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On projections in spaces of bounded analytic functions with applications

by

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Abstract. Projections in spaces A and H_∞ are investigated. It is shown that H_∞ is isomorphic to its l_∞-sum and has a contractible linear group. Certain generalizations to spaces of bounded analytic functions of several complex variables are presented. Norm-one finite rank projections in A and H_∞ are described. The description shows in particular that A and L_1/H_1 are not π_1-spaces. We also investigate isometric and isomorphic preduals of H_∞.

Introduction. In the present paper we consider projections in spaces of bounded analytic functions. Our main interest lies in the space $H_\infty(U)$, the space of bounded analytic functions in the unit disc U, but generalizations to $H_\infty(U^n)$ and $H_\infty(B_n)$ (the spaces of bounded analytic functions in n-polydisc and n-dimensional ball) are also presented. First we exhibit a class of elementary projections which play the crucial role in our paper. Those are projections given by linear extension operators from certain subsets of the fibres of $\mathfrak{M}(H_\infty)$. Using those projections, we show that H_∞ is isomorphic to its direct sum in the sense of l_∞. Applying the result of Bočkariov [2], we infer that H_∞ is isomorphic to a second conjugate space, thus answering the question of Rickart, asked in [22]. We show the isomorphic character of this result, proving that isometrically H_∞ has a unique predual space (this result answers the question of Porcelli [24] problem 59) and is not isometric to the second conjugate space of any Banach space. This is done in Section 1.

Section 2 contains the proof that the group of linear isomorphisms of $H_\infty(U)$ is contractible. This is done by using the general scheme elaborated by B. S. Mitiagin [17]. We show that this scheme is applicable by a detailed analysis of certain elementary projections, used also in Section 1.

In Section 3 we consider spaces of bounded analytic functions in polydiscs $U^n \subset C^n$ and balls $B_n \subset C^n$. We are able to generalize our main results to polydiscs. The space $H_\infty(U^n)$ is isomorphic to its direct sum in the sense of l_∞ and has a contractible linear group. As regards the space $H_\infty(B_n)$, we show that it is isomorphic to its l_∞-sum.

In Section 4 we consider finite dimensional norm-one projections in A and H_∞. It follows from our results that A is not a π_1-space and that every norm-one finite dimensional projection in the space L_1/H_1^0 is one-dimensional.

Section 5, the last, contains some open problems and certain easy observations about the fibre algebra, i.e. the restriction of H_∞ to a fibre.

Definitions and notation. If $V \subset C^n$ is an open bounded set, then $H_\infty(V)$ is the Banach space of all bounded analytic functions in V considered with the supremum norm. By $A(V)$ we will mean the space of functions continuous on $\bar{V}$ (the closure of V), and analytic in V, equipped with the supremum norm. In our paper the set V will be either the polydisc

$$U^n = \{(z_1, z_2, \ldots, z_n) \in C^n \colon |z_i| < 1 \text{ for } i = 1, 2, \ldots, n\}$$

or the ball

$$B_n = \Big\{(z_1, z_2, \ldots, z_n) \in C^n : \sum_{i=1}^{n} |z_i|^2 < 1\Big\}.$$

The set $U^1 = B_1$ will be denoted by U. $H_\infty(U)$ will often be denoted by H_∞ and $A(U)$ will often be denoted by A. It is well known (cf. [11], [20]) that $H_\infty(U^n)$ can be isometrically identified with a subalgebra of $L_\infty(S^n, \lambda^n)$ where $S = \{z \in C\colon |z| = 1\}$, S^n is the Cartesian product of n copies of S and λ^n denotes the normalized Lebesgue measure on S^n. In the same way $A(U^n)$ can be identified with a subalgebra of $C(S^n)$. The set of all linear multiplicative functionals on $H_\infty(V)$ will be denoted by $\mathfrak{M}(H_\infty(V))$. It follows from the Gelfand representation theorem that $H_\infty(V)$ is isometric to a subalgebra of $C(\mathfrak{M}H_\infty(V))$. This isometry will be denoted by "$\wedge$", i.e. if $f \in H_\infty(V)$, then $\hat{f}$ denotes the same function considered as a function on $\mathfrak{M}H_\infty(V)$. Since V can be embedded into $\mathfrak{M}H_\infty(V)$ (the point $t \in V$ corresponds to the linear multiplicative functional "value at t"), one can think of $\hat{f}$ as a certain extension of f to $\mathfrak{M}H_\infty(V)$.

If $t = (t_1, t_2, \ldots, t_n) \in \bar{V} - V$, then the fibre $\mathfrak{M}_t$ over t is defined as

$$\mathfrak{M}_t = \{\varphi \in \mathfrak{M}H_\infty(V)\colon \varphi(z_i) = t_i \text{ for } i = 1, 2, \ldots, n\}.$$

The continuous map $\Phi\colon \mathfrak{M}H_\infty(V) \to \mathfrak{M}H_\infty(V)$ is said to be *analytic* if for every $f \in H_\infty(V)$ the composition $\hat{f} \circ \Phi$ is in $H_\infty(V)$.

In this paper we will frequently use the following Rudin–Carleson theorem (cf. [5], [11]): *Let $\Delta \subset S$ be a closed set of Lebesgue measure* 0 *and let $f \in C(\Delta)$. Then there exists an $\tilde{f} \in A$ such that $\tilde{f}|\Delta = f$ and $\|\tilde{f}\| = \|f\|$. Moreover, $\tilde{f}$ can be chosen in such a way that for every $z \in \bar{U}$, $z \notin \Delta$, we have $|\tilde{f}(z)| < \|f\|$.*

Our general reference about analytic functions of one complex variable is Hoffman [11]. For functions of several complex variables the reader can consult Rudin [20]. The reference about Banach spaces is Lindenstrauss–Tzafriri [14].

