Subspaces of $L^1$ containing $L^1$

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Abstract. An analytically defined class of operators on $L^1(0, 1)$ called E-operators is introduced. It is proven that a bounded linear operator $T: L^1 \to L^1$ is an E-operator if and only if there is a closed linear subspace $X$ isomorphic (i.e., linearly homeomorphic) to $L^1$ such that $TX$ is a homeomorphism (into). If $T$ is an E-operator, then there exists a subspace $X$ isometrically isomorphic to $L^1$ with $T|_X$ a homeomorphism (into) and $TX$ complemented in $L^1$.

As a corollary it is shown that every subspace $Y$ of $L^1$ isomorphic to $L^1$ contains a subspace which is isomorphic to $L^1$ and complemented in the whole space. From this it follows that if a complemented closed linear subspace $X$ of $L^1$ contains a subspace isomorphic to $L^1$, then $X$ is isomorphic to $L^1$.

Another corollary of the main theorem is that if $L^1$ is isomorphic to an unconditional sum of a sequence of Banach spaces, then one of the spaces is isomorphic to $L^1$. In particular, $L^1$ is primary.

It is shown that an operator $T$ on $L^1$ is an E-operator if and only if $|T|$ is an E-operator.

1. Introduction. This paper contains a study of certain bounded linear operators $T: L^1 \to L^1$ called E-operators. This class of operators is defined analytically.

Theorem 4.1 states that an operator on $L^1$ is an E-operator if and only if the operator carries some subspace isomorphic to $L^1$ isomorphically. It is shown in Theorem 4.2 that an E-operator actually possesses an apparently stronger property: if $T: L^1 \to L^1$ is an E-operator, then there exists a subspace $Y$ of $L^1$ with $Y$ isometric to $L^1$, with $T|_Y$ an isomorphism, and with $TY$ complemented. ($Y$ is also automatically complemented; see (9).)

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** The research of the second author forms his thesis submitted to the University of California at Berkeley. Much of the research was done while he was visiting the University of Illinois at Urbana — Champaign.
As a corollary of Theorem 4.3 we prove, using a result of Lindenstrauss and Pełczyński [8], that if $D'$ is isomorphic to an unconditional sum of a sequence of Banach spaces, then one of the spaces is isomorphic to $L^1$ (Corollary 5.5). In particular $L^1$ is primary. Maurey [11] adapted unreported techniques of some of the present authors (Enflo) to give one proof that $L^p$ is primary for all $p$, $1 \leq p < \infty$. The result asserts that if $L^p$ is written as the direct sum of two Banach spaces, then at least one of them is isomorphic to $L^p$. The question whether this was true was raised by Lindenstrauss and Pełczyński in [9], where they proved that $C[0,1]$ is primary. For $p > 1$, an alternative proof that $L^p$ is primary, based on a result of Casazza and Lin [2], is presented by Alspach, Enflo, and Odell in [1].

Another corollary (Corollary 5.2) of Theorem 4.3 is that any isomorphism of $L^1$ in $L^1$ contains a subspace isomorphic to $L^1$ which is complemented in the whole space. This yields some information about complemented subspaces of $L^1$, for it implies, by the Pełczyński decomposition method [12], that if a complemented subspace $X$ of $L^1$ contains a subspace isomorphic to $L^1$, then $X$ itself is isomorphic to $L^1$ (Corollary 5.3). It is an open question whether every complemented infinite dimensional subspace of $L^1$ is isomorphic either to $L^1$ or to $l_1$. Lewis and Specker's results [7] (see also [15]) show that a complemented infinite dimensional subspace $Z$ of $L^1$ has the Radon--Nikodym property if and only if any projection onto $Z$ factors through $l_1$; hence $Z$ is isomorphic to $l_1$. Corollary 5.2 implies that if the projection onto $Z$ is an $B$-operator, then $Z$ is isomorphic to $L^1$. It is known that there are operators which do not factor through $l_1$ and yet are not $B$-operators ([8] and [13]). If such a projection exists, then the above open question would be answered in the negative.

It may be possible to reduce some questions about bounded linear operators on $L^1$ to questions about positive operators by using Proposition 7.1. Proposition 7.1 states that $T$ is an $E$-operator if and only if $|T|$ is an $E$-operator. (With every bounded linear operator $T$: $E \to L^1$ can be associated a positive bounded linear operator $|T|$, the absolute value of $T$. See the remarks preceding Proposition 7.1.) One tends to regard $E$-operators as "big" in that they carry a big subspace (i.e., one isomorphic to $L^1$) isomorphically. Proposition 7.1 then implies that if $|T|$ is big in this sense, then $T$ is already big.

As an application of the theorems, we answer affirmatively the following question of A. Pełczyński: Suppose

$$S = \int T \nu S d \theta$$

is an isomorphism on a subspace isomorphic to $L^1$. Must $S$ have the same property? Here $\nu$ ranges over points in the circle group $G$, $T$ is translation by $\nu$, and $S$ is an operator on $L^1(G)$. Proposition 7.2 gives the affirmative answer. Of course, in view of Theorem 4.1, the result is that if $S$ is an $E$-operator, then so is $S$. The fact that the "average" operator $S$ being an $E$-operator implies that $S$ is an $E$-operator agrees with the intuition of thinking of an $E$-operator as "big".

The structure of $E$-operators as illuminated in Theorem 4.3 and in the more technical Theorem 4.3 forms the basis for most of the other results, including Theorem 4.1. We give now a brief intuitive summary of the methods used to prove Theorems 4.2 and 4.3. One fundamental idea is that a simple way to construct a subspace of $L^1$ isometric to $L^1$ is to divide a subset $E$ of $[0,1]$ of positive measure into two subsets, then divide each of these subsets into two subsets, and so on, in such a way that all the subsets eventually become smaller and smaller in measure. The collection of the characteristic functions of these subsets of $[0,1]$ has closed linear span isometric to $L^1$. (The closed linear span consists of all $\nu$-measurable functions in $L^1$, where $\nu$ is the $\nu$-subspace of subsets of $E$ generated by all the subsets into which $E$ has been divided. Since $\nu$ will be non-atomic, $L^1(\nu)$ is isometric to $L^1$ [10]. The operator of conditional expectation with respect to $\nu$ is a projection of norm 1 onto $L^1(\nu)$.

