under the assumption $K \in H_1 \cap \mathrm{CL}$ the conditions of Theorem III $(s = 1)$ are also equivalent to $\hat K \in L^{\infty}$. While the sufficiency here appears to be relatively straight forward (see Benedek–Calderón–Panzone), we do not know of a direct proof for the necessity of (24).

§ 5. Further necessary conditions for type (p, q). Here we discuss the case $p < q$. In dimension $n = 1$ condition (18) is necessary for our kernels according to Corollary I, but also sufficient since the Hardy–Littlewood–Sobolev kernels map. For dimension $n > 1$ there are further necessary conditions.

THEOREM IV. *Assume* $1 \leqslant p < q < \infty$, $1 \leqslant s \leqslant p'$, $0 < 1/s < (1/r) + (1/n)$, $K \in L_{\mathrm{loc}}(\boldsymbol{R}^n - 0)$ *and* $K \in W_s$. *Now, if* $K_s \in L^{\#}_{p,q}$ *uniformly, then*

$$(27)\qquad \|K\|_r^{\#} < \infty.$$

If also $K \in \mathrm{CL}(\boldsymbol{R}^n)$, *the same conclusion can be drawn from* $K \in L^{\#}_{p,q}$.

Observe that in case $n = 1$ the choice $s = 1$ is always possible, while in general at least the choice $s = r$ is possible.

Proof. By Theorem I we have (17), hence for $\varrho > 0$

$$(28)\qquad \int_{\varrho\leqslant|y|\leqslant 2\varrho} |K(y)|\,dy \leqslant B\varrho^{n/r'}, \qquad \int_{|y|\leqslant\varrho} |K(y)|\,dy \leqslant B\varrho^{n/r'}$$

since $r>1$, $r'<\infty$. Also, by Corollary II ($\sigma = s$)

$$(29)\qquad \Big(\int_{|y|\geqslant\varrho} |K(y+x)-K(y)|^s dy\Big)^{1/s} \leqslant B\varrho^{n((1/s)-(1/r))} \quad \text{for} \quad \varrho>0,\ |x|\leqslant \varrho/2.$$

Let $a=\varrho^n$ and $f\in C_0^\infty$, $|f|\leqslant 1$, $m\{\operatorname{supp} f\}\leqslant a$. Then, by (28) and (29), we obtain $K\in L_{\rm loc}$ and for $|x|\leqslant\varrho/2$

$$\Big|\int_{R^n}(K(y+x)-K(y))f(-y)\,dy\Big|$$
$$\leqslant \int_{|y|\leqslant\varrho}\big(|K(y+x)|+|K(y)|\big)\,dy+\Big(\int_{|y|\geqslant\varrho}|K(y+x)-K(y)|^s dy\Big)^{1/s}a^{1/s'}$$
$$\leqslant Ba^{1/r'}+Ba^{(1/s)-(1/r)}a^{1/s'}\leqslant B'a^{1/r'}.$$

We need only deal with the case $K\in L^{\#}_{p,q}$ (use Fatou), and we may assume that

$$\Big|\int_{R^n}K(y)f(-y)\,dy\Big| = 2\lambda,\qquad \lambda\geqslant B'a^{1/r'},$$

since we are finished otherwise in view of (7′). Hence,

$$|(K*f)(x)|\geqslant 2\lambda - B'a^{1/r'}\geqslant\lambda \quad\text{for}\quad |x|\leqslant\varrho/2,$$
$$m\{x:\ |(K*f)(x)|\geqslant\lambda\}\geqslant Aa.$$

But then

$$\lambda(Aa)^{1/q}\leqslant \|K*f\|_q^{\#}\leqslant B\|f\|_p\leqslant Ba^{1/p}$$

giving the conclusion $\lambda\leqslant Ba^{(1/p)-(1/q)}=Ba^{1/r'}$. In view of (7′) this completes the proof.

Remark III. In case $r<s\leqslant p'$ ($s<\infty$) the condition $K\in W_s$ in Theorem IV can be replaced by $K\in V_s$. By Theorem I condition (17) is still necessary and implies (23), (29) for $|x|\leqslant\delta\varrho$. So the same proof works. In case $s=r$ the condition $K\in W_s$ can be replaced by $K\in H_r$. Then (29) holds and also (17) according to Remark I, and there is no change in the proof.

§ 6. Necessary and sufficient conditions for type (p,q).

Here we discuss the case $p<q$ and show that our necessary conditions are sufficient as well even without any regularity conditions on K.

THEOREM V. *Assume* $1\leqslant p<q<\infty, 1\leqslant s\leqslant p', 0<1/s<(1/r)+(1/n)$, $K\in L_{\rm loc}(R^n-0)$, *and* $K\in W_s$. *Now* $K_\varepsilon\in L^{\#}_{p,q}$ *uniformly if and only if* $\|K\|_r^{\#}<\infty$. *If also* $K\in {\rm CL}(R^n)$, *the same conclusion holds with respect to* $K\in L^{\#}_{p,q}$.