1. In this section we prove the existence of linear extension operators from certain subsets of the fibre of H_∞. Our main results are based on the analysis of the conformal map $L\colon \bar{U} \to \bar{U}$ defined by

$$L(z) = \frac{z + i(z-1)}{1 + i(z-1)}.$$

We will consider iterations of this map, $L^n(z)$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ It is easily checked that

$$L^n(z) = \frac{z + ni(z-1)}{1 + ni(z-1)}.$$

The following lemma is known (cf. [11], [21]) and easy to prove.

LEMMA 1.0. (a) $L(1) = 1$;

(b) *for every $z \in \bar{U}$, $L^n(z) \to 1$ as $n \to \infty$ and $L^n(z) \to 1$ as $n \to -\infty$;*

(c) *for every compact $K \subset U$ there exists a constant C_K such that*

$$\sup\{|L^n(z) - 1|\colon z \in K\} \leqslant C_K |n|^{-1} \quad \textit{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

We begin our considerations with the following

LEMMA 1.1. *Let $(n_k)_{k=1}^\infty$ be a lacunary sequence of integers, i.e. $n_{k+1}/n_k \geqslant \lambda > 1$. Let us define $h(z) = z\prod_{k=0}^{\infty} L^{-n_k}(z)$. Then $h \in H_\infty$, $\|h\| = 1$ and*

(1) $\quad |h(L^{n_k}(z)) - z| \to 0$ *as* $k \to \infty$ *uniformly on each compact subset of the disc U.*

Proof. By Lemma 1.0 (c) we infer that the product defining h is almost uniformly convergent, and so h is not identically zero, belongs to H_∞ and $\|h\| \leqslant 1$. Let us fix a compact subset $K \subset U$. For $z \in K$ we have

$$\begin{aligned}
\big|h(L^{n_k}(z)) - z\big| &= \Big|L^{n_k}(z)\prod_{s=0}^{k-1} L^{n_k - n_s}(z)\, z \prod_{s=k+1}^{\infty} L^{n_k - n_s}(z) - z\Big| \\
&= |z|\,\Big|L^{n_k}(z)\prod_{s=0}^{k-1} L^{n_k - n_s}(z) \prod_{s=k+1}^{\infty} L^{n_k - n_s}(z) - 1\Big| \\
&\leqslant |z|\Big[|L^{n_k}(z) - 1| + \sum_{s=0}^{k-1} |L^{n_k - n_s}(z) - 1| + \sum_{s=k+1}^{\infty} |L^{n_k - n_s}(z) - 1|\Big] \\
&\leqslant |z|\Big(C_K|n_k|^{-1} + \sum_{s=0}^{k-1} C_K|n_k - n_s|^{-1} + \sum_{s=k+1}^{\infty} C_K|n_s - n_k|^{-1}\Big) \\
&\leqslant C_K\Big(|n_k|^{-1} + k|n_k - n_{k-1}|^{-1} + |n_k|^{-1}\sum_{s=k+1}^{\infty} |n_s/n_{k-1}|^{-1}\Big).
\end{aligned}$$

Since $|n_k - n_{k-1}| \geq |\lambda n_{k-1} - n_{k-1}| = |n_{k-1}|(\lambda-1)$ and $n_s/n_k \geq \lambda^{s-k}$, we have for $z \in K$

$$h\left|\left(L^{n_k}(z)\right) - z\right| \leq C_k\Big(|n_k|^{-1} + k\left(|n_{k-1}|(\lambda-1)\right)^{-1} + |n_k|^{-1} \sum_{s=k+1}^{\infty} (\lambda^{s-k}-1)^{-1}\Big).$$

But $|n_{k-1}| \geq \lambda^{k-1}$ and $\lambda > 1$, and so we see that the right-hand side of the inequality tends to zero as k tends to infinity; thus (1) is proved. It is easily seen that (1) implies $\|h\| \geq 1$; hence $\|h\| = 1$ and the lemma is proved.

PROPOSITION 1.2. *Let (n_k) be a lacunary sequence of integers and let $h(z)$ be defined as in Lemma 1.1 Then*

(a) *$A\colon H_\infty \to H_\infty$ defined by $A(f) = f \circ h$ is a multiplicative isometry of H_∞ into itself;*

(b) *There exists an operator $T\colon H_\infty \to H_\infty$ such that T is multiplicative and onto and for every $f \in H_\infty$ $T(f)$ is the almost uniform limit of some subsequence of the sequence $(f \circ L^{n_k})_{k=1}^{\infty}$;*

(c) *$P = AT$ is a multiplicative projection from H_∞ onto the image of A.*

(d) *$P^*|\mathfrak{M}(H_\infty)$ is an analytical retraction from $\mathfrak{M}(H_\infty)$ into the fibre $\mathfrak{M}_1(H_\infty)$ and $P^*(\mathfrak{M}(H_\infty))$ is homeomorphic to $\mathfrak{M}(H_\infty)$.*

Proof. Part (a) is clear since $h(U) = U$. Since for every $f \in H_\infty$ the sequence $(f \circ L^{n_k})$ is a uniformly bounded sequence, it contains (by Montel's theorem) an almost uniformly convergent subsequence. Hence, using the standard technique of generalized limits (cf. for example [16]), we infer that there exists a linear and multiplicative operator $T\colon H_\infty \to H_\infty$ such that, for every $f \in H_\infty$, Tf is the limit of an almost uniformly convergent subsequence of $(f \circ L^{n_k})$. To prove that T is onto let us consider the operator TA. For an $f \in H_\infty$ we have $T \circ A(f) = T(f \circ h) = \lim f \circ h \circ L^{n_{s_k}} = f$. The last equality follows from Lemma 1.1. Hence $T \circ A = \mathrm{id}$ and T is onto, which proves (b).