The proof of Theorem 4.3 shows that if $T$ is an $E$-operator, then there exist such a set $E$ and such a splitting process for $E$, generating a $\nu$-subspace $\nu$ of subsets of $E$, and there exists a subset $F$ of $[0,1]$ such that by making a single change of signs (by multiplying by a fixed $\{1,-1\}$ valued function $s$), the operator $sB_{E,T}[L^1(\nu)]$ is almost exactly a non-zero scalar multiple of a positive isometry. Here $B_{E,T}$ denotes the operator which restricts functions to the set $F$.

The proof accomplishes this result by finding a splitting process on $E$ such that when $T$ is applied to the characteristic functions of two disjoint subsets of $E$ in $\nu$, the two image functions are almost disjointly supported, whenever the set $F$.

There are actually two senses in which the image functions are "almost" disjointly supported on $F$. The one needed for the proof of Theorem 4.2 is the well-known concept of relative disjointness. Rosenthal proved in [14] that relatively disjoint collections of functions in $L^1$ span complemented isomorphs of $l_1$ of the appropriate dimension, either finite or infinite). His calculations of the bounds for the distance from $l_1$ of such isomorphs, and for the norm of a projection onto such subspaces are used in the proof of Theorem 4.2.

Another concept of "almost disjoint", stated in Theorem 4.3 (c), is useful for proving Theorem 4.1 and Proposition 7.2.

The method by which almost disjointly supported functions are recognized is to compare the integral of the maximum of the absolute
values of the functions with the integral of the sum of the absolute values of the functions. When these two quantities are (almost) equal, the functions are (almost) disjointly supported. The integral of the maximum of functions in $L^1$ was investigated by L. Dor [5] in connection with the still open problem of whether every subspace of $L^1$ isomorphic to $L^1$ is complemented. (He showed that this is true if the subspace is sufficiently close to $L^1$ in the Banach–Markov sense.) We use one of these results in the proofs of Theorem 4.1 and Corollary 5.2. In our terminology his result implies that if $T$ is an isomorphism, then $T$ is an $E$-operator.

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Corollary 5.5 was called to our attention in private correspondence by N. J. Kalton, who has proven independently the same result by a different method.

Format. The format of the paper is as follows: Section 2 gives the definitions of bush, tree, and $E$-operator. Section 3 presents some preliminary facts about operators and bushes. Section 4 states the main theorems and gives most of their proofs. Section 5 draws corollaries of Theorem 4.3. In Section 6 the proof of Theorem 4.1 is completed. Section 7 contains the propositions about $|T|$ and the average operator $E$, and contains the two open problems.

2. Definitions. We deal with $L^1 = L^1(0, 1)$, the Banach space of equivalence classes of Lebesgue integrable real-valued functions defined on $[0, 1]$. $E$ is Lebesgue measure. The notation $|E|$ will also be used to denote the Lebesgue measure of a measurable subset $E \subseteq [0, 1]$. $\chi_E$ denotes the characteristic function of $E$, where

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \notin E. \end{cases}$$

$E_A$ denotes the restriction operator on $L^1$ defined by

$$E_A(f)(t) = f(t)\chi_A(t).$$

If $T: L^1 \rightarrow L^1$ is a bounded linear operator, we sometimes write $TE$ in place of $T(x_E)$. An “isomorphism” is a linear homeomorphism into. If there exists an isomorphism from a Banach space $X$ onto a Banach space $Y$, then $X$ and $Y$ are “isomorphic”, and we write $X \sim Y$. An “isomorph of $L^1$” is a Banach space isomorphic to $L^1$.

If $\mathcal{A}$ is a collection of sets, $\mathcal{A}(z)$ denotes the ring generated by $\mathcal{A}$, and $\mathcal{A}(z)$ denotes the $\sigma$-ring generated by $z$.

Definitions.

1. A bush is a sequence of finite partitions of a measurable subset $E^i \subseteq [0, 1]$ of positive measure in which each partition refines the preceding partition, and in which the mesh of the partitions tends to zero. In symbols, $(E^i)$, $i = 1, \ldots, M_n$, $n = 0, 1, 2, \ldots$ is a bush if

- (1) $M_0 = 1$ and $|E^1| > 0$,
- (2) for each $n$, $\bigcup_{i=1}^{M_n} E^i = E^1$,
- (3) for each $n$, $E^i \cap E^j = \emptyset$ if $i \neq j$,
- (4) for each $n$ and each $j$, $1 \leq j \leq M_{n+1}$, there is an $i$, $1 \leq i \leq M_n$ with $E^i \cap E^j \neq \emptyset$,
- (5) $\sum_{1 \leq i \leq M_n} |E^i| \rightarrow 0$ as $n \rightarrow \infty$.

2. A tree is a bush $(E^i)$, $1 \leq i \leq M_n$, $n = 0, 1, 2, \ldots$ in which

- (1) $M_0 = 3$,
- (2) $E_i^1 = E_{i+1} \cup E_i+1$ for each $n$ and $i$, $1 \leq i \leq 3^n$.

3. Let $T: L^1 \rightarrow L^1$ be a bounded linear operator. $T$ is called an $E$-operator if there exist $\delta > 0$ and a bush $(E_i^i)$ with

$$\frac{1}{|E_i^i|} \int_{1 \leq i \leq M_n} |T(x_{E_i^i})| > \delta$$

for each $n$. If $T$ is an $E$-operator and $\delta > 0$, $T$ is called an $E$-operator of constant $\delta$ if

$$\delta \leq \sup_{n \geq 0} \frac{1}{|E_i^i|} \int_{1 \leq i \leq M_n} |T(x_{E_i^i})|,$$

where the supremum is taken over all bushes $(E_i^i)$.

Remark. It is shown in Section 3 (see the remark after Lemma 3.3) that for any bush $(E_i^i)$ the limit

$$\lim_{n \rightarrow \infty} \frac{1}{|E_i^i|} \int_{1 \leq i \leq M_n} |T(x_{E_i^i})|$$

exists; hence, in the above definition, the limit superior could be replaced by limit.
Remark. The letter \( E \) in the term \( E \)-operator is an abbreviation for “Enflo”. The phrase “Enflo operator” was coined by H. P. Rosenthal after Enflo began the study of these operators several years ago.