If one fixes s in the assumption, it follows that the weak mapping properties are equivalent for pairs (p,q) with the same r in the range $1\leqslant p\leqslant s'$ ($q<\infty$) and therefore also equivalent to the strong mapping properties for $1<p<s'$ ($q<\infty$, $s<\infty$). Some of the restrictions for p can be removed by duality.

Proof. The necessity has been shown already in Theorem IV. For the sufficiency note that $1<r<\infty$, and we may interpret (27) by means of (7′) since $K\in L_{\rm loc}$ follows easily. Hence we obtain for $f\in L$ and $|g|=1$ or 0 (g measurable) the inequality

$$(30)\qquad \iint_{R^n\times R^n} dx\,dy\,|K(x+y)g(-x)f(-y)|\leqslant A\int_{R^n}dy\,|f(-y)|\,\|K\|_r^{\#}\|g\|_{r'}$$
$$\leqslant B\|f\|_1\|g\|_{r'}.$$

By (30) in connection with (7′) we see that K is of weak type $(1,r)$ and of restricted type (r',∞). Therefore, by Stein–Weiss, K is also of strong type (p,q) for $p>1$, $q<\infty$, so, in particular, of weak type for all pairs in question. The same proof works with K_ε in place of K, uniformly in ε. In fact, we have shown $|K|\in L^{\#}_{p,q}$.

Remark IV. The condition $K\in W_s$ in Theorem V can be replaced by $K\in V_s$ if $s>r$ (additionally) and by $K\in H_r$ if $s=r$ (see Remark III). Note that the extra conditions were only needed in the necessary part. In case of dimension $n=1$ condition (27) can be replaced by the stronger condition (18), see Corollary I. By a slight extension of Hörmander's proof (in combination with convexity theorems as used in connection with (30)) we find that under the assumption $K\in H_r\cap{\rm CL}$ the condition $K\in L^{\#}_{p,q}$ is equivalent to $K\in L^{\#}_{1,r}$ for all pairs (p,q) with that same r. By comparison we see that $K\in L^{\#}_{1,r}$ is equivalent to $\|K\|_r^{\#}<\infty$ at least under the additional assumption $K\in H_r\cap{\rm CL}$. It turns out that the additional assumption $K\in H_r$ is superfluous for this conclusion ($1<r<\infty$). For the sufficient part this was already shown above. For the necessary part we refer to the fact that mappings from L can be handled in considerable generality. Further details will be discussed in a forthcoming paper by Fiedler, Jurkat, and Körner on L^p estimates for more general operators.

In conclusion we mention that the example at the end of § 1 with $a=-3/4r$ satisfies $\|K\|_r^{\#}<\infty$, but not (18), although K is a case for which our theory applies.

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Received January 28, 1977 (1253)

STUDIA MATHEMATICA, T. LXV. (1979)

A note on rotations in separable Banach spaces

by

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Abstract. We show that every separable Banach space X is a complemented subspace of a separable Banach space Y which has the following rotation property:

There is a dense subset S of the unit sphere of Y so that for every $x, y \in S$ there is an isometric automorphism $T: Y \to Y$ with $T(x) = y$. As a consequence, there is a separable Banach space satisfying this rotation property which, on the other side, fails to have the approximation property.

This paper is concerned with the following Banach space property:

(M) *Let X be a Banach space (real or complex). There is a dense subset S of the unit sphere of X such that for every $x, y \in S$ there is an isometric automorphism $T: X \to X$ with $T(x) = y$.*

An isometric automorphism of a Banach space X is sometimes called *rotation*. We obtain immediately:

Let X be a Banach space having (M). *For every $\varepsilon > 0$ and $x, y \in X$ with $\|x\| = \|y\| = 1$ there is an automorphism $T: X \to X$ with $(1-\varepsilon)\|z\| \leqslant \|T(z)\| \leqslant (1+\varepsilon)\|z\|$ for all $z \in X$ and $T(x) = y$.*

Clearly, the separable Hilbert space satisfies (M). In [7] it was shown that there is a separable Banach space G with property (M) whose dual space G^* is isometrically isomorphic to an abstract L-space (cf. [5]). It turns out that the rotation property of (M) holds on the set of all smooth points of G (i.e. on the points x with $\|x\| = 1$ and there is only one linear functional x^* with $x^*(x) = \|x^*\| = 1$). The set of smooth points is a dense G_δ-subset of the unit sphere of any separable Banach space (Mazur [9]).

On the other hand, the unit sphere of G contains points x, y which do not admit a rotation T of G carrying x onto y. Thus G is an example of a Banach space having (M) which is different from a Hilbert space. Exploiting Banach's characterization of the rotations in $L_p(0,1)$; $1 \leqslant p < \infty$; $p \neq 2$; ([1] Chap. XI), we obtain:

Let $f, g \in L_p(0,1)$; $1 \leqslant p < \infty$; $p \neq 2$; (with respect to the Lebesgue measure λ) so that $\|f\| = \|g\| = 1$.