To see (c) let us observe that $P^2 = ATAT = AT$ since $TA = \mathrm{id}$. Clearly $\operatorname{Im} P = \operatorname{Im} A$.

Since P is multiplicative, $P^*(\mathfrak{M}(H_\infty)) \subset \mathfrak{M}(H_\infty)$ and since $P = P^2$, $P^*|\mathfrak{M}(H_\infty)$ is a retraction, and obviously an analytic one. The image of $P^*|\mathfrak{M}(H_\infty)$ can be identified with $\mathfrak{M}(\operatorname{Im} P)$, but the algebra $\operatorname{Im} P = A(H_\infty)$ is homomorphically isometric with H_∞; hence $P^*(\mathfrak{M}(H_\infty))$ is homeomorphic to $\mathfrak{M}(H_\infty)$. Now we show that $P^*(\mathfrak{M}(H_\infty)) \subset \mathfrak{M}_1(H_\infty)$. Let us take $\varphi \in \mathfrak{M}(H_\infty)$. Then $P^*(\varphi)(z) = T^*A^*(\varphi)(z) = T^*\left(A^*(\varphi)\right)(z) = A^*(\varphi) \times \left(T(z)\right) = A^*(\varphi)(1) = 1$ since $T(z) = \lim_s L^{n_{s_k}} = 1$.

This completes the proof.

Since the rotations of U induce homeomorphisms of $\mathfrak{M}(H_\infty)$ onto $\mathfrak{M}(H_\infty)$ which transform fibres onto fibres, and for every $\alpha \in S$ there exists a rotation which transforms $\mathfrak{M}_\alpha(H_\infty)$ onto $\mathfrak{M}_1(H_\infty)$, we have

COROLLARY 1.3. *For every $\alpha \in S$ there is an analytic retraction $r_\alpha\colon \mathfrak{M}(H_\infty) \to \mathfrak{M}(H_\infty)$ such that $\operatorname{Im} r_\alpha \subset \mathfrak{M}_\alpha(H_\infty)$ and $\operatorname{Im} r_\alpha$ is homeomorphic to $\mathfrak{M}(H_\infty)$.*

Remark 1.4. (a) Since for $\varphi \in \mathfrak{M}(H_\infty)$ and $f \in H_\infty$, $P(f)(\varphi) = \hat{f}(P^*(\varphi))$ and $P^*|\mathfrak{M}(H_\infty)$ is a retraction on $\mathfrak{M}(H_\infty)$, P is a multiplicative linear extension operator from $P^*(\mathfrak{M}(H_\infty))$ to $\mathfrak{M}(H_\infty)$. This point of view will be employed in the rest of the present paper.

(b) Proposition 1.2 is an improvement of the result of Schark [21] (cf. also [11], pp. 167–168). Our proof is a modification of Schark's proof.

Suppose that we have an analytic retraction $r\colon \mathfrak{M}(H_\infty) \to \mathfrak{M}(H_\infty)$ such that $\operatorname{Im} r \subset \mathfrak{M}_\alpha(H_\infty)$ for a certain $\alpha \in S$ and such that $\hat{H}_\infty|\operatorname{Im} r$ is isometric to H_∞. Suppose also that we have a function $g \in H_\infty$, $\|g\| = 1$, such that $\hat{g}|\operatorname{Im} r = 1$. Then we can define a projection P by $P(f) = \hat{g} \cdot \hat{f} \circ r$. Such projections will be called *elementary projections*. If we want to emphasise how the projection is built, we say that it is an elementary projection over a point α or an elementary projection given by the retraction r and the function g.

PROPOSITION 1.5. *The space H_∞ contains a complemented subspace isomorphic to $\left(\sum H_\infty\right)_\infty$.*

Proof. Let us choose a sequence of points $(\alpha_n) \subset S$ with $\alpha_n \to 1$. Let $\Delta_n = \{z \in \overline{U}\colon |z-\alpha_n| \leq \varepsilon_n\}$ where ε_n are chosen in such a way that the Δ_n's are pairwise disjoint. Let us take a sequence of functions $(f_n) \in A$ such that

$$f_n(\alpha_m) = \begin{cases} 1 & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \neq m, \end{cases} \tag{2}$$

$$\|f_n\| = 1, \tag{3}$$

$$|f_n(z)| < 2^{-n-1} \quad \text{for} \quad z \notin \Delta_n. \tag{4}$$

The possibility of such choice follows from the Rudin–Carleson theorem. Let r_n be the retraction given by Corollary 1.3 for the point α_n. Our projection will be the sum of elementary projections defined by r_n and f_n, i.e. for $f \in H_\infty$ we define

$$Pf = \sum_{n=1}^{\infty} f_n \cdot (\hat{f} \circ r_n).$$

Since any compact set $K \subset U$ intersects only a finite number of Δ_n's, condition (4) implies that the series $\sum_{n=1}^{\infty} f_n \cdot (\hat{f} \circ r_n)$ is almost uniformly convergent. Moreover,

$$|Pf(z)| \leq \sum_{n=1}^{\infty} |f_n(z)|\,|(\hat{f} \circ r_n)(z)| \leq \|f\| \sum_{n=1}^{\infty} |f_n(z)| \leq 2\|f\|, \tag{5}$$

and so $P(f) \in H_\infty$ and P is a linear operator of norm less than or equal to 2. We claim that P is a projection and $\operatorname{Im} P \sim (\sum H_\infty)_\infty$.