3. Properties of operators on \( L^1 \) with respect to bushes. For the whole of this section, let \( T : L^1 \to L^1 \) be a bounded linear operator and let \( \{E_i\} \), \( i = 1, 2, \ldots, M \); \( n = 1, 2, \ldots \) be a bush.

For \( n = 0, 1, 2, \ldots \) define

\[
\gamma_n = \max_{i \leq n} \|TE_i\|.
\]

(Here we have used the notation \( TE_i \) in place of \( T(\|E_i\|) \).

Then \( \gamma_n \in L^1 \).

Define the \( L^1 \)-valued measure \( v_n \) on the finite algebra \( \mathcal{A}(E_1, \ldots, E_M) \) by

\[
v_n(B) = \sum_{i \leq n} \|TE_i\|, \quad B \in \mathcal{A}(E_1, \ldots, E_M).
\]

Lemma 3.1. Given \( \mathcal{A} \in \mathcal{M}(\mathcal{A}(E_1, \ldots, E_M)) \),

1. \( v_n(B) \) is defined for sufficiently large \( n \). For such \( n \)
2. \( v_n(B) \geq v_n(B) \);
3. \( v_n(B) \leq \|T\|E_i \).
4. As \( n \to \infty \), \( v_n(B) \) converges both a.e. and in the \( L^1 \) norm to a function \( v(B) \in L^1 \).
5. \( v \) is \( \sigma \)-additive positive \( L^1 \)-valued measure on \( \mathcal{A}(E_1, \ldots, E_M) \) which extends to a \( \sigma \)-additive positive \( L^1 \)-valued measure on \( \sigma(\mathcal{A}(E_1, \ldots, E_M)) \). (We denote this extension by \( v \).)
6. For each \( G \in \sigma(\mathcal{A}(E_1, \ldots, E_M)) \), \( |\mathcal{E}| \leq |\mathcal{G}| \) a.e.

Proof. (1) is clear. (2) and (3) are applications of the triangle inequality. (4) follows from the monotone convergence theorem.

For (5), note that \( v \) is clearly finitely additive. Hence using (3), it is \( \sigma \)-additive on \( \mathcal{A}(E_1, \ldots, E_M) \) with \( v(B) \leq \|T\|E_i \). Therefore \( v \) extends to \( \sigma(\mathcal{A}(E_1, \ldots, E_M)) \).

For (6), suppose first that \( B \in \sigma(\mathcal{A}(E_1, \ldots, E_M)) \). Then \( |\mathcal{E}| \leq |\mathcal{G}| \) a.e. by the triangle inequality. Now suppose \( C \in \sigma(\mathcal{A}(E_1, \ldots, E_M)) \). Then choose \( B_n \in \sigma(\mathcal{A}(E_1, \ldots, E_M)) \) with \( |B_n \triangle C| \leq \theta \), that is \( B_n \to C \) in \( \mathcal{G} \), as \( n \to \infty \). Then \( B_n \to C \) in \( L^1 \). By passing to a subsequence if necessary, we may assume that \( |B_n \to C| \) a.e. and, similarly, that \( v(B_n) \to v(C) \) a.e. a.e.

Lemma 3.2. The sequence \( \gamma_n \) converges a.e. and in the \( L^1 \) norm to a function \( g \) in \( L^1 \).

Proof. We shall show below that for all \( n \)

\[
\gamma_{n+1} - v_{n+1}(E_i) \leq \gamma_n - v_n(E_i).
\]

Hence \( \lim_{n \to \infty} (\gamma_n - v_n(E_i)) \) exists, being the limit of a decreasing sequence.

By Lemma 3.1 (4), \( \lim v_n(E_i) \) exists and is finite a.e. Hence \( g = \lim v_n \).

Furthermore

\[
0 \leq \gamma_n \leq v_n(E_i) \leq v(E_i) \leq L^1,
\]

so by the dominated convergence theorem, \( \gamma_n \to g \) in the \( L^1 \) norm, and \( g \in L^1 \).

We now show that for fixed \( t \in (0, 1] \)

\[
\gamma_{n+1}(t) - v_{n+1}(E_i)(t) \leq \gamma_n(t) - v_n(E_i)(t).
\]

By the definition of \( \gamma_{n+1} \), there is a \( j \) with \( \gamma_{n+1}(t) = |\mathcal{E}_{n+1}^{(i)}(t)| \). By the properties of bushes, there is an \( i \) with \( E_i^{(n+1)} \in \mathcal{E}_{n+1} \). Then

\[
\gamma_{n+1}(t) - \gamma_n(t) \leq |\mathcal{E}_{n+1}^{(i)}(t)| - |\mathcal{E}_{n+1}^{(i)}(t)|
\]

\[
= \gamma_{n+1}(E_i^{(n+1)}(t)) - v(E_i^{(n+1)}(t))
\]

\[
\leq \gamma_{n+1}(E_i^{(n+1)}(t)) - v_n(E_i^{(n+1)}(t))
\]

\[
\leq \gamma_{n+1}(E_i^{(n+1)}(t)) - v(E_i^{(n+1)}(t))
\]

since \( v_{n+1} - v_n \geq 0 \) (so \( v_{n+1}(E_i^{(n+1)}(t)) - v(E_i^{(n+1)}(t)) \geq 0 \)).

Remark. Lemma 3.2 (3) implies that in the definition of an “\( E \)-operator of constant \( \delta \)”, the limit superior could be replaced by limit, or by limit inferior.

4. The main theorems. Our first major result is

Theorem 4.1. Let \( T : L^1 \to L^1 \) be a bounded linear operator. \( T \) is an \( E \)-operator if and only if there exists a subspace \( Y \) of \( L^1 \) with \( Y \) isomorphic to \( L^1 \) with \( Y \) an isomorphism (into).

The proof of Theorem 4.1 depends on the next two theorems which are our other main results.

Theorem 4.2. Suppose \( T \) is an \( E \)-operator of constant \( \delta \) and \( 0 < \epsilon < \frac{1}{4} \).

