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Interpolation of 2^n Banach spaces

by

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Abstract. In this paper we introduce a theory of interpolation for several Banach spaces via a generalization of the J and K-methods of J. Peetre. The special feature is to consider 2^n spaces. This enable us to prove the equivalence between the generalized J and K-methods and then to obtain a reiteration theorem. An application to Lorentz spaces, with mixed norms, is made.

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0. Introduction. Let X and Y be two measure spaces with σ-finite measures μ and ν, respectively. We say that a complex valued $\mu\times\nu$-measurable function f on $X\times Y$ *belongs to* L^q (L^p) if

$$\big\| \|f(x,y)\|_{L^p(X)} \big\|_{L^q(Y)} = \Big\{ \int_Y \Big\{ \int_X |f(x,y)|^p \, d\mu \Big\}^{q/p} d\nu \Big\}^{1/q} < \infty,$$

that is, if the iterated L^p norm of f is finite.

If g is a complex valued function on L^1+L^∞, the Hardy transform of g is defined by

$$g^{**}(t) = t^{-1}\|g\|_{L^1+tL^\infty} = t^{-1}K(t, g; L^1, L^\infty),$$

(see Oklander [10] and Peetre [8]).

Now, if f is a $\mu\times\nu$-measurable function on $X\times Y$, it is natural to put

$$f^{**}(s,t) = t^{-1}\big\| s^{-1}\|f\|_{L^1+sL^\infty} \big\|_{L^1+tL^\infty} = t^{-1}s^{-1}\|f\|_{(L^1+tL^\infty)(L^1+sL^\infty)}.$$

After a formal multiplication, it is natural to, also, consider

$$f^{****}(s,t) = t^{-1}s^{-1}\|f\|_{L^1+sL^1(L^\infty)+tL^\infty(L^1)+stL^\infty}.$$

The function norms f^{**} and f^{****} are equivalent (see [7]).

These ideas enable us to introduce the following.

Let E_0, E_1, E_2, and E_3 be four Banach spaces continuously embedded in the same topological vector space. If $f \in E_0+E_1+E_2+E_3$ and $s>0$, $t>0$, we put

$$K(s,t;f) = \|f\|_{E_0+sE_1+tE_2+stE_3} = \inf_{\substack{f=\Sigma f_k\\ f_k\in E_k}} \{\|f_0\|_{E_0}+s\|f_1\|_{E_1}+t\|f_2\|_{E_2}+st\|f_3\|_{E_3},$$

and we say that

$$f \in (E_0, E_1, E_2, E_3)_{\Theta,Q;K},$$

where $\Theta = (\theta_1, \theta_2)$ and $Q = (q_1, q_2) \geqslant 1$, if and only if

$$s_1^{-\theta_1} s_2^{-\theta_2} K(s_1, s_2; f) \in L_*^{q_2}(L_*^{q_1}).$$

If $\Theta = (\theta, \theta)$, $Q = (q, q)$, $E_0 = E_1$, and $E_2 = E_3$, we will have

$$(E_1, E_2)_{\theta,q;K} = (E_1, E_1, E_2, E_2)_{\Theta,Q;K}$$

with equivalence of norms; that is, our spaces are a generalization of Peetre's interpolation spaces (see [8]).

We will study first the theory for 2^2 spaces and then extend the theory to 2^n spaces.

The interpolation of several Banach spaces was suggested by C. Foiaş and J. L. Lions [4]. The means method of Lions–Peetre for several Banach spaces was developed by Yoshikawa [11], and the complex method of Calderón by Favini [3]. The extension of Peetre's K-method was introduced by Sparr [9] and the author in two notes, [5] and [6]. We give a recursive definition for the K function norm and this enables us to obtain the equivalence theorem (with the J-method) and then the reiteration theorem. We give also an application to Lorentz spaces with mixed norm (see [7]).

1. Preliminaries. We consider 4-tuples $\boldsymbol{E} = (E_0, E_1, E_2, E_3)$ of Banach spaces E_0, E_1, E_2, and E_3 algebraic and continuously embedded in a some Hausdorff topological vector space V.

If $\boldsymbol{E} = (E_0, E_1, E_2, E_3)$ is a 4-tuple of Banach spaces, we say that an element $f \in V$ *belongs to*

$$\Sigma\boldsymbol{E} = E_0+E_1+E_2+E_3$$

if there exists $f_k \in E_k$, $k = 0, 1, 2, 3$, such that

$$f = f_0+f_1+f_2+f_3.$$

If the linear subspace $E_0 \cap E_1 \cap E_2 \cap E_3$ of V is different from $\{0\}$, it will be denoted by

$$\bigcap\boldsymbol{E}.$$

The spaces $\Sigma\boldsymbol{E}$ and $\bigcap\boldsymbol{E}$ are Banach spaces, continuously embedded in V, under the norms

$$\|f\|_{\Sigma\boldsymbol{E}} = \inf_{\substack{f=\Sigma f_k\\ f_k\in E_k}} \{\|f_0\|_{E_0}+\|f_1\|_{E_1}+\|f_2\|_{E_2}+\|f_3\|_{E_3}\}$$

and

$$\|f\|_{\cap\boldsymbol{E}} = \max\{\|f\|_{E_0}, \|f\|_{E_1}, \|f\|_{E_2}, \|f\|_{E_3}\},$$

respectively.

We will call an *intermediate space* (with respect to $\boldsymbol{E}$) a Banach space E such that

$$\bigcap\boldsymbol{E} \subset E \subset \Sigma\boldsymbol{E}.$$

(in the sequel $\subset$ will denote continuous embeddings).

Let $\boldsymbol{E} = (E_0, E_1, E_2, E_3)$ be a 4-tuple of Banach spaces and $\boldsymbol{t} = (t_1, t_2) > 0$ (that is, $t_1 > 0$ and $t_2 > 0$). If $f \in \Sigma\boldsymbol{E}$, we set

$$K(t_1, t_2; f) = K(\boldsymbol{t}; f) = K(t; f; \boldsymbol{E}) = \inf_{\substack{f=\Sigma f_k\\ f_k\in E_k}} \{\|f_0\|_{E_0}+t_1\|f_1\|_{E_1}+t_2\|f_2\|_{E_3}+t_1t_2\|f_3\|_{E_4}\},$$

where $f_k \in E_k$, $k = 0, 1, 2, 3$.

If $g \in \bigcap\overline{\boldsymbol{E}}$, we set

$$J(t_1, t_2; f) = J(\boldsymbol{t}; f) = J_{\boldsymbol{E}}(t; f) = \max\{\|f\|_{E_0}, t_1\|f\|_{E_1}, t_2\|f\|_{E_2}, t_1t_2\|f\|_{E_3}\}.$$

It is easy to see that $K(\boldsymbol{t}; f)$ and $J(\boldsymbol{t}; f)$ are *function norms* (they depend on the parameter $\boldsymbol{t}$) on $\Sigma\boldsymbol{E}$ and $\bigcap\boldsymbol{E}$, respectively.

The following inequalities will be useful later.

Let $f \in \Sigma\boldsymbol{E}$, $g \in \bigcap\boldsymbol{E}$, $\boldsymbol{s} = (s_1, s_2) > 0$ and $\boldsymbol{t} = (t_1, t_2) > 0$; then

$$K(t_1, t_2; f) \leqslant \max\{1, s_1t_1^{-1}, s_2t_2^{-2}, s_1s_2t_1^{-1}t_2^{-2}\}K(s_1, s_2; f),$$
$$\min(1, s_1t_1^{-1}, s_2t_2^{-1}, s_1s_2t_1^{-1}t_2^{-1})K(t_1, t_2; f) \leqslant K(s_1, s_2; f),$$
$$\min(1, t_1, t_2, t_1, t_2)\|f\|_{\Sigma\boldsymbol{E}} \leqslant K(t_1, t_2; f),$$
$$K(t_1, t_2; f) \leqslant \max\{1, t_1, t_2, t_1t_2\}\|f\|_{\Sigma\boldsymbol{E}},$$
$$\min\{1, t_1^{-1}, t_2^{-1}, t_1^{-1}t_2^{-1}\}K(t_1, t_2; f) \leqslant \|f\|_{\Sigma\boldsymbol{E}},$$
$$\|f\|_{\Sigma\overline{\boldsymbol{E}}} \leqslant \max\{1, t_1^{-1}, t_2^{-1}, t_1^{-1}t_2^{-1}\}K(\boldsymbol{t}; f),$$
$$\min\{1, t_1s_1^{-1}, t_2s_2^{-1}, s_1^{-1}s_2^{-1}t_1t_2\}J(s_1, s_2; g) \leqslant J(t_1, t_2; g),$$
$$K(t_1, t_2; g) \leqslant \min\{1, t_1s_1^{-1}, t_2s_2^{-1}, s_1^{-1}s_2^{-1}t_1t_2\}J(s_1, s_2; g),$$
$$\max\{1, s_1t_1^{-1}, s_2t_2^{-1}, s_1s_2t_1^{-1}t_2^{-1}\}K(t_1, t_2; g) \leqslant J(s_1, s_2; g).$$

The proof is straightforward.

Let E be a Banach space and $1 \leqslant Q = (q_1, q_2) \leqslant \infty$. The space $L_*^Q(E) = L_*^Q(R_+^2, E)$ is the space of E-valued strongly measurable functions on R_+^2, with respect to the Haar measure

$$\frac{dt}{t} = \frac{dt_1}{t_1}\frac{dt_2}{t_2},$$

and such that

$$\|g\|_{L_*^Q(E)} = \big\| \|g\|_{L_*^{q_1}(E)} \big\|_{L_*^{q_2}} < \infty;$$

when $E = \boldsymbol{R}$ or $\boldsymbol{C}$ we will write L_*^Q instead of $L_*^Q(E)$.

1.1. Proposition. *Suppose that one of the following conditions holds:*

(1.1.1) $0 < \theta_1 < 1, \quad 0 < \theta_2 < 1, \quad 1 \leqslant q_1, \quad q_2 \leqslant \infty;$

(1.1.2) $0 \leqslant \theta_1 \leqslant 1, \quad 0 < \theta_2 < 1, \quad q_1 = \infty, \quad 1 \leqslant q_2 \leqslant \infty;$

(1.1.3) $0 < \theta_1 < 1, \quad 0 \leqslant \theta_2 \leqslant 1, \quad 1 \leqslant q_1 \leqslant \infty, \quad q_2 = \infty;$

(1.1.4) $0 \leqslant \theta_1 \leqslant 1, \quad 0 \leqslant \theta_2 \leqslant 1, \quad q_1 = q_2 = \infty.$

Then

$$\|t_1^{-\theta_1} t_2^{-\theta_2} \min(1, t_1, t_2, t_1 t_2)\|_{L_*^Q} < \infty.$$

Proof. The proof follows from direct calculation.

2. The K-method and J-method. Let $0 \leqslant \boldsymbol{\Theta} = (\theta_1, \theta_2) \leqslant 1$ and $1 \leqslant Q = (q_1, q_2) \leqslant \infty$.

We define the space

$$\boldsymbol{E}_{\Theta,Q;K} = (E_0, E_1, E_2, E_3)_{\Theta,Q;K}$$

to be the space of all elements $f \in \Sigma \boldsymbol{E}$ for which

$$t_1^{-\theta_1} t_2^{-\theta_2} K(t_1, t_2; f) \in L_*^Q = L_*^{q_2}(L_*^{q_1}).$$

The spaces $\boldsymbol{E}_{\Theta,Q;K}$ are Banach spaces under the norms

$$\|f\|_{\Theta,Q;K} = \|\boldsymbol{t}^{-\Theta} K(\boldsymbol{t}; f)\|_{L_*^Q} = \big\| \|t_1^{-\theta_1} t_2^{-\theta_2} K(t_1, t_2; f)\|_{L_*^{q_1}} \big\|_{L_*^{q_2}}$$

when one of conditions (1.1.1), (1.1.2), (1.1.3) or (1.1.4) holds. In all other cases the space reduces to $\{0\}$.