We know by Corollary 1.3 that, for every n, $\hat{H}_\infty | \operatorname{Im} r_n$ is isometric with H_∞, and so it is enough to prove that $\operatorname{Im} P \sim (\sum \hat{H}_\infty | \operatorname{Im} r_n)_\infty$. We define an operator $i\colon (\sum \hat{H}_\infty | \operatorname{Im} r_n)_\infty \to H_\infty$ by $i((g_n)) = \sum_{n=1}^{\infty} f_n \cdot (g_n \circ r_n)$. As in (5), we show that $\|i\| \leqslant 2$. Moreover, $\operatorname{Im} P \subset \operatorname{Im} i$, because $P(f) = i((\hat{f} | \operatorname{Im} r_n))$ for every $f \in H_\infty$. Our claim will be proved if we show that $P(i((g_n))) = i((g_n))$.

$$P\big(i((g_n))\big) = P\Big(\sum_{n=1}^{\infty} f_n \cdot (g_n \circ r_n)\Big) = \sum_{k=1}^{\infty} f_k \Big(\overline{\sum_{n=1}^{\infty} f_n \cdot (g_n \circ r_n)}\Big) \circ r_k$$
$$= \sum_{k=1}^{\infty} f_k \cdot \Big(\hat{f}_k \cdot (\widehat{g_k \circ r_k}) + \overline{\sum_{\substack{n=1 \\ n \neq k}}^{\infty} f_n \cdot (g_n \circ r_n)}\Big) \circ r_k$$
$$= \sum_{k=1}^{\infty} f_k \cdot \big(\hat{f}_k \cdot (\widehat{g_k \circ r_k}) \circ r_k\big) + \sum_{k=1}^{\infty} f_k \Big(\overline{\sum_{\substack{n=1 \\ n \neq k}}^{\infty} f_n \cdot (g_n \circ r_n)}\Big) \circ r_k.$$

For $z \in \Delta_k$ we have

$$\Big| \sum_{n \neq k, n=1}^{\infty} f_n(z) (g_n \circ r_n)(z) \Big| \leqslant \sum_{n \neq k, n=1}^{\infty} |f_n(z)| \, \|g_n\| \leqslant \sup \|g_n\| \sum_{k \neq n=1}^{\infty} |f_n(z)|.$$

Conditions (2) and (4) imply that $v(z) = \sum_{k \neq n=1}^{\infty} |f_n(z)|$ is a continuous function on Δ_k and $v(a_k) = 0$. Hence for an arbitrary sequence $z_j \to a_k$ we have

$$\lim_j \sum_{k \neq n=1}^{\infty} f_n(z_j) (g_n \circ r_n)(z_j) = 0.$$

But by [11], p. 162, it implies that

$$\overline{\sum_{k \neq n=1}^{\infty} f_n \cdot (g_n \circ r_n)} | \mathfrak{M}_{a_k} = 0.$$

On the other hand, $\widehat{g_k \circ r_k} | \operatorname{Im} r_k = g_k$ and $\hat{f}_k | \mathfrak{M}_{a_k} = 1$, and so

$$P\big(i((g_n))\big) = \sum_{k=1}^{\infty} f_k (g_k \circ r_k) = i\big((g_n)\big).$$

This completes the proof of the proposition.

The standard decomposition method gives us (cf. [15])

COROLLARY 1.6. *The space* H_∞^n *is isomorphic to* $(\sum H_\infty)_\infty$.

In [2] S. V. Bočkariov constructed an orthonormal system (b_n) on S which is a Schauder basis for A. The orthogonality implies that (b_n) is also a basis for the space $L_1(\lambda)/H_1^0(\lambda)$. It is well known (cf. [10]) that $[L_1(\lambda)/H_1^0(\lambda)]^* = H_\infty$. Let us define the space $B = (\sum X_n)_1$, where $X_n = \operatorname{span}\{b_1, \ldots, b_n\}$ in $L_1(\lambda)/H_1^0(\lambda)$. Now we can define the operator $T\colon B \xrightarrow{\text{onto}} L_1(\lambda)/H_1^0(\lambda)$ by the formula $T((x_n)) = \sum_{n=1}^{\infty} x_n$. Let us define for every n an operator τ_n which isometrically transforms X_n considered as a subspace of $L_1(\lambda)/H_1^0(\lambda)$ onto X_n naturally embedded into B, and satisfies the condition $T\tau_n(x) = x$ for every $x \in X_n \subset L_1(\lambda)/H_1^0(\lambda)$. In this situation Proposition 1 of [12] (cf. also [23]) implies that $T^*([L_1(\lambda)/H_1^0(\lambda)]^*)$ is norm 1-complemented in B^*. But T^* is an isometric embedding, and hence we infer that $B^* = (\sum X_n^*)_\infty$ contains a 1-complemented copy of H_∞. On the other hand, X_n^* are uniformly complemented in H_∞ because (b_n) is a basis for the predual of H_∞. This implies that $(\sum X_n^*)_\infty$ is complemented in $(\sum H_\infty)_\infty$ by coordinatewise projection. Corollary 1.6 and the decomposition method give us $H_\infty \sim (\sum X_n^*)_\infty$. But X_n^* are uniformly isomorphic to $H_\infty^n = \operatorname{span}\{b_1, b_2, \ldots, b_n\}$ considered as a subspace of A. We can summarise our observations in

THEOREM 1.7. *The space* H_∞ *is isomorphic to* $(\sum H_\infty)_\infty$ *and to* $(\sum H_\infty^n)_\infty$ *where* H_∞^n *is the span of the first* n *elements of the Bočkariov basis for the disc algebra* A.

Remark 1.8. The proof of Proposition 1 of [12] is a compactness argument. If we neglect the good estimates for the norm of the projection, we can make R a cluster point of the sequence τ_n^* in the topology of w^*-pointwise convergence of operators from B^* into H_∞ (here we consider τ_n as an operator from $L_1(\lambda)/H_1^0(\lambda)$) and T^*R is the required projection.