Then there exists a purely non-atomic \( \sigma \)-ring \( \mathcal{A} \) of Lebesgue measurable sets such that

1. \( L^1(\mathcal{A}) \) is isometric to \( L^1 \);
2. \( T(L^1(\mathcal{A})) \) is an isomorphism; for \( f \in L^1(\mathcal{A}) \),

\[
|Tf| \geq (1 - \epsilon)^f \delta ||f||;
\]

3. The image \( T|L^1(\mathcal{A}) \) is complemented; it is the range of a projection of norm at most

\[
\frac{||T||}{(1 - \epsilon)^f \delta}.
\]

Theorem 4.3 follows immediately from the more technical
Theorem 4.3. Suppose $T$ is an $E$-operator of constant $\delta$, and $0 < \varepsilon < \frac{1}{2}$. Then there exists a tree $(A_i^t)$, $i = 1, \ldots, 2^n$, $n = 0, 1, \ldots$ of measurable subsets of $[0, 1]$ with

(a) \[ |A_i^t| = \frac{|A_i^t|}{2^n} \]

and there exists a tree $(F_i^t)$ of measurable subsets of $[0, 1]$ such that for each $n$, and $i$, $1 \leq i \leq 2^n$, 

(b) \[ (1 - \varepsilon) \delta |A_i^t| \leq \int \int |TA_i^t| \leq (1 + \varepsilon) \int \int |TA_i^t|, \]

and

(c) \[ \sum_{j \in 2^n} |\mathcal{A}_j^t(t)| \leq \varepsilon |\mathcal{A}_j^t(t)| \]

for almost all $t \in F_i^t$; and such that

(d) if $B_1, \ldots, B_m$ are disjoint members of $\mathcal{A}_\varepsilon([A_i^t])$, then

\[ \int \max_{j \leq m} |TB_j| \geq (1 - \varepsilon) \delta \int |B_i| \]

Remark. Notice that conclusions 1 and 2 of Theorem 4.2 assert a strong form of the direct implication claimed in Theorem 4.1. The proof of the other direction of Theorem 4.1 will be given in Section 6.

In this section we first show that Theorem 4.3 implies Theorem 4.2. Then we prove Theorem 4.3.

Proof of Theorem 4.3 assuming the truth of Theorem 4.3. We actually need only conclusions (a) and (b) stated in Theorem 4.3 in order to derive Theorem 4.2. (Conclusions (c) and (d) will be used in Sections 6 and 7.)

Suppose there exist trees $(A_i)$ and $(F_i)$ with properties (a) and (b).

Let $\mathcal{A}_\varepsilon = \sigmaZF([A_i^t])$. Then by (a) $\mathcal{A}_\varepsilon$ is purely non-atomic, so by Maharam's Theorem [10], $L'(\mathcal{A}_\varepsilon)$ is isometric to $L^2$. This gives conclusion 1 of Theorem 4.3.

The right-hand inequality of (b) implies that the collection $(TA_i^t, \ldots, TA_{2^n}^t)$ is a relative disjoint collection [see [14]]. It follows that the finite sequence $TA_i^t, \ldots, TA_{2^n}^t$ is a basis sequence equivalent to the usual basis of $L^2$. Because of (a) and (b) and the fact that $\varepsilon$ and $\delta$ are independent of $n$, the constant of equivalence for this sequence is bounded by a constant independent of $n$. Specifically, for any scalars $a_i, 1 \leq i \leq 2^n$,

\[ \left\| \sum_{i=1}^{2^n} a_i TA_i^t \right\| \geq \frac{(1 - \varepsilon)^2}{1 + \varepsilon} \left\| \sum_{i=1}^{2^n} a_i X_i^t \right\| \]

This gives conclusion 2 of Theorem 4.2.

Another property implied by the relative disjointness of $TA_i^t, \ldots, TA_{2^n}^t$ is that their linear span is complemented with the projection constant bounded by

\[ \frac{|F_i|}{\delta} \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \]

It follows by a familiar compactness argument (see [4]) that the closed linear span of $(TA_i^t)_{n \to i}$ is complemented. This is conclusion 3 of Theorem 4.2.

Proof of Theorem 4.3. We shall use the functions $g_n$ and $g_0$ and the $L^1$-valued measures $\nu_n$ and $\nu$ defined in Section 3. The assumption that $T$ is an $E$-operator of constant $\delta$ means that there is a bush $(B_i^t)$ with

\[ \int \frac{1}{|B_i^t|} g \geq \delta (1 - \varepsilon) \]

for infinitely many values of $n$. Lemma 3.3 says that $g_0 \to g$ in $L^1$. Hence (1) is actually true for all sufficiently large $n$ and

\[ \int \frac{1}{|B_i^t|} g \geq \delta (1 - \varepsilon) \]

Hence we may use Egoroff's Theorem to find a subset $F \subset [0, 1]$ such that

\[ \int \frac{1}{|B_i^t|} g \geq \delta (1 - \varepsilon) \]

(2) $g_n \to g$ uniformly on $F$,

and

(3) $g_n \to g$ uniformly on $F$.

We may also assume

(4) $\nu_n(E_i^t) \to \nu(E_i^t)$ uniformly on $F$.

Choose $\beta$ so small that

\[ (1 - \varepsilon) \delta (1 - 6\beta) \geq (1 - \varepsilon) \]

(5) $|g_n - g| < \beta g$ on $F$.

Then using (3), (4) and (5), choose $N$ so large that for $n \geq N$,

(6) $|g_n - g| < \beta g$ on $F$.

and

(7) $0 \leq v(F) - v_n(F) < \beta g$ on $F$.

(for any $v$ on which $v_n$ is defined).
The next stage is to select a "subbush", by choosing one of the sets $E_i$, $i = 1, \ldots, M_N$ and all of its subsets in the original bush, with the property that the image functions of disjoint elements in this subbush are almost disjoint when restricted to a fixed set (to be called $E_i$).

For each $i$, $1 \leq i \leq M_N$, let

$$G_i = \{ t \in E_i \}$$

for infinitely many values of $n$, there exists

$$E_i^N \subset E_i^N$$

with $g_n(t) = |TE_i^N(t)|$.

Then (since $M_N$ is finite),

$$\bigcup_{i=1}^{M_N} G_i = F.$$

Thus there exists $i_0 \leq M_N$ such that

$$\int g \geq \delta (1 - \frac{1}{2}\alpha) |E_i^N|.$$

(If not, $\int g \leq \delta (1 - \frac{1}{2}\alpha) |E_i^N|$ for all $i$; summation over $i$ contradicts (3)).