We define the space

$$\boldsymbol{E}_{\Theta,Q;J} = (E_0, E_1, E_2, E_3)_{\Theta,Q;J}$$

to be the space of all elements $f \in \Sigma \boldsymbol{E}$ for which there exists a strongly measurable function $u = u(s_1, s_2)$ with values in $\bigcap \boldsymbol{E}$ such that

$$f = \int_0^\infty \int_0^\infty u(s_1, s_2) \frac{ds_1}{s_1}\frac{ds_2}{s_2}$$

and

$$s_1^{-\theta_1} s_2^{-\theta_2} J\big(s_1, s_2; u(s_1, s_2)\big) \in L_*^{q_2}(L_*^{q_1}).$$

The spaces $\boldsymbol{E}_{\Theta,Q;J}$ are meaningful when one of conditions (1.1.1), (1.1.2), (1.1.3) or (1.1.4) holds. In all other cases the space reduces to $\{0\}$.

They are Banach spaces under the norms

$$\|f\|_{\Theta,Q;J} = \inf\Big\{ \big\| s_1^{-\theta_1} s_2^{-\theta_2} J\big(s_1, s_2; u(s_1, s_2)\big) \big\|_{L_*^{q_2}(L_*^{q_1})} \Big\},$$

where the infimum is taken on all u such that

$$f = \int_0^\infty \int_0^\infty u(s_1, s_2) \frac{ds_1}{s_1}\frac{ds_2}{s_2}.$$

2.1. Proposition. *The spaces $\boldsymbol{E}_{\Theta,Q;K}$ and $\boldsymbol{E}_{\Theta,Q;J}$ are intermediate spaces with respect to $\boldsymbol{E} = (E_0, E_1, E_2, E_3)$, that is, we have*

(2.1.1) $$\bigcap \boldsymbol{E} \subset \boldsymbol{E}_{\Theta,Q;K} \subset \Sigma \boldsymbol{E}$$

and

(2.1.2) $$\bigcap \boldsymbol{E} \subset \boldsymbol{E}_{\Theta,Q;J} \subset \Sigma \boldsymbol{E}.$$

Proof. Let $f \in \bigcap \boldsymbol{E}$; then

$$\|f\|_{\Sigma \boldsymbol{E}} \leqslant \|f\|_{\cap \boldsymbol{E}}$$

and

$$K(t_1, t_2; f) \leqslant \min(1, t_1, t_2, t_1 t_2) \|f\|_{\Sigma \boldsymbol{E}} \leqslant \min(1, t_1, t_2, t_1 t_2) \|f\|_{\cap \boldsymbol{E}}.$$

This implies

$$\|f\|_{\Theta,Q;K} \leqslant \|t_1^{-\theta_1} t_2^{-\theta_2} \min(1, t_1, t_2, t_1 t_2)\|_{L_*^Q} \|f\|_{\cap \boldsymbol{E}},$$

that is, the first embedding in (2.1.1). The second embedding in (2.1.1) follows from the inequality

$$\|f\|_{\Sigma \boldsymbol{E}} \leqslant \{\|t_1^{-\theta_1} t_2^{-\theta_2} \min(1, t_1, t_2, t_1 t_2)\|_{L_*^Q}\}^{-1} \|f\|_{\Theta,Q;K}.$$

Now, let $f \in \boldsymbol{E}_{\Theta;Q;J}$, $\boldsymbol{s} = (s_1, s_2) > 0$ and $\boldsymbol{t} = (t_1, t_2) > 0$. Then by the Hölder inequality (see [2]) we have

$$\|f\|_{\Sigma \boldsymbol{E}} \leqslant \big\| \|u(s_1, s_2)\|_{\Sigma \boldsymbol{E}} \big\|_{L_*^1}$$
$$\leqslant \|\{\min(1, s_1^{-1}, s_2^{-1}, s_1^{-1} s_2^{-1})\} J(s_1, s_2; f)\|_{L_*^1}$$
$$\leqslant \|s_1^{\theta_1} s_2^{\theta_2} \min(1, s_1^{-1}, s_2^{-1}, s_1^{-1} s_2^{-1})\|_{L_*^{Q'}} \big\| s_1^{\theta_1} s_2^{\theta_2} J\big(s_1, s_2; u(s_1, s_2)\big) \big\|_{L_*^Q},$$

where $Q + Q' = QQ'$. But

$$\|s_1^{\theta_1} s_2^{\theta_2} \min(1, s_1^{-1}, s_2^{-1}, s_1^{-1} s_2^{-1})\|_{L_*^{Q'}} = \|s_1^{-\theta_1} s_2^{-\theta_2} \min(1, s_1, s_2, s_1 s_2)\|_{L_*^{Q'}} < \infty,$$

then

$$\|f\|_{\Sigma E} \leqslant C\|f\|_{E_{\Theta,Q;J}},$$

where C is a constant that depends only on Θ and Q.

To prove the continuity of the second embedding consider a non-negative function $\psi = \psi(t_1, t_2)$ such that

$$\|t_1^{-\theta_1} t_2^{-\theta_2} \psi(t_1, t_2)\|_{L_*^Q} = 1.$$

Putting

$$u(t_1, t_2) = \frac{\psi(t_1, t_2)\min(1, t_1^{-1}, t_2^{-1}, t_1^{-1}t_2^{-1})}{\|\psi(s_1, s_2)\min(1, s_1^{-1}, s_2^{-1}, s_1^{-1}s_2^{-1})\|_{L_*^1}} f,$$

we have

$$\|\psi(s_1, s_2)\min(1, s_1^{-1}, s_2^{-1}, s_1^{-1}s_2^{-1})\|_{L_*^1} \|f\|_{\Theta,Q;J}$$

$$\leqslant \|\psi(s_1, s_2)\min(1, s_1^{-1}, s_2^{-1}, s_1^{-1}s_2^{-1})\|_{L_*^1} \big\|t_1^{-\theta_1}t_2^{-\theta_2} J(t_1, t_2; u(t_1, t_2))\big\|_{L_*^Q}$$

$$\leqslant \|t^{-\theta_1}t^{-\theta_2}\psi(t_1, t_2)\min(1, t_1^{-1}, t_2^{-1}, t_1^{-1}t_2^{-1}) J(t_1, t_2; f)\|_{L_*^Q}$$

$$\leqslant \|t_1^{-\theta_1}t_2^{-\theta_2}\psi(t_1, t_2)\|_{L_*^Q} \|f\|_{\cap E}$$

$$\leqslant \|f\|_{\cap E}.$$

Now, taking the supremum with respect to ψ, we get

$$\|f\|_{\Theta,Q;J} \leqslant \{\|t_1^{\theta_1}t_2^{\theta_2}\min(1, s_1^{-1}, s_2^{-1}, s_1^{-1}s_2^{-1})\|_{L_*^{Q'}}\}^{-1}\|f\|_{\cap E},$$

and this inequality completes the proof of (2.1.2).

We have also the following

2.2. Proposition. *Let $f \in E_{\Theta,Q;K}$. Then*

$$K(s_1, s_2; f) \leqslant s_1^{\theta_1}s_2^{\theta_2}\{\|t_1^{-\theta_1}t_2^{-\theta_2}\min(1, t_1, t_2, t_1t_2)\|_{L_*^{Q'}}\}^{-1}\|f\|_{\Theta,Q;K}.$$

Proof. We have

$$\min(1, t_1s_1^{-1}, t_2s_2^{-1}, t_1t_2s_1^{-1}s_2^{-1})K(s_1, s_2; f) \leqslant K(t_1, t_2; f).$$

Then

$$\|t_1^{\theta_1}t_2^{\theta_2}\min(1, t_1s_1^{-1}, t_2s_2^{-1}, t_1t_2s_1^{-1}s_2^{-1})\|_{L_*^{Q}} K(s_1, s_2; f) \leqslant \|f\|_{\Theta,Q;K}.$$

Now, a change of variables gives the result.

3. On the equivalence of the K and J-methods. We start with the following theorem.

3.1. Proposition. *For $0 < \Theta = (\theta_1, \theta_2) < 1$, $1 \leqslant P = (p_1, p_2) \leqslant Q = (q_1, q_2) \leqslant \infty$,*

$$E_{\Theta,P;J} \subset E_{\Theta,Q;K}. \tag{3.1.1}$$

Proof. If $f \in E_{\Theta,P;J}$, we have

$$K(t_1, t_2; f) \leqslant \int_0^\infty\int_0^\infty K\big(t_1, t_2; u(s_1, s_2)\big)\frac{ds_1}{s_1}\frac{ds_2}{s_2}$$

$$\leqslant \int_0^\infty\int_0^\infty \min(1, t_1s_1^{-1}, t_2s_2^{-1}, t_1s_1^{-1}t_2s_2^{-1})\ J\big(s_1, s_2; u(s_1, s_2)\big)\frac{ds_1}{s_1}\frac{ds_2}{s_2}$$

and

$$t_1^{-\theta_1}t_2^{-\theta_2}K(t_1, t_2; f) \leqslant t_1^{-\theta_1}t_2^{-\theta_2}\int_0^\infty\int_0^\infty \min(\ldots)J(\ldots)\frac{ds_1}{s_1}\frac{ds_2}{s_2}.$$

Now, putting

$$\min(1, \boldsymbol{t}, \boldsymbol{s}) = \min(1, s_1t_1, s_2t_2, s_1t_1s_2t_2),$$

by Young's inequality (see [1]) we obtain

$$\|\boldsymbol{t}^{-\theta}K(\boldsymbol{t}; f)\|_{L_*^Q} \leqslant \Big\|\big\|\boldsymbol{s}^{-\theta}\boldsymbol{t}^{-\theta}\min(1, \boldsymbol{t}, \boldsymbol{s}^{-1})\boldsymbol{s}^{\theta}J(\boldsymbol{s}; u(\boldsymbol{s}))\big\|_{L_*^1}\Big\|_{L_*^Q}$$

$$\leqslant \|\boldsymbol{t}^{-\theta}m(1, 1, \boldsymbol{t}^{-1})\|_{L_*^R}\|\boldsymbol{t}^{-\theta}J(\boldsymbol{t}; u(\boldsymbol{t}))\|_{L_*^P},$$

where $1/R = 1-(1/P-1/Q)$, for all representations.

$$f = \int_0^\infty\int_0^\infty u(t_1, t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

Thus

$$\|f\|_{\Theta,Q;K} \leqslant \|\boldsymbol{t}^{-\theta}\min(1, 1, \boldsymbol{t})\|_{L_*^R}\|f\|_{\Theta,P;J}$$

which proves the proposition.

In the next proposition we show that inclusion (3.1.1) may be reversed. For this purpose we need the following non-trivial extension of Peetre's lemma ([8]).