COROLLARY 1.9. H_∞ *is isomorphic to the second dual of the Banach space* $(\sum H_\infty^n)_{c_0}$.

This corollary answers the question of Rickart asked in [22].

Now we want to show that isometrically H_∞ is not a second dual space. We will consider H_∞ as a subalgebra of L_∞ and we will identify L_∞ with $C(\mathfrak{M}(L_\infty))$, and so H_∞ is a closed subalgebra of $C(\mathfrak{M}(L_\infty))$ separating points. We will identify the Lebesgue measure λ with the measure it induces on $\mathfrak{M}(L_\infty)$.

A subset $K \subset \mathfrak{M}(L_\infty)$ is called a *peak set for* H_∞ if there exists a function $f \in H_\infty$ such that $f(t) = 1$ for $t \in K$ and $|f(t)| < 1$ for $t \notin K$. A point which is the intersection of a family of peak sets is called a *p-point*. The following three known facts will be used in the proof of Theorem 1.11.

PROPOSITION 1.10. (a) *If* $t \in \mathfrak{M}(L_\infty)$ *is a p-point for* H_∞ *and* V *is an* *open neighbourhood of* t, *then there exists a function* $f \in H_\infty$ *such that* $f(t) = 1 = \|f\|$ *and* $|f(s)| < 1$ *for* $s \notin V$.

(b) *If* K *is a closed subset of* $\mathfrak{M}(L_\infty)$ *with* $\lambda(K) = 0$, *then there exists a peak set for* H_∞, $\tilde{K}$, *such that* $\tilde{K} \supset K$ *and* $\lambda(\tilde{K}) = 0$.

(c) *If* $f \in H_\infty$ *and* $f(t) = 0$ *for all* p-*points* t, *then* $f = 0$.

Part (a) is a special case of Theorem II.11.1 of [5] (it is explained in [5], II, § 12, that this theorem is valid also for p-points), part (b) is a result of Amar–Lederer [1] and part (c) is a special case of Theorem II.12.10 of [5].

THEOREM 1.11. *The predual of* H_∞ *unique up to isometry is* $L_1(\lambda)/H_1^0(\lambda)$.

This theorem answers the question of P. Porcelli, Studia Mathematica 38, problem 59.

Proof. If X^* is isometric to H_∞, then we can identify X with a certain subspace of H_∞^*. Let us take an arbitrary $\mu \in X$, $\|\mu\| = 1$ and by the Hahn–Banach and the Riesz representation theorems let us extend it to a measure $\tilde{\mu}$ on $\mathfrak{M}(L_\infty)$, $\|\tilde{\mu}\| = 1$. Let us consider the Lebesgue decomposition $\tilde{\mu} = fd\lambda + \tilde{\mu}_s$, where $\tilde{\mu}_s$ is singular with respect to the Lebesgue measure and

$$1 = \|\tilde{\mu}\| = \int |f|\,d\lambda + \|\tilde{\mu}_s\|. \tag{6}$$

Let us suppose $\tilde{\mu}_s \neq 0$. Since $X^* = H_\infty$, there exists a $g \in H_\infty$, $\|g\| = 1$ such that $\mu(g) = 1$. By the regularity of $\tilde{\mu}_s$ we can find a compact set $K \subset \mathfrak{M}(L_\infty)$ with $\lambda(K) = 0$ such that $\|\tilde{\mu}_s | K\| \geqslant \|\tilde{\mu}_s\|/2$. In view of Proposition 1.10(b) we may assume that K is a peak set for H_∞. Let $\varphi \in H_\infty$ be a function peaking on K and let r be a natural number such that

$$\Big|\int g\varphi^r\,d\tilde{\mu}_s\Big| \geqslant \|\tilde{\mu}_s\|/2, \tag{7}$$

$$\Big|\int g\varphi^r f\,d\lambda\Big| < \|\tilde{\mu}_s\|/2. \tag{8}$$

Such choice is possible because $\int g\varphi^n f\,d\lambda \to 0$ and

$$\int g\varphi^n\,d\tilde{\mu}_s \to \int_K g\,d\tilde{\mu}_s = \|\tilde{\mu}_s | K\|.$$

The last equality follows from (6).

The sequence $\varphi^n g$ is contained in the unit ball of H_∞, and so it has a $\sigma(H_\infty, X)$-cluster point ζ, $\zeta \in H_\infty$ and $\|\zeta\| \leqslant 1$. Observe that

$$\tilde{\mu}(\varphi^n g) = \int \varphi^n g f\,d\lambda + \int \varphi^n g\,d\tilde{\mu}_s \to \|\tilde{\mu}_s | K\| \geqslant \|\tilde{\mu}_s\|/2 \quad \text{as} \quad n \to \infty.$$

This implies that $\zeta \neq 0$.