Let $E_i = G_{i_0}$, $E_i = E_i^N$ and $E_i = E_i \cap E_i$ (for $n > 0$). We now restrict our attention to the bush $(E_i)$. We begin by showing that for $t \in E_i$,

$$g(t) \leq v(E_i^N)(t) \leq (1 + 2\beta) g(t).$$

(As if this means that on $E_i$ the images of disjoint elements of the bush $(E_i)$ are almost disjointly supported.)

To see the left-hand inequality in (10), let $E_i^N$ be a subset of $E_i$ such that $g_n(t) = |TE_i^N(t)|$ (see the definition of $g_n$). Then

$$g_n(t) = |TE_i^N(t)| = v_n(E_i^N)(t) \leq v(E_i)(t).$$

Since this is true for infinitely many values of $n$, the left-hand inequality of (10) follows.

For the right-hand inequality of (10), note that for $t \in F_i \subset E_i$,

$$v(E_i^N)(t) \leq v(E_i)(t) + \beta g(t)$$

(by (8))

$$= |TE_i^N(t)| + \beta g(t)$$

(since $E_i^N = E_i^N$)

$$\leq \beta g(t) + \beta g(t)$$

(by (7)).

This gives (10).

**Lemma 4.4.** For each $C$ in $\sigma(\mathcal{A})(E_i^N)$ let

$$\Phi(C) = \{ t \in E_i : |TC(t)| > (1 - \beta) v(E_i^N)(t) \}.$$
definition of $P_k = G_k$. Then

$$|TB_k(t)| \geq |TB_k(t)| - \sum_{t_k \leq t} |TB_k(t_k)| = 2|TB_k(t)| - \sum_{t_k \leq t} |TB_k(t_k)| = 2|TB_k(t)| - \sum_{t_k \leq t} v_n|TB_k(t_k)| = 2|TB_k(t)| - \sum_{t_k \leq t} v_n(t_k)$$

$$\geq (2 - 2\beta)\inf_{t_k \leq t} \|v(t_k) - v(B_k(t_k))\| \geq \frac{2}{1 + 2\beta} \inf_{t_k \leq t} \|v(t_k) - v(B_k(t_k))\| \text{ (by (10))}$$

$$= \frac{1 - 4\beta}{1 + 2\beta} \inf_{t_k \leq t} \|v(t_k) - v(B_k(t_k))\|.$$ 

Hence $t \in \mathcal{V}(B_k)$. So

$$\bigcup_{k=1}^m \mathcal{V}(B_k) = \mathcal{V}.$$ 

Note that if $G_k, B_k \in \mathcal{A}(B^*)$ and $|B_k \Delta C| \to 0$ as $k \to \infty$, then for some subsequence $k_i$, $\lim_{k_i \to \infty} \mathcal{V}(B_{k_i}) = \mathcal{V}$. Therefore, there exists a subsequence $G_{k_i} \to G$ a.e. Since $1 - 6\beta < \frac{1 + 4\beta}{1 + 2\beta}$, $\mathcal{V}(B_{k_i}) \to \mathcal{V}$ a.e. By the dominated convergence theorem

$$\int_{L^2} v(B_{k_i}) = \int_{L^2} v(B_{k_i}) - \mathcal{V} \to \int_{L^2} v(B_k) = \mathcal{V} \to 0.$$ 

We now show that if $G_1, \ldots, G_m \in \mathcal{A}(B_k)$ and $\bigcup_{i=1}^m G_i = B_1$, then

$$\bigcup_{i=1}^m \mathcal{V}(G_i) = \mathcal{V}.$$ 

Given $\eta > 0$, choose $B_k, B_{k_1}, \ldots, B_{k_m} \in \mathcal{A}(B_k)$ with

$$\bigcup_{i=1}^m B_{k_i} = B_1$$

and $\|v(B_{k_i}) - \mathcal{V}(G_i)\| < \eta$ a.e. Then

$$\int_{L^2} \left| \bigcup_{i=1}^m \mathcal{V}(G_i) \right| = \int_{L^2} \left| \bigcup_{i=1}^m \mathcal{V}(B_{k_i}) \right| = \int_{L^2} \left| \bigcup_{i=1}^m \left( v(B_{k_i}) - \mathcal{V}(G_i) \right) \right|$$

$$= \int_{L^2} \left| \bigcup_{i=1}^m \left( v(B_{k_i}) - \mathcal{V}(G_i) \right) \right|$$

$$\leq \int_{L^2} \sum_{i=1}^m \|v(B_{k_i}) - \mathcal{V}(G_i)\| < \eta.$$ 

Since $\eta$ was arbitrary, $\mathcal{V} = \bigcup_{i=1}^m \mathcal{V}(G_i)$ a.e.

This together with (13) shows that finite $\mathcal{A}(B_k)$-partitions of $B_1$ are sent by $\mathcal{V}$ to (essential) partitions of $\mathcal{V}$.

Next note that $\mathcal{V}(G) = \bigcup_{i=1}^m \mathcal{V}(G_i)$ if $m \geq 1$. Then $\mathcal{V}(G_i) \to 0$ in $\mathcal{V}$, and hence $\|v(G_i)\| \to 0$.

To finish the proof of the lemma, suppose $G_1, G_2, \ldots$ is a disjoint sequence in $\mathcal{A}(B_k)$. Then $\mathcal{V}(\bigcup_{i=1}^m G_i) \to 0$ as $m \to \infty$ so for some subsequence $m_k, X_{i=1}^{m_k} c_{i=1}^{m_k} 0 \rightarrow 0$. Then

$$X_{i=1}^{m_k} c_{i=1}^{m_k} 0 \rightarrow 0 \text{ a.e. Then}$$

$$X_{i=1}^{m_k} c_{i=1}^{m_k} 0 \rightarrow 0 \text{ a.e.}$$

So $\mathcal{V}(C) \to \mathcal{V}(C) \text{ a.e.},$ and $\mathcal{V}$ is a $\sigma$-homomorphism. 

We are now in a position to construct the trees $(A_1^*)$ and $(F_k^*)$ with the properties listed in the conclusion of Theorem 4.3.