3.2. Lemma. *Let $f \in \Sigma E$, for which there exist constants*

$$0 < \Theta = (\theta_1, \theta_2) < 1 \quad \textit{and} \quad C = C(f),$$

such that

$$K(\boldsymbol{s}; f) \leqslant C(f)\boldsymbol{s}^{\Theta}. \tag{3.2.1}$$

Then there exists a strongly measurable function $u = u(s_1, s_2)$ in $\cap E$ satisfying

$$f = \int_0^\infty\int_0^\infty u(s_1, s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2} \tag{3.2.2}$$

and

$$J\big(s_1, s_2; u(s_1, s_2)\big) \leqslant (4e)^2 K(s_1, s_2; f). \tag{3.2.3}$$

Proof. Let $\boldsymbol{E} = (E_0, E_1, E_2, E_3)$ and $E_{01} = E_0 + E_1$ and $E_{23} = E_2 + E_3$ with the norms

$$\|\cdot\|_{01} = \|\cdot\|_{E_0+E_1} \quad \text{and} \quad \|\cdot\|_{23} = \|\cdot\|_{E_2+E_3},$$

respectively. We will have

$$K_{01}(s_1; f_{01}) + s_2 K_{23}(s_1; f_{23}) \leqslant K(s_1, s_2; f),$$

where $f = f_{01} + f_{23}$ with $f_{01} \in E_{01}$ and $f_{23} \in E_{23}$. Now, taking $s_2 = 1$ and using (3.2.1), we have

$$K_{01}(s_1; f_{01}) \leqslant C s_1^{\theta_1} \quad \text{and} \quad K_{23}(s_1, f_{23}) \leqslant C s_1^{\theta_1}.$$

Now, by Peetre's lemma there exist strongly measurable functions $u = u(s_1) \in E_0 \cap E_1$ and $v = v(s_1) \in E_2 \cap E_3$ such that

$$f_{01} = \int_0^\infty u(s_1) \frac{ds_1}{s_1} \quad (\text{in } E_0 + E_1)$$

and

$$f_{23} = \int_0^\infty v(s_1) \frac{ds_1}{s_1} \quad (\text{in } E_2 + E_3).$$

Furthermore,

$$J_{01}\big(s_1; u(s_1)\big) \leqslant 4e K_{01}(s_1; f_{01})$$

and

$$J_{23}\big(s_1; u(s_1)\big) \leqslant 4e K_{23}(s_1; f_{23}).$$

Putting $w = u + v$, we see that there exists a strongly measurable function $w = w(s_1)$ in $E_0 \cap E_1 + E_2 \cap E_3$ such that

$$f = \int_0^\infty w(s_1) \frac{ds_1}{s_1} \quad (\text{in } \Sigma \boldsymbol{E}).$$

Now, putting $E^{01} = E_0 \cap E_1$ and $E^{23} = E_2 \cap E_3$ under norms

$$\|\cdot\|^{01} = \|_{E_0 \cap s E_1} \quad \text{and} \quad \|\cdot\|^{23} = \|\cdot\|_{E_2 \cap s E_3},$$

we have

$$\begin{aligned} K\big(s_2, w(s_1)\big) &\leqslant J_{01}\big(s_1; u(s_1)\big) + s_2 J_{23}\big(s_1; v(s_1)\big) \\ &\leqslant 4e\{K_{01}(s_1; f_{01}) + s_2 K(s_1; f_{23})\} \\ &\leqslant 4e K(s_1, s_2; f). \end{aligned}$$

Fixing s_1, we have

$$K\big(s_2; w(s_1)\big) \leqslant C'(f) s_2^{\theta_2}.$$

Again, by Peetre's lemma, there exists a strongly measurable function $u = u(s_1, s_2)$ in $E^{01} \cap E^{23}$ such that

$$w(s_1) = \int_0^\infty u(s_1, s_2) \frac{ds_2}{s_2}$$

and

$$J\big(s_2; u(s_1, s_2)\big) \leqslant 4e K\big(s_2; w(s_1)\big).$$

Now we have

$$\begin{aligned} J\big(s_1, s_2; u(s_1, s_2)\big) &= J\big(s_2; u(s_1, s_2)\big) \\ &\leqslant 4e K\big(s_2; w(s_1)\big) \\ &\leqslant (4e)^2 K(s_1, s_2; f) \end{aligned}$$

and

$$f = \int_0^\infty \int_0^\infty u(s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2}.$$

3.3. Proposition. *Let* $0 < \Theta = (\theta_1, \theta_2) < 1$ *and* $1 \leqslant Q = (q_1, q_2) \leqslant \infty$. *Then*

$$\boldsymbol{E}_{\Theta,Q;K} \subset \boldsymbol{E}_{\Theta,Q;J}.$$

Proof. Taking $f \in \boldsymbol{E}_{\Theta,Q;K}$ and $\boldsymbol{t} = (t_1, t_2) \leqslant \boldsymbol{s} = (s_1, s_2)$, we have

$$K(t_1, t_2; f) \leqslant K(s_1, s_2; f).$$

If $E(t) = E(t_1, t_2) = (t_1, \infty) \times (t_2, \infty)$, then

$$K(\boldsymbol{t}; f) \|\boldsymbol{s}^{-\Theta} \chi_{E(\boldsymbol{t})}(\boldsymbol{s})\|_{L_*^Q} \leqslant \|\boldsymbol{s}^{-\Theta} K(\boldsymbol{s}; f) \chi_{E(\boldsymbol{t})}(\boldsymbol{s})\|_{L_*^Q},$$

and this implies

$$K(\boldsymbol{t}; f) \leqslant \boldsymbol{t}^\Theta \|f\|_{\Theta,Q;K}.$$

Now, by Lemma 3.2, there exists a strongly measurable function $u = u(\boldsymbol{t}) = u(t_1, t_2)$ in $\bigcap \boldsymbol{E}$ such that

$$f = \int_0^\infty \int_0^\infty u(t_1, t_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2},$$

$$J\big(t_1, t_2; u(t_1, t_2)\big) \leqslant (4e)^2 K(t_1, t_2; f).$$

Hence

$$\|t^{-\theta}J(t;u(t))\|_{L_*^Q} \leqslant (4e)^2 \|t^{-\theta}K(t;f)\|_{L_*^Q}$$

and we see also that

$$\|u\|_{L_*^1(\Sigma E)} \leqslant \|t^{-\theta}m(1,t)\|_{L_*^{Q'}} \|t^{-\theta}J(t;u(t))\|_{L_*^Q} < \infty,$$

where $1/Q+1/Q'=1$ and $m(1,t)=\min(1,t_1,t_2,t_1t_2)$. This completes the proof.

Combining the results of the latter two propositions, we obtain the following theorem.

3.4. Theorem. *For the intermediate spaces* $E_{\Theta,Q;K}$ *and* $E_{\Theta,Q;J}$; *we have*

(3.4.1) $$E_{\Theta,Q;K} = E_{\Theta,Q;J}$$

with equivalence of norms.

3.5. Corollary. *For* $0<\Theta<1$ *and* $1\leqslant P\leqslant Q$ *we have*

$$E_{\Theta,P;K} \subset E_{\Theta,Q;K},$$

in particular,

$$E_{\Theta,1,K} \subset E_{\Theta,Q;K} \subset E_{\Theta,\infty,K}$$

(here $\mathbf{1}=(1,1)$ and $\infty=(\infty,\infty)$).

4. Reiteration theorems.

4.1. Definition. We say that an intermediate space E of $\boldsymbol{E} = (E_0, E_1, E_2, E_3)$ *belongs to*

(i) *the class* $K(\Theta,\boldsymbol{E})$, $0\leqslant\Theta=(\theta_1,\theta_2)\leqslant 1$ if

(4.1.1) $$K(t_1,t_2;f) \leqslant C_1 t_1^{\theta_1} t_2^{\theta_2} \|f\|_E \quad (f\in E),$$

(ii) *the class* $J(\Theta,\boldsymbol{E})$, $0\leqslant\Theta=(\theta_1,\theta_2)\leqslant 1$ if

$$\|f\|_E \leqslant C_2 t_1^{-\theta_1} t_2^{-\theta_2} J(t_1,t_2;f) \quad (f\in \bigcap \boldsymbol{E})$$

(iii) *the class* $H(\Theta,\boldsymbol{E})$, $0\leqslant\Theta=(\theta_1,\theta_2)\leqslant 1$, if it belongs to the class $K(\Theta,\boldsymbol{E})$ as well as to $J(\Theta,\boldsymbol{E})$.

The next proposition gives necessary and sufficient conditions for an intermediate space E of $\boldsymbol{E}$ to belong to one of the class defined above.

4.2. Proposition. *An intermediate space* E *of* $\boldsymbol{E}$ *belongs to*

(a) $K(\Theta;\boldsymbol{E})$, $0\leqslant\Theta\leqslant 1$, *if and only if*

(4.2.1) $$E \subset E_{\Theta,\infty;K},$$

(b) $J(\Theta;\boldsymbol{E})$, $0\leqslant\Theta\leqslant 1$, *if and only if*

(4.2.2) $$E_{\Theta,1;J} \subset E,$$

(c) $H(\Theta;\boldsymbol{E})$, $0\leqslant\Theta\leqslant 1$, *if and only if*

(4.2.3) $$E_{\Theta,1;J} \subset E \subset E_{\Theta,\infty;K}.$$

Proof. Part (a) follows by the definition of $E_{\Theta,\infty;K}$ and $K(\Theta;\boldsymbol{E})$.

To prove part (b), we first show that if $E\in J(\Theta;\boldsymbol{E})$, then $E_{\Theta,1;J}\subset E$. Indeed, if $f\in E_{\Theta,1;J}$, then there is a strongly measurable function $u=u(t_1,t_2)$ in $\bigcap\boldsymbol{E}$ such that

(4.2.4) $$f = \int_0^\infty\int_0^\infty u(t_1,t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

and

$$t_1^{-\theta_1}t_2^{-\theta_2}J(t_1,t_2;u(t_1,t_2)) \in L_*^1.$$

Hence

$$\|f\|_E \leqslant \int_0^\infty\int_0^\infty \|u(t_1,t_2)\|_E \frac{dt_1}{t_1}\frac{dt_2}{t_2} \leqslant C\int_0^\infty\int_0^\infty t_1^{-\theta_1}t_2^{-\theta_2}J(t_1,t_2;u(t_1,t_2))\frac{dt_1}{t_1}\frac{dt_2}{t_2},$$

and this holds for all $u=u(t_1,t_2)$ that satisfy (4.2.4), and thus

(4.2.5) $$\|f\|_E \leqslant C\|f\|_{\Theta,1;J}.$$

Now, assume (4.2.5). For $n=1,2,\ldots$, and $t=(t_1,t_2)>0$ set

$$\psi_n(s_1,s_2) = \begin{cases} n^2 & \text{if } t_i e^{1/n} \leqslant s_i < t_i,\ i=1,2; \\ 0 & \text{otherwise.}\end{cases}$$

For each $f\in\bigcap\boldsymbol{E}$, let $u_n(s_1,s_2)=\psi_n(s_1,s_2)f$. Thus

$$f = \int_0^\infty\int_0^\infty u_n(s_1,s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}$$

and

$$\|f\|_{\Theta,1;J} \leqslant \int_0^\infty\int_0^\infty s_1^{-\theta_1}s_2^{-\theta_2}J(s_1,s_2;u_n(s_1,s_2))\frac{ds_1}{s_1}\frac{ds_2}{s_2} \leqslant n^2\int_{t_2e^{1/n}}^{t_2}\int_{t_1e^{1/n}}^{t_1} s_1^{-\theta_1}s_2^{-\theta_2}J(s_1,s_2;f)\frac{ds_1}{s_1}\frac{ds_2}{s_2} \leqslant t_1^{-\theta_1}t_2^{-\theta_2}J(t_1,t_2;f).$$

Hence

$$\|f\|_E \leqslant \|f\|_{\Theta,1;J} \leqslant C\, t_1^{-\theta_1} t_2^{-\theta_2} J(t_1, t_2; f)$$

proving part (b).

Parts (a) and (b) give part (c).