Let t be a p-point, $t \notin K$. We claim that $\zeta(t) = 0$. Suppose $\zeta(t) \neq 0$. Then there exist an open neighbourhood V of t, $V \cap K = \emptyset$ and a function

$p \in H_\infty$ such that $\|p\| = p(t) = 1$ and $|p(s)| < \|\tilde{\mu}_s\|/2$ for all $s \notin V$ (by Proposition 1.10(a)). Since the unit ball of X is $\sigma(H_\infty^*, H_\infty)$ dense in the unit ball of H_∞^*, for an arbitrarily small $\beta > 0$ we can find $\gamma \in X$ with $\|\gamma\| \leqslant 1$ such that

$$|\delta_t(1) - \gamma(1)| = |1 - \gamma(1)| < \beta, \tag{9}$$

$$|\delta_t(p) - \gamma(p)| = |1 - \gamma(p)| < \beta, \tag{10}$$

$$|\delta_t(\zeta) - \gamma(\zeta)| = |\zeta(t) - \gamma(\zeta)| < \beta. \tag{11}$$

Let us extend γ to a measure $\tilde{\gamma}$ with $\|\tilde{\gamma}\| = \|\gamma\|$. Then we have

$$\begin{aligned} 1-\beta \leqslant |\gamma(p)| &= \Big|\int p\,d\tilde{\gamma}\Big| \leqslant \Big|\int_V p\,d\tilde{\gamma}\Big| + \Big|\int_{\mathfrak{M}(L_\infty)\setminus V} p\,d\tilde{\gamma}\Big| \\ &\leqslant \|\tilde{\gamma} | V\| + \tfrac{1}{2}\|\tilde{\mu}_s\|\,\|\tilde{\gamma} | \mathfrak{M}(L_\infty)\setminus V\| \\ &\leqslant 1 - \|\tilde{\gamma} | \mathfrak{M}(L_\infty)\setminus V\| + \|\tilde{\mu}_s\|\tfrac{1}{2}\|\tilde{\gamma} | \mathfrak{M}(L_\infty)\setminus V\|; \end{aligned}$$

so

$$\|\tilde{\gamma} | \mathfrak{M}(L_\infty)\setminus V\| \leqslant \beta(1 - \|\tilde{\mu}_s\|/2)^{-1}.$$

Now observe that since $\gamma \in X$, the sequence $\gamma(\varphi^n g) = \int \varphi^n g\,d\tilde{\gamma}$ has a subsequence convergent to $\gamma(\zeta) = \int \zeta\,d\tilde{\gamma}$. On the other hand,

$$\int \varphi^n g\,d\tilde{\gamma} \to \int_K g\,d\tilde{\gamma} \quad \text{as} \quad n \to \infty,$$

so

$$\int_K g\,d\tilde{\gamma} = \int \zeta\,d\tilde{\gamma}.$$

But

$$\Big|\int_K g\,d\tilde{\gamma}\Big| \leqslant \int_{\mathfrak{M}(L_\infty)\setminus V} |g|\,d|\tilde{\gamma}| \leqslant \|\tilde{\gamma} | \mathfrak{M}(L_\infty)\setminus V\| \leqslant \beta(1 - \|\tilde{\mu}_s\|/2)^{-1},$$

and so

$$|\gamma(\zeta)| = \Big|\int \zeta\,d\tilde{\gamma}\Big| \leqslant \beta(1 - \|\tilde{\mu}_s\|/2)^{-1}.$$

For small β this contradicts (11). This contradiction shows that $\zeta(t) = 0$ for an arbitrary p-point outside K.

Now the function $\zeta(\varphi - 1)$ is zero at all p-points, and so $\zeta(\varphi - 1) = 0$ by Proposition 1.10(c). Since $\varphi - 1 \neq 0$, we infer that $\zeta = 0$. This contradiction shows that $\tilde{\mu}_s = 0$; hence every norm-preserving extension of $\mu \in X$ is a measure that is absolutely continuous with respect to the Lebesgue measure. This implies that X is a subspace of $L_1(\lambda)/H_1^0(\lambda)$, but obviously it has to be equal to the whole $L_1(\lambda)/H_1^0(\lambda)$.

COROLLARY 1.12. *H_∞ is not isometric to the second dual of any Banach space.*

Proof. The corollary follows from the fact that $L_1(\lambda)/H_1^0(\lambda)$ is not isomorphic to a conjugate space. The argument for this fact runs as follows: Since $H_1^0(\lambda)$ is a separable dual, it does not contain a subspace isomorphic to L_1 by Gelfand's theorem (cf. [8]). Corollary 4 of [25] shows that $L_1(\lambda)/H_1^0(\lambda)$ does not have the Radon–Nikodym property, and so it is not a subspace of a separable conjugate space.

Remark 1.13. The results of this section show the analogy between H_∞ and L_∞. It is well known (cf. [13]) that L_∞ has isometrically a unique predual L_1 which is not a subspace of a separable conjugate space. On the other hand, L_∞ is isomorphic to l_∞, which is the second dual of the space c_0. This indicates that the space $(\sum H_\infty^n)_{c_0}$ should play in the linear theory of spaces of analytic functions an analogous role to that played by c_0 in the theory of classical Banach spaces. Recently F. Delbaen has proved that the disc algebra A contains a complemented subspace isomorphic to $(\sum H_\infty^n)_{c_0}$.

Remark 1.14. Using the fact that there exists an uncountable number of non-isomorphic separable spaces of the form $C(\alpha)$, α being a countable ordinal number, $C(\alpha)^* = l_1$, one can easily construct an uncountable family of non-isomorphic spaces whose second duals are isomorphic to H_∞.

2. In this section we will prove that the general linear group of H_∞, $\mathrm{GL}(H_\infty)$, is contractible. The general scheme for proving the contractibility of the general linear group of a given Banach space was developed by B. S. Mitiagin in [17]. He proved that $\mathrm{GL}(X)$ is contractible if X has two properties called by him ID and SOB. The property ID is satisfied, in particular, if $X \sim (\sum X)_\infty$; so the space H_∞ satisfies the ID by Theorem 1.7. This section will be devoted to the proof that H_∞ satisfies the SOB.