Define a measure $\mu: \mathcal{A}(B_1^*) \to \mathbb{R}^2$ by

$$\mu(C) = \left( \int_{B_1^*} v(B_1^*), \int_{B_1^*} v(C) \right).$$

This measure is non-atomic since $|C|$, $|\mathcal{V}(C)|$, and $\|v(C)\| \to 0$ as $|C| \to 0$. Hence, by Liapunov's convexity theorem (see [7]), the range of $\mu$ is convex. Then

$$\int \mathcal{V}(A_1^*) = \int \mathcal{V}(A_1^*) = \int \mathcal{V}(A_1^*)$$

Thus we can find a tree $(A_1^*)$ with

$$A_1^* = B_1^*$$

and

$$\mu(A_1^*) = \frac{1}{2 \mathcal{V}(A_1^*)}.$$ 

i.e.,

$$\int \mathcal{V}(A_1^*) = \int \mathcal{V}(A_1^*) = \int \mathcal{V}(A_1^*)$$

and

$$\int \mathcal{V}(A_1^*) = \int \mathcal{V}(A_1^*) = \int \mathcal{V}(A_1^*)$$

Define $F_1^* = \mathcal{V}(A_2^*)$. (In particular, $F_1^* = F_0 = G_k^*$.) We now verify conclusions (b), (c), and (d) listed in the statement of Theorem 4.3. 

$$\int |\mathcal{V}(A_1^*)| \geq (1 - 0\beta) \int \mathcal{V}(A_1^*)$$

$$= (1 - 0\beta) \frac{1}{2 \mathcal{V}(A_1^*)} \int \mathcal{V}(A_1^*) \text{ (by 16)}$$
5. Corollaries of Theorem 4.2. We have proven Theorem 4.3, and hence have proven Theorem 4.2 and the direct implication in Theorem 4.1. Before proving the reverse implication in Theorem 4.1, we deduce some corollaries of Theorem 4.2. Recall first the result of Dor [5]:

**Proposition 5.1 (Dor).** If \( T : L^1 \to L^1 \) is an (into) isomorphism, then for any partition \( (E_i) \), \( i = 1, \ldots, M \), of a set \( E \) of positive measure, we have

\[
\frac{1}{\|T^{-1}\|^2} \leq \frac{1}{|E|} \int \max_i |T E_i|.
\]

**Corollary 5.2.** Let \( Z = L^1 \). If \( Z \) is a subspace of \( Z \) and \( X \sim L^1 \), then there exists a subspace \( Y \) of \( X \) with \( Y \sim L^1 \) and with \( Y \) complemented in \( Z \).

**Proof.** Let \( T : L^1 \to Z \) be any isomorphism onto \( X \). Proposition 5.1 implies that \( T \) is an \( E \)-operator. Theorem 4.2 gives the result.

**Corollary 5.3.** If a complemented subspace \( X \) of \( L^1 \) contains a subspace isomorphic to \( L^1 \), then \( X \sim L^1 \).

**Proof.** Apply Corollary 5.2 and Pelczynski's decomposition method [12].

Before stating the next corollary we make the following

**Remark.** If \( T_1 + T_2 \) is an \( E \)-operator, then either \( T_1 \) or \( T_2 \) must be an \( E \)-operator. For

\[
\int \max_i |(T_1 + T_2) E_i| \leq \int \max_i |T_1 E_i| + \int \max_i |T_2 E_i|
\]

for each \( s \) and for any bush \( (E_i) \). If neither \( T_1 \) nor \( T_2 \) is an \( E \)-operator, then as \( s \to \infty \) the limit of the right-hand side, and hence the left-hand side, of (17) is 0. Since this is true for all bush \( s \), \( T_1 + T_2 \) cannot be an \( E \)-operator. A continuous version of this observation is given in Proposition 7.2.

**Corollary 5.4.** \( L^1 \) is primary; i.e., if \( L^1 \sim X \oplus Y \), then either \( X \sim L^1 \) or \( Y \sim L^1 \) (or both).

**Proof.** Consider \( X \) and \( Y \) as complementary subspaces in \( L^1 \) with projections \( P \) and \( I - P \) onto \( X \) and \( Y \). Since these two operators sum to the identity operator, which is certainly an \( E \)-operator, one of them (let us say \( P \)) is an \( E \)-operator. Theorem 4.2 then asserts the existence of a complemented subspace of \( X \), (the range of \( P \)) isomorphic to \( L^1 \). Pelczynski's decomposition method [12] implies that \( X \sim L^1 \).

A generalization of Corollary 5.4 is

**Corollary 5.5.** Suppose \( L^1 \) is isomorphic to an unconditional sum of a sequence of Banach spaces \( X_i \). Then there is a \( j \) such that \( X_j \sim L^1 \).
Proof. By a result of Lindenstrauss and Pełczyński [8] the hypothesis implies that the $i$th sum of the spaces $X_i$ is isomorphic to $L_i$. Hence we may regard $X_i$ as a subspace of $L_i^*[1, 2^i, 1/2^i]$, $i = 1, 2, \ldots$; let $X$ denote the $i$th sum of these $X_i$, so $X \sim L_i^*[0, 1]$. We are given that $X \sim L_i$. Let $T_i : L_i \rightarrow L_i$ be an isomorphism from $L_i$ onto $X$. By Proposition 5.1, $T$ is an $E$-operator, i.e., for all $n$

$$
\int \max_{i \leq n} |TE_i^n| \geq \delta
$$

for some $\delta > 0$, where $E_i^n = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]$. Hence there is a $k$ such that

$$
\int_{[0, 1]} \max_{i \leq n} |TE_i^n| \geq \delta/2
$$

for all sufficiently large $n$ (since by Lemma 3.2 $\max_{i \leq n} |TE_i^n|$ converges in $L_i$ as $n \rightarrow \infty$). Thus the operator $B_{\sum \delta_i} \mathcal{F} \mathcal{G}$ is an $E$-operator. By Theorem 4.2 its range, which is $(\sum \delta_i X_i)_h$ (a complemented subspace of $X \sim L_i$) contains a complemented subspace isomorphic to $L_i$. Thus by Pełczyński's decomposition method (Corollary 5.4), there exists $f_j, 1 \leq j \leq h$, with $X_j \sim L_i$. 