4.3. PROPOSITION. *Let* $\Theta_0 = (\theta_0^1, \theta_0^2)$, $\Theta_1 = (\theta_2^1, \theta_0^2)$, $\Theta_2 = (\theta_0^1, \theta_2^2)$ *and* $\Theta_3 = (\theta_2^1, \theta_2^2)$ *such that* $0 < \theta_0^1 < \theta_2^1 < 1$ *and* $0 < \theta_0^2 < \theta_2^2 < 1$. *Put* $\Theta = (\theta^1, \theta^2)$ *where* $\theta^1 = (1-\lambda^1)\theta_0^1 + \lambda^1\theta_2^1$ *and* $\theta^2 = (1-\lambda^2)\theta_0^2 + \lambda^2\theta_2^2$, *for* $0 < \lambda = (\lambda^1, \lambda^2) < 1$. *Let* $F_\varkappa$ *be an intermediate space that belongs to* $K(\Theta_\varkappa; E)$, $\varkappa = 0, 1, 2, 3$, *and* $F = (F_0, F_1, F_2, F_3)$. *Then*

$$F_{\lambda,Q;K_F} \subset E_{\Theta,Q;K_E}.$$

Proof. Let $f = f_0+f_1+f_2+f_3 \in F_{\lambda,Q;K}$ and $t = (t_1, t_2) > 0$. Then

$$K(t; f_0) \leqslant C_0 t^{\Theta_0}\|f_0\|_{F_0} = C_0 t_0^{\theta_0^1} t_2^{\theta_0^2}\|f_0\|_{F_0},$$
$$K(t; f_1) \leqslant C_1 t^{\Theta_1}\|f_1\|_{F_1} = C_1 t_1^{\theta_1^1} t_2^{\theta_0^2}\|f_1\|_{F_1},$$
$$K(t; f_2) \leqslant C_2 t^{\Theta_2}\|f_2\|_{F_2} = C_2 t_1^{\theta_0^1} t_2^{\theta_2^2}\|f_2\|_{F_2},$$
$$K(t; f_3) \leqslant C_3 t^{\Theta_3}\|f_3\|_{F_3} = C_3 t_1^{\theta_1^1} t_2^{\theta_1^2}\|f_3\|_{F_3}.$$

Since $t^{\Theta_1-\Theta_0} = t_1^{\theta_2^1-\theta_0^1}$, $t^{\Theta_2-\Theta_0} = t_2^{\theta_2^2-\theta_0^2}$ and $t^{\Theta_3-\Theta_0} = t_1^{\theta_2^1-\theta_0^1} t_2^{\theta_2^2-\theta_0^2}$, we have

$$\begin{aligned} K(t; f) &\leqslant K(t; f_0) + K(t; f_1) + K(t; f_2) + K(t; f_3) \\ &\leqslant C_0 t^{\Theta_0}\|f_0\|_{F_0} + C_1 t^{\Theta_1}\|f_1\|_{F_1} + C_2 t^{\Theta_2}\|f_2\|_{F_2} + C_3 t^{\Theta_3}\|f_3\|_{F_3} \\ &\leqslant C t_1^{\theta_0^1} t_2^{\theta_0^2}\{\|f_0\|_{F_0} + t_1^{\theta_2^1-\theta_0^1}\|f_1\|_{F_1} + t_2^{\theta_2^2-\theta_0^2}\|f_2\|_{F_2} + \\ &\qquad + t_1^{\theta_2^1-\theta_0^1} t_2^{\theta_2^2-\theta_0^2}\|f_3\|_{F_3}\}, \end{aligned}$$

where

$$C = \max\{C_0, C_1, C_2, C_3\}.$$

Hence

$$K(t_1, t_2; f) \leqslant C t_1^{\theta_0^1} t_2^{\theta_0^2} K(t_1^{\theta_2^1-\theta_0^1}, t_2^{\theta_2^2-\theta_0^2}; f)$$

and

$$t_1^{-\theta^1} t_2^{-\theta^2} K(t_1, t_2; f) \leqslant C t_1^{\theta_0^1-\theta^1} t_2^{\theta_0^2-\theta^2} K(t_1^{\theta_2^1-\theta_0^1}, t_2^{\theta_2^2-\theta_0^2}; f).$$

Putting

$$s_1 = t_1^{\theta_2^1\theta_0^1} \quad \text{and} \quad s_2 = t_2^{\theta_2^2\theta_0^2}$$

yields

$$\|f\|_{\Theta,Q;K} \leqslant C(\theta_2^1-\theta_0^1)^{1/q_1}(\theta_2^2-\theta_0^2)^{1/q_2}\left\|s_1^{\frac{\theta_0^1-\theta^1}{\theta_2^1-\theta_0^1}} s_2^{\frac{\theta_0^2-\theta^2}{\theta_2^2-\theta_0^2}} K(s_1, s_2; f)\right\|_{L_*^Q}.$$

For $0 < \lambda = (\lambda^1, \lambda^2) < 1$, define

$$\theta^1 = (1-\lambda^1)\theta_0^1 + \lambda^1\theta_2^1 \quad \text{and} \quad \theta^2 = (1-\lambda^2)\theta_0^2 + \lambda^2\theta_2^2.$$

Then

$$\|f\|_{\Theta,Q;K} \leqslant C^1\|s_1^{\lambda^1} s_2^{\lambda^2} K(s_1, s_2; f)\|_{L_*^Q} \leqslant C^1\|f\|_{\lambda,Q;K}.$$

4.4. PROPOSITION. *Let* $\Theta_0 = (\theta_0^1, \theta_0^2)$, $\Theta_1 = (\theta_2^1, \theta_0^2)$, $\Theta_2 = (\theta_0^1, \theta_2^2)$ *and* $\Theta_3 = (\theta_2^1, \theta_2^2)$ *such that* $0 \leqslant \theta_0^1 < \theta_2^1 \leqslant 1$ *and* $0 \leqslant \theta_0^2 < \theta_2^2 \leqslant 1$. *Put* $\Theta = (\theta^1, \theta^2)$ *where* $\Theta^1 = (1-\lambda^1)\theta_0^1 + \lambda^1\theta_2^1$ *and* $\Theta^2 = (1-\lambda^2)\theta_0^2 + \lambda^2 + \theta_2^2$. *For* $0 < \lambda = (\lambda^1, \lambda^2) < 1$. *Let* $F_\varkappa$ *be an intermediate space that belongs to* $J(\Theta_\varkappa; E)$, $\varkappa = 0, 1, 2, 3$, *and* $F = (F_0, F_1, F_2, F_3)$. *Then*

$$E_{\Theta,Q;J} \subset F_{\lambda,Q;J}.$$

Proof. Let $f \in E_{\Theta,Q;J}$. Then there exists $u = u(s_1, s_2) \in \bigcap E$ such that

$$f = \int_0^\infty\int_0^\infty u(s_1, s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}$$

and

$$s_1^{-\theta_1} s_2^{-\theta_2} J(s_1, s_2; u(s_1, s_2)) \in L_*^Q.$$

Since $F_\varkappa \in J(\theta_\varkappa, E)$, we have

$$\|u(s)\|_{F_0} \leqslant C_0 t^{\Theta_0} J(t, u(s)) = C_0 t_1^{\theta_0^1} t_2^{\theta_0^2} J(t_1, t_2; u(s_1, s_2)),$$
$$\|u(s)\|_{F_1} \leqslant C_1 t^{\Theta_1} J(t, u(s)) = C_1 t_1^{\theta_2^1} t_2^{\theta_0^2} J(t_1, t_2; u(s_1, s_2)),$$
$$\|u(s)\|_{F_2} \leqslant C_2 t^{\Theta_2} J(t, u(s)) = C_2 t_1^{\theta_0^1} t_2^{\theta_2^2} J(t_1, t_2; u(s_1, s_2)),$$
$$\|u(s)\|_{F_3} < C_3 t^{\Theta_3} J(t, u(s)) = C_3 t_1^{\theta_2^1} t_2^{\theta_2^2} J(t_1, t_2; u(s_1, s_2)),$$

and since

$$t^{\Theta_0-\Theta_1} = t_1^{\theta_0^1-\theta_2^1}, \quad t^{\Theta_0-\Theta_2} = t_2^{\theta_0^2-\theta_2^2}, \quad t^{\Theta_0-\Theta_3} = t_1^{\theta_0^1-\theta_2^1} t_2^{\theta_0^2-\theta_2^2},$$

we have also

$$\begin{aligned} J'(s, u(s)) &= \max\{\|u(s)\|_{F_0}, s_1\|u(s)\|_{F_1}, s_2\|u(s)\|_{F_2}, s_1 s_2\|u(s)\|_{F_3}\} \\ &\leqslant \max\{C_0 t^{\Theta_0}, s_1 C_1 t^{\Theta_1}, s_2 C_2 t^{\Theta_2}, s_1 s_2 C_3 t^{\Theta_3}\} J(t; u(s)) \\ &\leqslant C t^{\Theta_0}\max\{1, s_1 t^{\Theta_0-\Theta_1}, s_2 t^{\Theta_0-\Theta_2}, s_1 s_2 t^{\Theta_0-\Theta_3}\} J(t, u(s)) \\ &\leqslant C t^{\Theta}\max\{1, s_1 t_1^{\theta_0^1-\theta_2^1}, s_2 t_2^{\theta_0^2-\theta_2^2}, s_1 s_2 t_1^{\theta_0^1-\theta_2^1} t_2^{\theta_0^2-\theta_2^2}\} J(t, u(s)), \end{aligned}$$

where $C = \max(C_0, C_1, C_2, C_3)$.

Let

$$s_1 = t_1^{\theta_2^1-\theta_0^1}, \quad s_2 = t_2^{\theta_2^2-\theta_0^2};$$

then

$$J'(s; u(s)) \leqslant C\, s_1^{\frac{-\theta_0^1}{\theta_2^1-\theta_0^1}} s_2^{\frac{-\theta_0^2}{\theta_2^2-\theta_0^2}} J\big(s_1^{\frac{1}{(\theta_2^1-\theta_0^1)}}, s_2^{\frac{1}{(\theta_2^2-\theta_0^2)}}; u(s_1, s_2)\big)$$

and putting

$$\Theta = (\theta^1, \theta^2) = \big((1-\lambda^1)\theta_0^1+\lambda^1\theta_2^1, (1-\lambda^2)\theta_0^2+\lambda^2\theta_2^2\big) = (1-\lambda)\Theta_0+\lambda\Theta_3$$

yields

$$s^\lambda J'(S; u(s)) \leqslant C s_1^{\frac{\theta^1}{\theta_2^1-\theta^1}} s_2^{\frac{\theta^2}{\theta_2^2-\theta^2}} J\big(s_1^{\frac{1}{\theta_2^1-\theta_0^1}}, s_2^{\frac{1}{\theta_2^2-\theta_0^2}}; u(s_1, s_2)\big),$$

and

$$\|s^\lambda J'(s, u(s))\|_{L_*^Q} \leqslant C \|s_1^{-\theta^1} s_2^{-\theta^2} J(s_1, s_2; u(s_1^{\theta_2^1-\theta_0^1}, s_2^{\theta_2^2-\theta_0^2}))\|_{L_*^Q}.$$

But

$$\frac{f}{(\theta_2^1-\theta_0^1)(\theta_2^2-\theta_0^2)} = \frac{1}{(\theta_2^1-\theta_0^1)(\theta_2^2-\theta_0^2)} \int_0^\infty\int_0^\infty u(s_1, s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}$$
$$= \int_0^\infty\int_0^\infty u(s_1^{\theta_2^1-\theta_0^1}, s_2^{\theta_2^2-\theta_0^2})\frac{ds_1}{s_1}\frac{ds_2}{s_2},$$

and this gives finally

$$\|f\|_{\lambda,Q,J'} \leqslant C\|f\|_{\Theta,Q,J'}.$$

As a corollary of equivalence theorem and Propositions 4.3 and 4.4 we have the main result of this section: the so-called *reiteration theorem*.