DEFINITION 2.1. The Banach space X *satisfies the property* SOB if for every compact $\mathcal{K} \subset L(X, X)$, where $L(X, X)$ is the space of all linear bounded operators on X, and for an arbitrary $\varepsilon > 0$ there exist projections $Q_1, Q_2\colon X \to X$ such that

$$Q_1Q_2 = Q_2Q_1 = 0, \tag{1}$$

$$\mathrm{Im}\,Q_1 \sim \mathrm{Im}\,Q_2 \sim X, \tag{2}$$

$$\|Q_1BQ_2\| < \varepsilon \quad \text{for every } B \in \mathcal{K}. \tag{3}$$

Let us introduce the following property SOB′:

DEFINITION 2.2. The Banach space X *satisfies the property* SOB′ if for every compact $\mathcal{K} \subset L(X, X)$ there exists a constant C such that for every $\varepsilon > 0$ there exist projections $Q_1, Q_2\colon X \to X$ such that

$$\mathrm{Im}\,Q_1 \sim \mathrm{Im}\,Q_2 \sim X, \tag{4}$$

$$\|Q_1\| \leqslant C \quad \text{and} \quad \|Q_2\| \leqslant C, \tag{5}$$

$$Q_2Q_1 = 0 \quad \text{and} \quad \|Q_1Q_2\| \leqslant \varepsilon, \tag{6}$$

$$\|Q_1BQ_2\| < \varepsilon \quad \text{for every } B \in \mathcal{K}. \tag{7}$$

LEMMA 2.3. *The property* SOB′ *implies the property* SOB.

Proof. We may assume that $\varepsilon < 1$. Then let us introduce the projection

$$\tilde{Q}_1 = (I + Q_1Q_2)Q_1(I - Q_1Q_2) = Q_1 - Q_1Q_2.$$

We show that the projections $\tilde{Q}_1$ and Q_2 satisfy (1)–(3).

$$\tilde{Q}_1Q_2 = Q_1Q_2 - Q_1Q_2 = 0,$$
$$Q_2\tilde{Q}_1 = Q_2Q_1 - Q_2Q_1Q_2 = 0$$

by (6).

$\mathrm{Im}\,\tilde{Q}_1$ is isomorphic to $\mathrm{Im}\,Q_1$ because $I - Q_1Q_2$ is an isomorphism of X. For $B \in \mathcal{K}$ we have

$$\|\tilde{Q}_1BQ_2\| = \|(Q_1 - Q_1Q_2)BQ_2\| \leqslant \|Q_1BQ_2\| + \|Q_1Q_2BQ_2\| \leqslant \varepsilon(1 + C^3 \sup_{B \in \mathcal{K}} \|B\|).$$

This completes the proof of the lemma.

For technical reasons we prefer to check that H_∞ has the property SOB′ rather than the property SOB.

We start with a detailed analysis of the behaviour of the maps L^n restricted to the circle S. This analysis leads to Theorem 2.14. Then we use Theorem 2.14 to prove the property SOB′.

It is clear that $|L^n(e^{i\theta})| = 1$ and $L^n|S$ is a homeomorphism of S onto itself. We define a function $\varphi_n\colon [0, 2\pi] \to [0, 2\pi]$ by the formula

$$L^n(e^{i\theta}) = e^{i\varphi_n(\theta)}. \tag{8}$$

LEMMA 2.4. *We have*

$$\varphi_n'(\theta) = \left(1 + 2n^2 - \sqrt{4n^4 + 4n^2}\cos(\theta - \gamma_n)\right)^{-1}$$

where

$$\sin\gamma_n = \frac{1}{\sqrt{n^2+1}} \quad \textit{and} \quad \cos\gamma_n = \frac{n}{\sqrt{n^2+1}};$$

so $\gamma_n = \operatorname{sgn} n \arcsin\left(\sqrt{n^2+1}\right)^{-1}$.

Proof. We write

$$\frac{e^{i\theta} + ni(e^{i\theta} - 1)}{1 + ni(e^{i\theta} - 1)} = e^{i\varphi_n(\theta)} \tag{9}$$

and differentiate both sides with respect to θ. We get

$$\frac{d}{d\theta}\left[\frac{e^{i\theta}+ni(e^{i\theta}-1)}{1+ni(e^{i\theta}-1)}\right] = i\varphi_n'(\theta)e^{i\varphi_n(\theta)}.$$

If we substitute for $e^{i\varphi_n(\theta)}$ the value given by equation (9), differentiate the left-hand side and solve the equation with respect to $\varphi_n'(\theta)$, we get, after standard computations, the required result.

COROLLARY 2.5. $\max\limits_{\theta}\varphi_n'(\theta) \leqslant 9n^2$.

Proof. This maximum is clearly equal to

$$\left(1+2n^2-\sqrt{4n^4+4n^2}\right)^{-1} = \left(\sqrt{n^2+1}-|n|\right)^{-2} = \left(\sqrt{n^2+1}+|n|\right)^2 \leqslant 9n^2.$$

LEMMA 2.6. *We have*

$$\lim_{n\to\infty}\frac{\gamma_n-1/n}{1/n^2} = \lim_{n\to-\infty}\frac{\gamma_n-1/n}{1/n^2} = 0.$$

Proof. Since $\gamma_{-n} = -\gamma_n$, it is enough to check that $\lim\limits_{n\to\infty}\dfrac{\gamma_n-1/n}{1/n^2} = 0$. But this limit equals

$$\lim_{x\to\infty}\frac{\arcsin\dfrac{1}{\sqrt{x^2+1}}-\dfrac{1}{x}}{1/x^2} = 0$$

by the application of de l'Hospital's rule.