6. The proof of Theorem 4.1. This section is devoted to the proof of the converse implication in Theorem 4.1, and to one simple corollary.

**Lemma 6.1.** Let $S : L_i \rightarrow L_i$ be a bounded linear operator. Then gives $f_1, \ldots, f_m, h, \ldots, h_m$ in $L_i$,

$$
\left| \int \max_i |Sf_i| - \int \max_i |Sh_j| \right| \leq \|S\| \sum_i |f_i - h_i|.
$$

Proof.

$$
|Sf_i| \leq |Sh_j| + |S(f_i - h_i)| \leq \max_i |Sh_j| + \sum_i |S(f_i - h_i)|.
$$

So

$$
\max_i |Sf_i| \leq \max_i |Sh_j| + \sum_i |S(f_i - h_i)|.
$$

Integration gives the result. 

The next lemma, a consequence of Lemma 6.1, shows that the class of $E$-operators is invariant under the natural isometries of $L_i$ determined by a change of sign and density.

$B_E$ denotes the operator on $L_i$ which restricts functions to the set $E$.

**Lemma 6.2.** Let $S : L_i \rightarrow L_i$ be a bounded linear operator. Suppose there is a function $f$ a $L_i$ and a bush $(F_i^n), i = 1, \ldots, M_i$; $n = 0, 1, \ldots$, such that

$$
\lim_{n \rightarrow \infty} \int \max_{i \leq n} |SR_{F_i^n}(f)| > 0.
$$

Then $S$ is an $E$-operator.

Proof. By approximating $f$ sufficiently closely by a step function

$$
g = \sum_{i=1}^{m} \delta_i Z_i
$$

we have by Lemma 6.1 that

$$
0 < \lim_{n \rightarrow \infty} \int \max_{i \leq n} |SR_{F_i^n}(\sum_{i=1}^{m} \delta_i Z_i)|
$$

$$
= \lim_{n \rightarrow \infty} \int \max_{i \leq n} \sum_{i=1}^{m} \delta_i |SR_{F_i^n}(Z_i)|.
$$

This shows that the operator $\sum_{i=1}^{m} \delta_i SR_{F_i^n}$ is an $E$-operator. Then there exists a $j$ with $SR_{F_j^n}$ an $E$-operator (see the Remark following Corollary 5.3). Indeed

$$
0 < \lim_{n \rightarrow \infty} \int \max_i |SR_{F_j^n}(f)| = \lim_{n \rightarrow \infty} \int \max_i |S(G_i \cap F_i^n)|,
$$

which shows that $S$ is an $E$-operator.

**Proof of Theorem 4.1.** The direct implication is given by Theorem 4.2. To prove the reverse, let $S : L_i \rightarrow L_i$ be a bounded linear operator and assume that there is a subspace $X$ isomorphic to $L_i$ with $S|_X$ an isomorphism. We may assume without loss of generality that $|S| = 1$.

Let $T : L_i \rightarrow L_i$ be an isomorphism of $L_i$ onto $X$ with $\|T\| = 1$. Then $ST : L_i \rightarrow L_i$ is an isomorphism, and by Dor's result (Proposition 5.1) there is a number $a > 0$ with

$$
\int \max_i |ST(B_i)| \geq a^i |B_i|
$$

for any finite disjoint collection $(B_i)$. Also by Dor's result $T$ is an $E$-operator. Choose $\delta > 0$ such that $T$ is of constant $\delta$, but not of constant $\delta + \frac{1}{2}(a^i/2)^i$. Let $\epsilon > 0$ be such that $\epsilon < a^i/8$ and

$$
(1 - \epsilon) \delta + \frac{a^i}{2} > \delta + \frac{1}{2} \left(\frac{a^i}{2}\right).
$$

Find trees $(A_i^n)$ and $(F_i^n)$ with properties (a), (b), (c) and (d) listed in Theorem 4.3.
We claim that

\[ \lim_{n \to \infty} \max_{\gamma \in \nu_\gamma} |RE_{\gamma}(TA^*_\gamma)| = \frac{d^2}{2} |A^*_\gamma|. \]

For if (19) fails, then by (18)

\[ \lim_{n \to \infty} \max_{1 \leq \gamma \leq n} |RE_{\gamma}(TA^*_\gamma)| > \frac{d^2}{2} |A^*_\gamma|. \]

Define \( V : L^1 \to L^1 \) by

\[ Vf = \lambda f \in A^*_\gamma \]

Statement (20) asserts that \( V \) is an \( E \)-operator of constant larger that \( d^2/2 \), so by Theorem 4.2 there is a non-atomic \( \sigma \)-ring \( \mathcal{A} \subseteq \sigma \mathcal{A}((A^*_\gamma)) \) such that

\[ |Vf| \geq \frac{d^2}{2} |f| \quad \text{for all } f \in L^1(\mathcal{A}). \]

Since \( |\mathcal{A}| = 1 \),

\[ |RE_{\gamma}(TA^*_\gamma)| \geq \frac{d^2}{2} |f| \]

for all \( f \in L^1(\mathcal{A}) \).

By Dör's Theorem (Proposition 5.1), for any finite disjoint collection

\[ \{B_i \} \subset \mathcal{A}, \]

\[ \int \max_{B_i} |TB_i| \geq \frac{d^2}{2} |\cup B_i|. \]

By property (d) in Theorem 4.3,

\[ \int \max_{B_i} |TB_i| \geq (1 - \delta) |\cup B_i|. \]

Addition of the last two inequalities would give that \( T \) is an \( E \)-operator of constant

\[ (1 - \delta) \delta + \left( \frac{d^2}{2} \right) > \frac{d^2}{2} \left( \frac{d^2}{2} \right) \]

which is a contradiction. Hence (19) is true.