4.5. Theorem. *Let $\Theta_0 = (\theta_0^1, \theta_0^2)$, $\Theta_1 = (\theta_2^1, \theta_0^2)$, $\Theta_2 = (\theta_0^1, \theta_2^2)$ and $\Theta_3 = (\theta_2^1, \theta_2^2)$ such that $0 < \theta_0^1 < \theta_2^1 < 1$ and $0 < \theta_0^2 < \theta_2^2 < 1$. Put, for $0 < \lambda = (\lambda^1, \lambda^2) < 1$, $\Theta^1 = (1-\lambda^1)\theta_0^1+\lambda^1\theta_2^1$ and $\Theta^2 = (1-\lambda^2)\theta_0^2+\lambda^2\lambda_2^2$. Let $F_\varkappa$ be an intermediate space that belongs to $H(\theta_\varkappa, \mathbf{E})$, $\varkappa = 0, 1, 2$, and 3. If $\mathbf{F} = (F_0, F_1, F_2, F_3)$, then*

$$\mathbf{E}_{\Theta,Q,K} = \mathbf{F}_{\lambda,Q;K}.$$

5. Interpolation theorems. Let $\mathbf{E}_0 = (E_0, E_1, E_2, E_3)$ to $\mathbf{F} = (F_0, F_1, F_2, F_3)$ be two 4-tuples of Banach spaces in X and Y, respectively.

5.1. Proposition. *Let T be a linear transformation from $\Sigma\mathbf{E}$ to $\Sigma\mathbf{F}$ such that the restriction of T to each $E_\varkappa$ is a bounded linear transformation from $E_\varkappa$ to $F_\varkappa$. Then, the restriction of T to $\mathbf{E}_{\Theta,Q;K}$ is a bounded linear transformation from $\mathbf{E}_{\Theta,Q;K}$ to $\mathbf{F}_{\Theta,Q;K}$.*

Proof. Let $C_\varkappa > 0$, $\varkappa = 0, 1, 2, 3$, be given such that for $g \in E_\varkappa$ we have

$$\|Tg\|_{F_\varkappa} \leqslant C_\varkappa \|g\|_{E_\varkappa}.$$

If $f = f_0+f_1+f_2+f_3 \in \mathbf{E}_{\Theta,Q;K}$, where $f_\varkappa \in E_\varkappa$, we have

$$\|Tf_\varkappa\|_{F_\varkappa} \leqslant C\|f_\varkappa\|_{E_\varkappa} \quad (\varkappa = 0, 1, 2, 3).$$

Now, putting $C = \max(C_0, C_1, C_2, C_3)$ and taking $f \in \mathbf{E}_{\Theta,Q;K}$, we obtain

$$K_F(t; Tf) \leqslant \|Tf_0\|_{F_0}+t_1\|Tf_1\|_{F_1}+t_2\|Tf_2\|_{F_2}+t_1t_2\|Tf_3\|_{F_3}$$
$$\leqslant C_0\|f_0\|_{E_0}+C_1t_2\|f_1\|_{E_1}+C_2t_2\|f_2\|_{E_2}+C_3t_1t_2\|f_3\|_{E_3}$$
$$\leqslant C\{\|f_0\|_{E_0}+t_1\|f_1\|_{E_1}+t_2\|f_2\|_{E_2}+t_1t_2\|f_3\|_{E_3}\}.$$

Hence

$$K_F(t; Tf) \leqslant CK_E(t; f)$$

and

$$\|Tf\|_{\mathbf{F}_{\Theta,Q;K}} \leqslant C\|f\|_{\mathbf{E}_{\Theta,Q;K}}.$$

5.2. Corollary. *Let $\mathbf{X} = (\mathbf{E}_{\Theta_\varkappa,Q;K})_{\varkappa=0,1,2,3}$ and $\mathbf{Y} = (\mathbf{F}_{\Theta_\varkappa,Q;K})_{\varkappa=0,1,2,3}$ where $\Theta_0 = (\theta_0^1, \theta_0^2)$, $\Theta_1 = (\theta_2^1, \theta_0^2)$, $\Theta_2 = (\theta_2^1, \theta_0^2)$, and $\Theta_3 = (\theta_2^1, \theta_2^2)$. Let T be a linear transformation from $\Sigma\mathbf{X}$ to $\Sigma\mathbf{Y}$, such that the restriction of T to each $\mathbf{E}_{\Theta_\varkappa,Q;K}$ is a bounded linear transformation from $\mathbf{E}_{\Theta_\varkappa,Q;K}$ to $\mathbf{F}_{\Theta_\varkappa,Q;K}$, $\varkappa = 0, 1, 2, 3$, and*

$$\|Tf\|_{\mathbf{F}_{\Theta_\varkappa,Q;K}} \leqslant C_\varkappa\|f\|_{\mathbf{E}_{\Theta_\varkappa,Q;K}},$$

where $f \in \mathbf{E}_{\Theta_\varkappa,Q;K}$ and $\varkappa = 0, 1, 2, 3$. Then there exists a constant $C > 0$ such that

$$\|Tf\|_{\mathbf{F}_{\lambda,Q;K}} \leqslant C\|f\|_{\mathbf{E}_{\Theta,Q;K}}$$

where $0 < \lambda = (\lambda^1, \lambda^2) < 1$ and $\Theta = (\theta^1, \theta^2)$ is defined by

$$\Theta^1 = (1-\lambda^1)\theta_0^1+\lambda^1\theta_2^1 \quad \text{and} \quad \Theta^2 = (1-\lambda^2)\theta_0^2+\lambda^2\theta_2^2.$$

Proof. The proof follows from 5.1 and 4.4.

5.3. Proposition. *Let T be a linear transformation from $\Sigma\mathbf{E}$ to $\Sigma\mathbf{F}$ such that the restriction of T to each $E_\varkappa$ is a bounded linear transformation from $E_\varkappa$ to $F_\varkappa$, $\varkappa = 0, 1, 2, 3$. Then, the restriction of T to $\mathbf{E}_{\Theta,Q;J}$ is a bounded linear transformation from $\mathbf{E}_{\Theta,Q;J}$ to $\mathbf{F}_{\Theta,Q;J}$.*

Proof. For each $u = u(s)$ in $\bigcap\mathbf{E}$, there exists a constant $C > 0$ such that for s in R_+^2 we have

$$J(s, Tu) \leqslant CJ(s; u).$$

Let $f \in \mathbf{E}_{\Theta,Q;J}$ and $u = u(s_1, s_2)$ be such that

$$f = \int_0^\infty\int_0^\infty u(s_1, s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}.$$

Putting $v = Tu$, we have

$$Tf = \int_0^\infty \int_0^\infty Tu(s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2} = \int_0^\infty \int_0^\infty v(s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2}.$$

Hence

$$\|Tf\|_{F_{\Theta,Q;J}} \leqslant \left\| s^{-\Theta} J(s, v(s)) \right\|_{L_*^Q} \leqslant C \left\| s^{-\Theta} J(s; u(s)) \right\|_{L_*^Q}$$

and

$$\|Tf\|_{F_{\Theta,Q;J}} \leqslant C \|f\|_{E_{\Theta,Q;J}}.$$

5.4. Corollary. *Let* $X = (E_{\Theta_\varkappa,Q;J})_{\varkappa=0,1,2,3}$ *and* $Y = (F_{\Theta_\varkappa,Q;J})_{\varkappa=0,1,2,3}$ *where* $\Theta_0 = (\theta_0^1, \theta_0^2)$, $\Theta_1 = (\theta_2^1, \theta_0^2)$, $\Theta_2 = (\theta_0^1, \theta_2^2)$ *and* $\Theta_3 = (\theta_2^1, \theta_2^2)$. *Let* T *be a linear transformation from* ΣX *to* ΣY *such that the restriction of* T *to each* $E_{\Theta_\varkappa,Q;J}$ *is a bounded linear transformation from* $E_{\Theta_\varkappa,Q;J}$ *to* $F_{\Theta\varkappa,Q;J}$, *that is, there exist constants* $C_\varkappa$ *such that*

$$\|Tf\|_{F_{\Theta_\varkappa,Q;J}} \leqslant C_\varkappa \|f\|_{E_{\Theta_\varkappa,Q;J}}$$

where $f \in E_{\Theta_\varkappa,Q;J}$, $\varkappa = 0, 1, 2, 3$. *Then there exists a constant* $C > 0$ *such that*

$$\|Tf\|_{Y_{\Theta_\varkappa,Q;J}} \leqslant C \|f\|_{E_{\Theta,Q;J}}$$

where $\Theta = (\theta^1, \theta^2)$ *is defined by*

$$\theta^1 = (1-\lambda^1)\theta_0^1 + \lambda^1 \theta_2^1 \quad \text{and} \quad \theta^2 = (1-\lambda^2)\theta_0^2 + \lambda^2 \theta_2^2$$

with $0 < \lambda = (\lambda^1, \lambda^2) < 1$.

Proof. The proof follows from 5.3 and 4.5.

Now, the main interpolation theorem can be stated.

5.5. Theorem. *Let* $X = (E_{\Theta_\varkappa,Q;K})_{\varkappa=0,1,2,3}$ *and* $Y = (F_{\Theta_\varkappa,Q;K})_{\varkappa=0,1,2,3}$ *and let* T *be a linear transformation from* ΣX *to* ΣY *for which there exist constants* $C_\varkappa > 0$ *such that*

$$\|Tf\|_{F_{\Theta_\varkappa,Q;K}} \leqslant C_\varkappa \|f\|_{E_{\Theta_\varkappa,Q;K}},$$

where $f \in E_{\Theta_\varkappa,Q;K}$. *Suppose that* $\Theta_0 = (\theta_0^1, \theta_0^2)$, $\Theta_1 = (\theta_2^1, \theta_0^2)$, $\Theta_2 = (\theta_0^1, \theta_2^1)$, $\Theta_3 = (\theta_2^1, \theta_2^2)$. *Then, if* $\Theta = (\theta^1, \theta^2)$ *is defined by*

$$\theta^1 = (1-\lambda^1)\theta_0^1 + \lambda^1 \theta_2^1 \quad \text{and} \quad \theta^2 = (1-\lambda^2)\theta_0^2 + \lambda^2 \theta_2^2$$

where $0 < \lambda = (\lambda^1, \lambda^2) < 1$, *there exists a constant* $C > 0$ *such that*

$$\|Tf\|_{F_{\Theta,Q;K}} \leqslant C \|f\|_{E_{\Theta,Q;K}}.$$

Furthermore, if $1 \leqslant P \leqslant Q$, *we have*

$$\|Tf\|_{F_{\Theta,Q;K}} \leqslant C \|f\|_{E_{\Theta,P;K}}.$$

Proof. The proof follows from the interpolation theorems 5.2 and 5.4 and the reiteration theorems 4.4 and 4.5.

6. Embedding in high dimensions. Let (E_0, E_1) be a Banach couple. The following result holds.

6.1. Theorem. *If* $0 \leqslant \Theta = (\theta, \theta) \leqslant 1$ *and* $1 \leqslant Q = (q, q) \leqslant \infty$, *then*

$$(E_0, E_1)_{\theta,q,K} = (E_0, E_0, E_1, E_1)_{\Theta,Q;K}$$

with equivalence of norms.

Proof. Let $f \in (E_0, E_0, E_1, E_1)_{\Theta,Q;K}$ and $f = f' + f''$, where $f' \in E_0$ and $f'' \in E_1$. Then

$$\begin{aligned} K(s, t; f) &= \inf\{K_1(s; f') + tK_2(s, f'')\} \\ &\geqslant \inf\{\min(1, s)\|f'\|_{E_0+E_0} + t\min(1, s)\|f''\|_{E_1+E_1}\} \\ &\geqslant \min(1, s)\inf\{\|f'\|_{E_0} + t\|f''\|_{E_1}\}, \end{aligned}$$

where the infimum are taken on all decompositions $f = f' + f''$. Now

$$s^{-\theta} t^{-\theta} K(s, t; f) \geqslant \{s^{-\theta}\min(1, s)\}\{t^{-\theta} K(t; f)\}$$

and so

$$\|s^{-\theta} t^{-\theta} K(s, t; f)\|_{L_*^q(L_*^q)} \geqslant \|s^{-\theta}\min(1, s)\|_{L_*^q} \|t^{-\theta} K(t; f)\|_{L_*^q},$$

that is,

$$(E_0, E_1)_{\theta,q;K} \supset (E_0, E_0, E_1, E_1)_{\Theta,Q;K}.$$

Reciprocally, let $f \in (E_0, E_1)_{\theta,q;K}$. Then, there exists an $E_0 \cap E_1$-valued function on $(0, \infty)$ such that

$$f = \int_0^\infty u(t) \frac{dt}{t}.$$

Now, let φ be a non-negative function in $C_c(0, \infty)$ with $\operatorname{supp}\varphi \subset (1, \infty)$ and such that

$$\int_0^\infty \varphi(s) \frac{ds}{s} = 1.$$

Defining $v(s, t) = \varphi(s) u(t)$, we will have

$$f = \int_0^\infty \int_0^\infty v(s, t) \frac{ds}{s} \frac{dt}{t}$$

and

$$\begin{aligned} J(s, t; v(s, t)) \leqslant \max\{&\max\{\|v(s, t)\|_{E_0}, s\|v(s, t)\|_{E_0}\}, \\ &t\max\{\|v(s, t)\|_{E_1}, s\|v(s, t)\|_{E_1}\}\} = \varphi(s)\max(1, s)J(t; u). \end{aligned}$$

Now

$$s^{-\theta}t^{-\theta}J(s,t;v(s,t)) \leqslant s^{-\theta}\varphi(s)\max(1,s)\, t^{-\theta}J(t;u(t)),$$

and so

$$\|s^{-\theta}t^{-\theta}J(s,t;v(s,t))\|_{L^q_*(L^q_*)} \leqslant \|s^{-\theta}\varphi(s)\max(1,s)\|_{L^q_*}\|t^{-\theta}J(t;u(t))\|_{L^q_*}.$$

Taking infimum, we obtain

$$\|f\|_{\theta,Q;J} \leqslant C\|f\|_{\theta,q;J}.$$

That is

$$(E_0,E_1)_{\theta,q;J} \subset (E_0,E_0,E_1,E_1)_{\Theta,Q;K}.$$

II. GENERALIZATION

We have studied the theory of interpolation for 4-tuples of Banach spaces. Now we deal with the 2^n-tuples case. Except for notation problems the only difficulty is the well setting of the "hiper-parallelepiped" condition.

7. General definitions. A Banach m-tuple is a family $\boldsymbol{E} = (E_1, \ldots, E_m)$ of Banach spaces embedded in a Hausdorff vector topological space V.

The space $\Sigma\boldsymbol{E}$ is the linear hull of this family and the space $\bigcap\boldsymbol{E}$ is defined in an obvious way. They are Banach spaces under the norms

$$x \in \Sigma\boldsymbol{E} \Rightarrow \|x\|_{\Sigma\boldsymbol{E}} = \inf\Big\{\sum_{\varkappa=1}^{m}\|x_\varkappa\|_{E_\varkappa};\ x = \Sigma X_\varkappa,\ x_\varkappa \in E_\varkappa\Big\},$$

and

$$x \in \bigcap\boldsymbol{E} \Rightarrow \|x\|_{\cap\boldsymbol{E}} = \max_{1\leqslant\varkappa\leqslant m}\|x\|_{E_\varkappa},$$

respectively. Furthermore, the spaces $\bigcap\boldsymbol{E}$ and $\Sigma\boldsymbol{E}$ are continuously embedded in V.

A Banach space E is an *intermediate space* with respect to the m-tuple $\boldsymbol{E} = (E_1, \ldots, E_m)$ if the following (algebraic and topological) inclusions hold:

$$\bigcap\boldsymbol{E} \subset E \subset \Sigma\boldsymbol{E}.$$

8. The generalized K and J-function norms. We will define the K and J-function norms by recurrence.

Let $\boldsymbol{E} = (E_1, \ldots, E_m)$ be a Banach m-tuple and $\boldsymbol{t} = (t_1, \ldots, t_n)$ a n-tuple of positive numbers.

Case $m = 2$. If $f \in E_1+E_2$ and $g \in E_1\cap E_2$, we put

$$K(t;f) = \inf\{\|f_1\|_{E_1}+t\|f_2\|_{E_2} \mid f = f_1+f_2, f_i \in E_i,\ i = 1,2\}$$

and

$$J(t;g) = \max\{\|g\|_{E_1}, t\|g\|_{E_2}\}.$$

Case $m = 4$. If $f \in E_1+E_2+E_3+E_4$ and $g \in E_1\cap E_2\cap E_3\cap E_4$, let

$$E' = (E_1, E_2) \quad\text{and}\quad E'' = (E_3, E_4).$$

Now put

$$\begin{aligned}K(t_1,t_2;f) &= \inf\{K'(t_1;f')+t_2K''(t_1;f'') \mid f = f'+f'',\ f' \in E',\ f'' \in E''\}\\ &= \inf\{\|f_1\|_{E_1}+t_1\|f_2\|_{E_2}+t_2\|f_3\|_{E_3}+t_1t_2\|f_4\|_{E_4} \mid\\ &\qquad f = \Sigma f_i, f_i \in E_i,\ i = 1,2,3,4\}\end{aligned}$$

and

$$\begin{aligned}J(t_1,t_2;g) &= \max\{J'(t_1;g), t_2J'(t_1;g)\}\\ &= \max\{\|g\|_{E_1}, t_1\|g\|_{E_2}, t_2\|g\|_{E_3}, t_1t_2\|g\|_{E_4}\}.\end{aligned}$$

Case $m = 2^n$. If $f \in \Sigma\boldsymbol{E}$ and $g \in \bigcap\boldsymbol{E}$, let

$$\boldsymbol{E}' = (E_1, \ldots, E_{2^{n-1}}) \quad\text{and}\quad \boldsymbol{E}'' = (E_{2^{n-1}+1}, \ldots, E_{2^n}).$$

Now put

$$\begin{aligned}K(\boldsymbol{t};f) &= K(t_1,\ldots,t_n;f)\\ &= \inf\{K'(t_1,\ldots,t_{n-1};f')+t_nK''(t_1,\ldots,t_{n-1};f'') \mid\\ &\qquad f = f'+f'', f' \in \Sigma\boldsymbol{E}', f'' \in \Sigma\boldsymbol{E}''\}\end{aligned}$$

and

$$\begin{aligned}J(\boldsymbol{t};g) &= J(t_1,\ldots,t_n;g)\\ &= \max\{J'(t_1,\ldots,t_{n-1};g), t_nJ''(t_1,\ldots,t_{n-1};g)\}.\end{aligned}$$

To generalize Proposition 1.1 we will need to define the following. Let $\boldsymbol{1} = (1, \ldots, 1)$ and $\boldsymbol{t} = (t_1, \ldots, t_n) > 0$. If $n = 1$, put

$$m_1(1,t) = \min(1,t),$$

$$M_1(1,t) = \max(1,t).$$

If $n = 2$, put

$$m_2(\boldsymbol{1},\boldsymbol{t}) = \min(m_1(1,t_1), t_2m_1(1,t_1)),$$

$$M_2(\boldsymbol{1},\boldsymbol{t}) = \max(M_1(1,t_1), t_2M_2(1,t_1)).$$

For the general case put $\tilde{t} = (t_1, \ldots, t_{n-1})$. Then

$$m_n(\boldsymbol{1},\boldsymbol{t}) = \min\{m_{n-1}(\boldsymbol{1},\tilde{\boldsymbol{t}}), t_nm_{n-1}(\boldsymbol{1},\tilde{\boldsymbol{t}})\},$$

$$M_n(\boldsymbol{1},\boldsymbol{t}) = \max\{\max(\boldsymbol{1},\tilde{\boldsymbol{t}}), t_n\max(\boldsymbol{1},\tilde{\boldsymbol{t}})\}.$$

If no confusion arises we will write $m_n(\boldsymbol{1},\boldsymbol{t}) = m(\boldsymbol{1},\boldsymbol{t})$ and $M_n(\boldsymbol{1},\boldsymbol{t}) = M(\boldsymbol{1},\boldsymbol{t})$.

Now it is not hard to see that the following proposition holds.

8.1. PROPOSITION. *If* $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{R}_+^n$, $\boldsymbol{st} = (s_1t_1, \ldots, s_nt_n)$, $s^{-1} = (s_1^{-1}, \ldots, s_n^{-1})$, $f \in \Sigma\boldsymbol{E}$ *and* $g \in \bigcap\boldsymbol{E}$, *then*

(8.1.1) $$m(\boldsymbol{1}, \boldsymbol{ts}^{-1})K(\boldsymbol{s}; f) \leqslant K(\boldsymbol{t}; f) \leqslant M(\boldsymbol{1}, \boldsymbol{st}^{-1})K(\boldsymbol{s}; f),$$

(8.1.2) $$m(\boldsymbol{1}, \boldsymbol{t}^{-1})\|f\|_{\Sigma E} \leqslant K(\boldsymbol{t}; f) \leqslant M(\boldsymbol{1}, \boldsymbol{t})\|f\|_{\Sigma E},$$

(8.1.3) $$m(\boldsymbol{1}, \boldsymbol{t}^{-1})K(\boldsymbol{t}; f) \leqslant \|f\|_{\Sigma E} \leqslant M(1, \boldsymbol{t}^{-1})K(\boldsymbol{t}; f),$$

(8.1.4) $$m(\boldsymbol{1}, \boldsymbol{st}^{-1})J(\boldsymbol{s}; g) \leqslant J(\boldsymbol{t}; g) \leqslant M(\boldsymbol{1}, \boldsymbol{ts}^{-1})J(\boldsymbol{s}; g),$$

(8.1.5) $$m(\boldsymbol{1}, \boldsymbol{t}^{-1})\|g\|_{\cap E} \leqslant J(\boldsymbol{t}; g) \leqslant M(\boldsymbol{1}, t)\|g\|_{\cap E},$$

(8.1.6) $$m(1, \boldsymbol{t}^{-1})J(\boldsymbol{t}; g) \leqslant \|g\|_{\cap E} \leqslant M(\boldsymbol{1}, \boldsymbol{t}^{-1})J(\boldsymbol{t}; g),$$

(8.1.7) $$K(\boldsymbol{t}; g) \leqslant m(\boldsymbol{1}, \boldsymbol{ts}^{-1})J(\boldsymbol{s}; g),$$

(8.1.8) $$M(\boldsymbol{1}, \boldsymbol{st}^{-1})K(\boldsymbol{t}; g) \leqslant J(\boldsymbol{s}; g).$$

8.2. PROPOSITION. *Let* $0 \leqslant \Theta = (\theta_1, \ldots, \theta_n) \leqslant 1$ *and* $1 \leqslant Q = (q_1, \ldots, q_n) \leqslant \infty$. *Suppose* $q_j = \infty$ *if and only if* $\theta_j = 0$ *or* $\theta_j = 1$. *Then*

$$\|\boldsymbol{t}^{-\Theta}m(\boldsymbol{1}, \boldsymbol{t})\|_{L_*^Q} = \Big\| \ldots \|t_1^{-\theta_1} \ldots t_n^{-\theta_n} m(1, t_1, \ldots, t_n)\|_{L_*^{q_1}} \ldots \Big\|_{L_*^{q_n}} < \infty.$$

Proof. Cases $n = 1$ and $n = 2$ follow by direct calculation and the general case by induction.

Observe that Proposition 8.2 generalises Proposition 1.1.