Next we use Lemma 6.1 to show that

\[ \lim_{n \to \infty} \max_{1 \leq \gamma \leq n} |RE_{\gamma}(TA^*_\gamma)| \geq \frac{d^2}{4} |A^*_\gamma|. \]

For each \( n \) and \( i \),

\[ |R_{\gamma_i}(TA^*_\gamma) - R_{\gamma_i}(TA^*_\gamma)| \]

\[ = |R_{\gamma_i}(TA^*_\gamma) + \sum_{\gamma \in \nu} R_{\gamma_i}(TA^*_\gamma) - R_{\gamma_i}(TA^*_\gamma) - R_{\gamma_i}(TA^*_\gamma)| \]

\[ \leq \sum_{\gamma \in \nu} |R_{\gamma_i}(TA^*_\gamma)| + |R_{\gamma_i}(TA^*_\gamma)| \]

\[ \leq \epsilon |A^*_\gamma| + \epsilon \int |TA^*_\gamma| \quad \text{(by Theorem 4.3 (c), (b))} \]

\[ \leq 2 \epsilon |A^*_\gamma| \leq \frac{d^2}{4} |A^*_\gamma| \leq \frac{d^2}{4} |A^*_\gamma|. \]

Summation over \( i \) and application of Lemma 6.1 and (19) give (21).

Lemma 6.2 and (21) complete the proof that \( S \) is an \( E \)-operator. ■

**Corollary 6.3.** Let \( T : L^1 \to L^1 \) be a bounded linear operator. If there exists a subspace \( \mathcal{F} \) isomorphic to \( L^1 \) with \( T|_{\mathcal{F}} \) an isomorphism, then there exists a subspace \( \mathcal{G} \) isometric to \( L^1 \) with \( T|_{\mathcal{G}} \) an isomorphism and with \( \mathcal{G} \) complemented

**Proof.** Combine Theorems 4.1 and 4.2. ■

**7. Further propositions and open problems.** Our next proposition deals with the absolute value of an operator on \( L^1 \). If \( T : L^1 \to L^1 \) is a bounded linear operator, its absolute value \( |T| \) is the operator on \( L^1 \) defined for \( f \geq 0 \), \( f \in L^1 \), by

\[ |Tf|(t) = \sup \left\{ \sum_{i=1}^{m} |Tf_i(t)| : \sum_{i=1}^{m} f_i = f, f_i \geq 0 \right\} \]

for all \( t \in [0, 1] \), and defined for general \( f \in L^1 \) by linearity, writing \( f \) as the difference of two positive functions. It is a fact that \( |T| \) is bounded with norm \( |T| \). (See Chapter IV of [16] for a general discussion of \( |T|).)

**Proposition 7.1.** Let \( T : L^1 \to L^1 \) be a bounded linear operator. \( T \) is an \( E \)-operator if and only if \( |T| \) is an \( E \)-operator.

**Proof.** Suppose \( T \) is an \( E \)-operator. Since for any measurable set \( E \), \( \mathcal{F} \) is non-negative, we have \( |T|E > |TE| \). Hence any basis which shows \( T \) as an \( E \)-operator shows also that \( |T| \) is an \( E \)-operator.

Suppose now that \( |T| \) is an \( E \)-operator. From Theorem 4.3 (c) (using \( \varepsilon = \frac{1}{4} \)) there exist trees \( (A^*_\gamma) \) and \( (F^*_\gamma) \) with

\[ \int \frac{\sum_{i=1}^{n} |T|A^*_i}{n} \leq \frac{1}{4} \int |T|A^*_i \leq \frac{1}{4} \int |T|A^*_i. \]
Summation over $i$ and reversal of the order of summation gives

$$
\sum_{k=1}^{m} \sum_{i=1}^{n} |T_iA_k^\alpha| \leq \frac{1}{4} \sum_{i=1}^{n} |T_iA^\alpha|.
$$

From the definition of $|T_i|$, we can find functions $f_k \geq 0$, $1 \leq k \leq m$, with

$$
\sum_{k=1}^{m} f_k = A_i^\alpha
$$

and with

$$
\sum_{k=1}^{m} \frac{1}{\lambda} \int |T_i f_k| \geq \frac{1}{2} \int |T_i A^\alpha|.
$$

We shall show that there is a constant $C > 0$ such that

$$
\sum_{k=1}^{m} \frac{1}{\lambda} \int \max_{j \in \mathbb{Z}} |T_j A^\alpha| f_k \geq C
$$

for all $n$. This is enough to complete the proof, for if $T$ were not an $E$-operator, then for each $k$,

$$
\max_{j \in \mathbb{Z}} |T_j A^\alpha| f_k \to 0
$$

as $n \to \infty$, by Lemma 6.2.

We now show (24). $\sum_{j=1}^{n} |T_j A^\alpha| \geq |T_i f_k|$. Hence

$$
\sum_{j=1}^{n} \int |T_j A^\alpha| \geq \sum_{j=1}^{n} |T_i f_k| \geq \frac{1}{2} \int |T_i A^\alpha|
$$

by (23). Notice that by the definition of $|T_i|$,

$$
\sum_{j=1}^{n} |T_j A^\alpha| \leq |T_i A^\alpha|.
$$

So by (22),

$$
\sum_{j=1}^{n} \sum_{k=1}^{m} \int |T_j A^\alpha| f_k \leq \frac{1}{4} \sum_{i=1}^{n} |T_i A^\alpha|.
$$

Subtraction of (26) from (25) yields

$$
\sum_{i=1}^{n} \sum_{j=1}^{m} \int |T_j A^\alpha| f_k = \frac{1}{4} \sum_{i=1}^{n} |T_i A^\alpha| = C > 0.
$$

This implies (24).
Problem 1. Suppose a projection $P: L^1 \to L^1$ fails to have the Dunford–Pettis property. Must $P$ be an $\mathcal{N}$-operator?

H. P. Rosenthal [15] has constructed an example of an operator on $L^1$ which, when restricted to the span of the Rademacher functions, is the identity operator (and hence fails to have the Dunford–Pettis property), but which is not an $\mathcal{N}$-operator.

The second problem concerns a local version of the property which defines an $\mathcal{N}$-operator.

Problem 2. Let $T$ be an operator on $L^1$. Suppose there exists a constant $\delta > 0$ such that for each $n$, $$\sup_{\omega \in \omega_n} \max_{1 \leq i \leq n} |TE_i| \geq \delta$$

where the supremum is taken over all partitions $(E_i)_i$, $i = 1, \ldots, n$, of $[0, 1]$ with $|E_i| = 1/n$. Must $T$ be an $\mathcal{N}$-operator?

References


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