9. The K- and J-methods. Let $\boldsymbol{E} = (E_1, \ldots, E_{2^n})$ a Banach 2^n-tuple, $0 \leqslant \Theta = (\theta_1, \ldots, \theta_n) \leqslant 1$, and $1 \leqslant Q = (q_1, \ldots, q_n) < \infty$. Suppose further $\theta_j = 0$ or $\theta_j = 1$ if and only if $q_j = \infty$. We define the space

$$\boldsymbol{E}_{\Theta,Q;K} = (E_1, \ldots, E_{2^n})_{\Theta,Q;K}$$

to be the space of elements $f \in \Sigma\boldsymbol{E}$ for which

$$\boldsymbol{t}^{-\Theta}K(\boldsymbol{t}; f) = t_1^{-\theta_1} \ldots t_n^{-\theta_n} K(t_1, \ldots, t_n) \in L_*^Q = L_*^{q_n}(\ldots(L_*^{q_1})\ldots).$$

The spaces $\boldsymbol{E}_{\Theta,Q;K}$ are Banach space under the norms

$$\|f\|_{\Theta,Q;K} = \|\boldsymbol{t}^{-\Theta}K(\boldsymbol{t}; f)\|_{L_*^Q} = \Big\| \ldots \|t^{-\theta_1} \ldots t_n^{-\theta_n} K(t_1, \ldots, t_n; f)\|_{L_*^{q_1}} \ldots \Big\|_{L_*^{q_n}} < \infty.$$

We define the space

$$\boldsymbol{E}_{\Theta,Q;J} = (E_1, \ldots, E_{2^n})_{\Theta,Q;J}$$

to be the space of all elements $f \in \Sigma\boldsymbol{E}$ for which there exists a strongly measurable function $u = u(s) = u(s_1, \ldots, s_n)$ with values in $\bigcap\boldsymbol{E}$ and such that

$$f = \int_0^\infty \ldots \int_0^\infty u(s_1, \ldots, s_n)\frac{ds_1}{s_1}\frac{ds_n}{s_n}$$

and

$$s_1^{-\theta_1} \ldots s_n^{-\theta_n} J(s_1, \ldots, s_n) \in L_*^Q.$$

The spaces $\boldsymbol{E}_{\Theta,Q;J}$ are meaningful. They are Banach space under the norms

$$\|f\|_{\Theta,Q;J} = \inf\left\{\left\|s_1^{-\theta_1} \ldots s_n^{-\theta_n} J(s_1, \ldots, s_n; u(s_1, \ldots, s_n))\right\|_{L_*^Q}\right\},$$

where the infimum is taken on all representation

$$f = \int_0^\infty \ldots \int_0^\infty u(s_1, \ldots, s_n)\frac{ds_1}{s_1}\frac{ds_2}{s_2}.$$

Furthermore, we have

9.1. PROPOSITION. *The spaces* $\boldsymbol{E}_{\Theta,Q;K}$ *are intermediate spaces with respect to the Banach* 2^n*-tuple* $\boldsymbol{E} = (E_1, \ldots, E_{2^n})$. *That is,*

(9.1.1) $$\bigcap\boldsymbol{E} \subset \boldsymbol{E}_{\Theta,Q;K} \subset \Sigma\boldsymbol{E},$$

(9.1.2) $$\bigcap\boldsymbol{E} \subset \boldsymbol{E}_{\Theta,Q;J} \subset \Sigma\boldsymbol{E}.$$

9.2. PROPOSITION. *Let* $f \in \boldsymbol{E}_{\Theta,Q;K}$ *and* $g \in \boldsymbol{E}_{\Theta,Q;J}$. *Then we have*

$$K(\boldsymbol{s}; f) \leqslant \boldsymbol{s}^\Theta\{\|\boldsymbol{t}^{-\Theta}m(\boldsymbol{1}, \boldsymbol{t})\|_{L_*^Q}\}^{-1}\|f\|_{\Theta,Q;K}.$$

10. On the equivalence.

10.1. PROPOSITION. *For* $0 < \Theta = (\theta_1, \ldots, \theta_n) < 1$ *and* $1 \leqslant P = (p_1, \ldots, p_n) \leqslant Q = (q_1, \ldots, q_n) \leqslant \infty$ *the following algebraic and topological inclusion holds:*

$$\boldsymbol{E}_{\Theta,P;J} \subset \boldsymbol{E}_{\Theta,Q;K}.$$

To reverse this inclusion the following lemma, that generalizes Lemma 3.2, will be useful.

10.2. LEMMA. *Let* $f \in \Sigma\boldsymbol{E}$ *for which there exist constants* $0 < \Theta = (\theta_1, \ldots, \theta_n) < 1$ *and* $C = C(f)$ *such that*

(10.2.1) $$K(s_1, \ldots, s_n; f) \leqslant C(f)s_1^{\theta_1} \ldots s_n^{\theta_n}.$$

Then there exists a strongly measurable function $u = u(s_1, \ldots, s_2)$ *in* $\bigcap\boldsymbol{E}$ *satisfying*

(10.2.2) $$f = \int_0^\infty \ldots \int_0^\infty u(s_1, \ldots, s_n)\frac{ds_1}{s_1}\frac{ds_n}{s_n}$$

and

(10.2.3) $$J(s_1, \ldots, s_n; u(s_1, \ldots, s_n)) \leqslant (4e)^n K(s_1, \ldots, s_n; f).$$

Proof. The case $n = 2$ is Proposition 3.2. The general case follows by induction.

Lemma 9.2 implies the following

10.3. PROPOSITION. *Let* $0 < \Theta = (\theta_1, \ldots, \theta_n) < 1$ *and* $1 \leqslant Q = (q_1, \ldots, q_n) \leqslant \infty$. *Then*

$$\boldsymbol{E}_{\Theta,Q;K} \subset \boldsymbol{E}_{\Theta,Q;J}.$$

Combining the results of Propositions 10.1 and 10.3, the following theorem arises.

10.4. THEOREM. *The intermediate spaces* $\boldsymbol{E}_{\Theta,Q;K}$ *and* $\boldsymbol{E}_{\Theta,Q;J}$ *coincide;*

(10.4.1) $$\boldsymbol{E}_{\Theta,Q;K} = \boldsymbol{E}_{\Theta,Q;J}$$

with equivalence of norms.

10.5. COROLLARY. *For* $0 < \Theta < 1$ *and* $1 \leqslant P \leqslant Q$ *we have*

$$\boldsymbol{E}_{\Theta,P;K} \subset \boldsymbol{E}_{\Theta,Q;K}.$$

In particular,

$$\boldsymbol{E}_{\Theta,\boldsymbol{1},K} \subset \boldsymbol{E}_{\Theta,Q;K} \subset \boldsymbol{E}_{\Theta,\infty,K}.$$

(Here $\boldsymbol{1} = (1, \ldots, 1)$ and $\infty = (\infty, \ldots, \infty)$.)

11. On the reiteration.

11.1. DEFINITION. We say that an intermediate space E of $\boldsymbol{E} = (E_1, \ldots, E_{2^n})$

(i) *belongs to the class* $K(\Theta; \boldsymbol{E})$, $0 \leqslant \Theta = (\theta^1, \ldots, \theta^n) \leqslant 1$, if

(11.1.1) $$K(\boldsymbol{t}; f) \leqslant C\boldsymbol{t}^{\Theta} \|f\|_E \quad (f \in E);$$

(ii) *belongs to the class* $J(\Theta; \boldsymbol{E})$, $0 \leqslant \Theta = (\theta^1, \ldots, \theta^n) \leqslant 1$, if

(11.1.2) $$\|f\|_E \leqslant \boldsymbol{t}^{-\Theta} J(\boldsymbol{t}; f) \quad (f \in \bigcap \boldsymbol{E});$$

(iii) *belongs to the class* $H(\Theta; \boldsymbol{E})$, $0 \leqslant \Theta = (\theta^1, \ldots, \theta^n) \leqslant 1$, if it belongs to the class $K(\Theta; \boldsymbol{E})$ as well as to the class $J(\Theta; \boldsymbol{E})$.

The next proposition gives necessary and sufficient conditions for an intermediate space E of $\boldsymbol{E}$ to belong to one of the above defined classes. The proof is analogous to the case $n = 2$.

11.2. PROPOSITION. *An intermediate space* E *of* $\boldsymbol{E} = (E_1, \ldots, E_{2^n})$

(i) *belongs to* $K(\Theta; \boldsymbol{E})$, $0 \leqslant \Theta = (\theta^1, \ldots, \theta^n) \leqslant 1$, *if and only if*

(11.2.1) $$E \subset \boldsymbol{E}_{\Theta,\infty,K};$$

(ii) *belongs to* $J(\Theta; \boldsymbol{E})$, $0 \leqslant \Theta = (\theta^1, \ldots, \theta^n) \leqslant 1$, *if and only if*

(11.2.2) $$\boldsymbol{E}_{\Theta,\boldsymbol{1};J} \subset E;$$

(iii) *belongs to* $H(\Theta; \boldsymbol{E})$, $0 \leqslant \Theta = (\theta^1, \ldots, \theta^n) \leqslant 1$, *if and only if*

(11.2.3) $$\boldsymbol{E}_{\Theta,\boldsymbol{1};J} \subset E \subset \boldsymbol{E}_{\Theta;\infty;K}.$$

11.3. To state reiterations theorems in the case 2^n we will need to define a sequence $(\Theta_1, \ldots, \Theta_{2^n})$ with the "hyper-parallelepiped" condition.

Let

$$\Theta_1 = (\theta_1^1, \ldots, \theta_1^j, \ldots, \theta_1^n)$$

and

$$\Theta_{2^n} = (\theta_{2^n}^1, \ldots, \theta_{2^n}^j, \ldots, \theta_{2^n}^n)$$

be two fixed n-tuples such that $0 \leqslant \theta_1^j < \theta_{2^n}^j \leqslant 1$, for $j = 1, \ldots, n$.

If $n = 2$ and $j = 1, 2, 3, 4$, let Θ_j be defined by

$$\Theta_1 = (\theta_1^1, \theta_1^2), \qquad \Theta_2 = (\theta_4^1, \theta_1^2), \qquad \Theta_3 = (\theta_1^1, \theta_4^2), \qquad \Theta_4 = (\theta_4^1, \theta_4^2).$$

Suppose, now, the sequence is already defined for $j > 2$.

To define it for $n = j+1$, let $(\Theta_{k,j})_{1 \leqslant k \leqslant 2^j}$ be the sequence yet defined for $n = j$ (here $\Theta_{1,j} = (\theta_1^1, \ldots, \theta_1^j)$ and $\Theta_{2^j,j} = (\theta_{2^n}^1, \ldots, \theta_{2^n}^j)$). Then, for $1 \leqslant k \leqslant 2^j$, put

$$\Theta_{k,j+1} = (\Theta_{k,j}, \theta_1^{j+1})$$

and

$$\Theta_{k+2^j,j+1} = (\Theta_{k,j}, \theta_{2^n}^{j+1}).$$

The sequence $(\Theta_1, \ldots, \Theta_{2^n})$ is defined for $n = j+1$.

For the sequence (Θ_k), just defined by the above process, it will be said that it *satisfies the hyper-parallelepiped condition associated with the* n*-tuples* Θ_1 and Θ_{2^n}.

11.4. PROPOSITION. *Let* $\Theta_1 = (\theta_1^1, \ldots, \theta_1^n)$ *and* $\Theta_{2^n} = (\theta_{2^n}^1, \ldots, \theta_{2^n}^n)$ *be two* n*-tuple such that* $0 \leqslant \theta_1^j < \theta_{2^n}^j \leqslant 1$, *for* $j = 1, \ldots, n$, *and assume that* $(\Theta_1, \ldots, \Theta_{2^n})$ *is the associated sequence which satisfies the "hiper-parallelepiped" condition. Also, let* $0 < \lambda = (\lambda_1, \ldots, \lambda_n) < 1$ *and* $\Theta = (\theta^1, \ldots, \theta^n)$ *be such that* $\theta^j = (1-\lambda^j)\theta_1^j + \lambda^j \theta_{2^n}^j$, *for* $j = 1, \ldots, n$.

Now, let $F = (F_1, \ldots, F_{2^n})$ *be a family of* 2^n *intermediate spaces with respect to* $E = (E_1, \ldots, E_{2^n})$ *such that, for* $k = 1, \ldots, 2^n$, *the space* $\boldsymbol{F}_k$ *belongs to*

(i) *the class* $K(\Theta_k; \boldsymbol{E})$; *then*

(11.4.1) $$\boldsymbol{F}_{\lambda,Q;K} \subset \boldsymbol{E}_{\Theta,Q;K};$$

(ii) *the class* $J(\Theta_k; \boldsymbol{E})$; *then*

(11.4.2) $$\boldsymbol{E}_{\Theta,Q;J} \subset \boldsymbol{F}_{\lambda,Q;J}.$$

This proposition and the Equivalence Theorem gives the following Reiteration Theorem.

11.4. THEOREM. *Let* $\Theta_1 = (\theta_1^1, \ldots, \theta_1^n)$ *and* $\Theta_{2^n} = (\theta_{2^n}^1, \ldots, \theta_{2^n}^n)$ *be two* n*-tuples such that* $0 < \theta_1^j < \theta_{2^n}^j < 1$, *for* $j = 1, \ldots, n$, *and* $(\Theta_1, \ldots, \Theta_{2^n})$, *and* λ *as in Proposition* 11.4.

Now, let $\boldsymbol{F} = (F_1, \ldots, F_{2^n})$ *be a family of* 2^n *intermediate spaces with respect to* $\boldsymbol{E} = (E_1, \ldots, E_{2^n})$ *such that, for* $k = 1, \ldots, 2^n$, *the space* F_k *belongs to the class* $H(\Theta_k; \boldsymbol{E})$. *Then*

(11.5.1) $$\boldsymbol{E}_{\Theta,Q;K} = \boldsymbol{F}_{\lambda,Q,K}.$$

12. Interpolation theorems. Let $E = (E_0, \ldots, E_{2n})$ and $F = (F_0, \ldots, F_{2n})$ be two Banach 2^n-tuples in V and W, respectively. The general form of Propositions 5.1 and 5.3 reads:

12.1. Proposition. *Let T be a linear transformation from ΣE into ΣF such that the restriction of T on each E_k is a bounded linear transformation from E_k into F_k. Then,*

(i) *the restriction of T on $\mathbf{E}_{\Theta,Q;K}$ is a bounded linear transformation from $\mathbf{E}_{\Theta,Q;K}$ into $\mathbf{F}_{\Theta,Q;K}$,*

(ii) *the restriction of T on $\mathbf{E}_{\Theta,Q;J}$ is a bounded linear transformation from $\mathbf{E}_{\Theta,Q;J}$ into $\mathbf{F}_{\Theta,Q;J}$.*

This proposition, the equivalence theorem 10.4 and the reiteration theorem, gives as a corollary

12.2. Theorem. *Suppose that $\Theta_\varkappa$, $\varkappa = 1, 2, \ldots, 2^n$, satisfy the "hiper-parallelepiped" condition. Let $\mathbf{X} = (\mathbf{E}_{\Theta_\varkappa,Q;K})_{\varkappa=1,\ldots,2^n}$ and $\mathbf{Y} = (\mathbf{F}_{\Theta_\varkappa,Q;K})$ and T a linear transformation from $\Sigma\mathbf{X}$ to $\Sigma\mathbf{Y}$ such that the restriction to each $\mathbf{E}_{\Theta_\varkappa,Q;K}$ is a bounded linear transformation from $\mathbf{E}_{\Theta_\varkappa,Q;K}$ on $\mathbf{F}_{\Theta_\varkappa,Q;K}$, $\varkappa = 1, \ldots, 2^n$. Then the restriction of T to $\mathbf{E}_{\Theta,Q;K}$ is a bounded linear transformation from $\mathbf{E}_{\Theta,Q;K}$ into $\mathbf{F}_{\Theta,Q;K}$, where $\Theta = (\theta^1, \ldots, \theta^n)$ is defined by $\theta^i = (1-\lambda^i)\theta + \lambda^i\theta$, $0 < \lambda^i < 1$, $i = 1, 2, \ldots, n$.*

III. APPLICATIONS

13. Lorentz spaces with mixed norms. Let (Z, ϱ) be an σ-finite measure space and consider ϱ-measurable real or complex function, for Z. The *distribution function* of f is defined by

$$m_f(\lambda) = \varrho\big(\{z \in Z \mid |f(z)| > \lambda > 0\}\big).$$

This is a non-negative, non-increasing and continuous from the right function of $\lambda > 0$. The non-increasing rearrangement of f onto $(0, \infty)$ is defined by

$$\begin{aligned} f^*(t) &= m_{m_f}(t) = |\{\lambda > 0 \mid m_f(\lambda) > t > 0\}| \\ &= \sup\{\lambda > 0 \mid m_f(\lambda) > t > 0\} = \inf\{\lambda > 0 \mid m_f(\lambda) \leqslant t\}. \end{aligned}$$

It is clear that f^* is a non-negative, non-increasing and continuous from the right function onto $(0, \infty)$. Finally, if $s > 0$, we define the *integral mean function* or the *Hardy transformation* by

$$f^{**}(s) = \frac{1}{s}\int_0^s f^*(t)\,dt.$$

J. Peetre [8] and E. T. Oklander [10] gave the following *abstract* characterization of the integral mean function. Let $L^1 + L^\infty$ be the space of ϱ-measurable functions on Z such that there exist functions $f_1 \in L^1(Z)$ and $f_2 \in L^\infty(Z)$ for which $f = f_1 + f_2$.

Now, putting for $s > 0$

$$\|f\|_{L^1+sL^\infty} = \inf\{\|f_1\|_{L^1} + s\|f_2\|_{L^\infty} \mid f = f_1 + f_2 \text{ and } f_1 \in L^1, f_2 \in L^\infty\},$$

we have

$$f^{**}(s) = s^1 \|f\|_{L^1+sL^\infty}.$$

We will introduce a generalization of this fact.

Let X and Y be two measure spaces with σ-finite measures μ and ν, respectively. The space of the complex valued $\mu\times\nu$-measurable functions on $X \times Y$ will be denoted by $M = M(X \times Y)$. Let $f \in M(X\times Y)$. Then, for each y fixed in Y, $f_y(x) = f(x, y)$ is an μ-measurable function on X and it makes sense to put

$$F(y) = \|f(\cdot, y)\|_{L^1+sL^\infty}$$

and we will have the following transformation suggested by E. T. Oklander

$$f^{**}(s, t) = \|F\|_{L^1+tL^\infty} = \big\|\, \|f(x, y)\|_{L^1+sL^\infty} \big\|_{L^1+tL^\infty} \qquad (s > 0,\ t > 0).$$

We say that $f \in (L^1 + tL^\infty)(L^1 + sL^\infty)$ if $f^{**}(s, t) < \infty$. The *function norm* $f^{**}(s, t)$ is a norm on $(L^1 + tL^\infty)(L^1 + sL^\infty)$. If we change the parameters s and t we obtain equivalent norms.

Also, we say that $f \in L^1 + sL^1(L^\infty) + tL^\infty(L^1) + stL^\infty$ if there exist $f_1(x, y) \in L^1$, $f_2(x, y) \in L^1(L^\infty)$, $f_3(x, y) \in L^\infty(L^1)$ and $f_4(x, y) \in L^\infty$ such that

$$f = f_1 + f_2 + f_3 + f_4.$$

This is a Banach space under the norm

$$\|f\|_{L^1+sL^1(L^\infty)+tL^\infty(L^1)+stL^\infty}$$

defined by

$$\inf_{f=\sum_{i=1}^4 f_i} \{\|f_1\|_{L^1} + s\|f_2\|_{L^1(L^\infty)} + t\|f_3\|_{L^\infty(L^1)} + st\|f_4\|_{L^\infty}\}.$$

We have the following result (see [7]):

13.1. Proposition. *The spaces $(L^1 + tL^\infty)(L^1 + sL^\infty)$ and $L^1 + L^1(L^\infty) + L^\infty(L^1) + L^\infty$ are equal and for all $s > 0$ and $t > 0$ we have*

$$\|f\|_{(L^1+tL^\infty)(L^1+sL^\infty)} \leqslant \|f\|_{L^1+sL^1(L^\infty)+tL^\infty(L^1)+stL^\infty} \leqslant 2\|f\|_{(L^1+tL^\infty)(L^1+sL^\infty)}.$$

Let now $P = (p_1, p_2)$, $Q = (q_1, q_2)$ with $1 \leqslant P, Q \leqslant \infty$ and $f \in M(X\times Y)$. We say that $f \in L^{PQ} = L^{PQ}(X \times Y)$ if

$$\|f\|_{L^{PQ}} = \|s^{1/p_1} t^{1/p_2} f^{**}(s, t)\|_{L^{q_2}(L^{q_1})} < \infty.$$

We can show that (see [7] for a complete treatment)

$$L^{PQ} = L^{p_2 q_2}(L^{p_1 q_1}),$$

with equivalence of norms, that is, the space L^{PQ} is the Lorentz space with mixed norm.

Our next result says that the Lorentz spaces with mixed norm are intermediate space with respect to L^1, $L^1(L^\infty)$, $L^\infty(L^1)$, and L^∞.

13.2. Proposition. *Let* $1 \leqslant P < \infty$ *and* $1 \leqslant Q < \infty$ *or* $1 \leqslant P \leqslant \infty$ *and* $Q = \infty$. *If* $\Theta = (\theta^1, \theta^2)$, *where* $\theta^1 = 1-1/p_1$ *and* $\theta^2 = 1-1/p_2$, *we have*

$$(L^1, L^1(L^\infty), L^\infty(L^1), L^\infty)_{\Theta,Q;K} = L^{PQ}$$

with equivalence of norms.

Proof. Putting $E_0 = L^1$, $E_1 = L^1(L^\infty)$, $E_2 = L^\infty(L^1)$, and $E_3 = L^\infty$, we have for $f \in \Sigma E$

$$K(s, t; f) = st f^{****}(s, t).$$

The following interpolation theorem of the Marcinkiewicz–Stein–Weiss–Calderón theorem type is an immediate consequence of Theorem 11.2 (see [1] and [7]).

Let $X \times Y$ and $V \times W$ be two σ-finite measure spaces.

13.3. Theorem. *Let* $1 < P_\varkappa < \infty$ *and* $1 < Q < \infty$, $\varkappa = 0, 1, 2, 3$ *such that*

$$P_0 = (p_0^1, p_0^2), \quad P_1 = (p_1^1, p_0^2), \quad P_2 = (p_0^1, p_1^2), \quad P_3 = (p_1^1, p_1^2),$$

and

$$Q_0 = (q_0^1, q_0^2), \quad Q_1 = (q_1^1, q_0^2), \quad Q_2 = (q_0^1, q_1^2), \quad Q_3 = (q_1^1, q_1^2)$$

and also $p_0^1 < p_1^1$, $p_0^2 < p_1^2$, $q_0^1 < q_1^1$ *and* $q_0^2 < q_1^2$. *Let* T *be a sublinear operator that maps* $L^{P_\varkappa 1} = L^{P_\varkappa 1}(X \times Y)$ *into* $L^{Q_\varkappa \infty} = L^{Q_\varkappa \infty}(V \times W)$ *and suppose that there exist constants* $C_\varkappa > 0$, *such that for all* $f \in L^{P_\varkappa 1}$

$$\|Tf\|_{L^{Q_\varkappa \infty}} \leqslant C \|f\|_{L^{P_\varkappa 1}},$$

for $\varkappa = 0, 1, 2, 3$. *Then there exists a constant* $C > 0$ *such that for all* $f \in L^{PR}$ *we have*

$$\|Tf\|_{L^{QS}} \leqslant C \|f\|_{L^{PR}},$$

where

$$\frac{1}{P} = \frac{1-\Theta}{P_0} + \frac{\Theta}{P_3}, \quad \frac{1}{Q} = \frac{1-\Theta}{Q_0} + \frac{\Theta}{Q_3},$$

for

$$0 < \Theta = (\theta^1, \theta^2) < 1 \quad \text{and} \quad 1 \leqslant P \leqslant R.$$

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