LEMMA 2.7. *For every positive constant C there exists a natural number k such that*

$$|\gamma_{\operatorname{sgn} n(|n|+k)}-\gamma_n| \geqslant Cn^{-2} \quad \textit{for } n \textit{ big enough}.$$

Proof. Let us take a natural number k and a positive ε such that $k-2C > 4\varepsilon$. To simplify the notation, we will assume that n is a natural number. Observe that for n big enough we have by Lemma 2.6

$$-\varepsilon n^{-2} \leqslant \gamma_n - n^{-1} \leqslant \varepsilon n^{-2};$$

hence

$$n^{-1}-\varepsilon n^{-2} \leqslant \gamma_n \leqslant n^{-1}+\varepsilon n^{-2}.$$

Using these inequalities, we have

$$|\gamma_{n+k}-\gamma_n| \geqslant \gamma_n-\gamma_{n+k} \geqslant n^{-1}-\varepsilon n^{-2}-(n+k)^{-1}-\varepsilon(n+k)^{-2}$$
$$= \frac{k}{n(n+k)}-\varepsilon\left(\frac{1}{n^2}+\frac{1}{(n+k)^2}\right) \geqslant \frac{k}{2n^2}-\frac{2\varepsilon}{n^2} = \left(\frac{k}{2}-2\varepsilon\right)n^{-2} \geqslant \frac{C}{n^2}.$$

LEMMA 2.8. *We have*

$$\int\frac{d\theta}{1+2n^2-\sqrt{4n^4+4n^2}\cos(\theta-\gamma_n)}$$
$$= 2\arctan\left[\left(1+2n^2+\sqrt{4n^4+4n^2}\right)\tan\frac{\theta-\gamma_n}{2}\right].$$

Proof. We first substitute $x = \theta-\gamma_n$ and next $t = \tan(x/2)$, and the integral reduces to

$$\int\frac{dt}{\left(1+2n^2+\sqrt{4n^4+4n^2}\right)t^2+1+2n^2-\sqrt{4n^4+4n^2}}.$$

This integral can easily be calculated.

Let us recall that a point $t\in S$ is a point of density of a set $\Delta\subset S$ if

$$\lim_{h\to 0}\frac{\lambda\left((t-h,t+h)\cap\Delta\right)}{2h} = 1.$$

PROPOSITION 2.9. *Let $\Delta\subset S$ be a measurable subset such that 1 is a point of density of Δ. Then* $\limsup\limits_{n\to\infty}\lambda\left(L^n(\Delta)\right) = 1$.

Proof. Suppose that our claim is false, i.e. that there exists a $\delta > 0$ such that $\lambda\left(L^n(\Delta)\right) < 1-2\delta$ for all natural n. We can find an ε_n such that

$$\lambda\left(L^n(\gamma_n-\varepsilon_n,\gamma_n+\varepsilon_n)\right) = 1-\delta.$$

Since

$$\lambda\left(L^n(\gamma_n-\varepsilon_n,\gamma_n+\varepsilon_n)\right) = \frac{1}{2\pi}\int_{\gamma_n-\varepsilon_n}^{\gamma_n+\varepsilon_n}\varphi_n'(\theta)\,d\theta,$$

we infer by Lemma 2.8 that ε_n is given by the condition

$$2\arctan\left[\left(1+2n^2+\sqrt{4n^4+4n^2}\right)\tan(\varepsilon_n/2)\right] = \pi(1-\delta).$$

If we denote $\tan\dfrac{\pi(1-\delta)}{2}$ by M, we can write

$$\tan\frac{\varepsilon_n}{2} = \frac{M}{1+2n^2+\sqrt{4n^4+4n^2}}.$$

This implies that there exists a constant C such that for all $n\neq 0$ $\varepsilon_n \leqslant C/n^2$. Lemma 2.7 shows that there exists a natural number k such that for n big enough the intervals $I_n = (\gamma_{nk}-\varepsilon_{nk},\gamma_{nk}+\varepsilon_{nk})$ are disjoint. We have $\lambda\left(L^{nk}(I_n)\right) = 1-\delta$ and $\lambda\left(L^{nk}(\Delta)\right) < 1-2\delta$; so $\lambda\left(L^{nk}(I_n\setminus\Delta)\right) > \delta$. Corollary 2.5 shows that $\lambda(I_n\setminus\Delta) > \delta/9k^2n^2$. Observe also that there exists a constant C_1 such that $\gamma_{nk}+\varepsilon_{nk} \leqslant C_1/n$.

Now we are ready to estimate

$$\frac{\lambda\big((0,\gamma_{nk}+\varepsilon_{nk})\cap\Delta\big)}{\gamma_{nk}+\varepsilon_{nk}} \leqslant \frac{\lambda\big((0,\gamma_{nk}+\varepsilon_{nk})\setminus(\bigcup_{r=1}^{\infty} I_r\setminus\Delta)\big)}{\gamma_{nk}+\varepsilon_{nk}}$$
$$\leqslant \frac{\gamma_{nk}+\varepsilon_{nk}-\sum_{r=n}^{\infty}\lambda(I_r\setminus\Delta)}{\gamma_{nk}+\varepsilon_{nk}} \leqslant 1-\frac{\sum_{r=n}^{\infty}\lambda(I_r\setminus\Delta)}{C_1/n}$$
$$\leqslant 1-\frac{n}{C_1}\sum_{r=n}^{\infty}\frac{\delta}{9k^2r^2} = 1-\frac{n\delta}{C_1 9k^2}\sum_{r=n}^{\infty}\frac{1}{r^2} \leqslant 1-\frac{\delta}{C_1 9k^2}\,\frac{n}{n+1}.$$

This estimate contradicts the fact that 1 is a point of density of the set Δ.

Remark 2.10. A fully analogous argument shows that under the assumptions of Proposition 2.9 $\limsup_{n\to-\infty}\lambda\big(L^n(\Delta)\big) = 1$.

Remark 2.11. Obviously Proposition 2.9 implies that there exists a lacunary sequence (n_k) such that $\lim_{k\to\infty}\lambda\big(L^{n_k}(\Delta)\big) = 1$.

Now we use the sequence $(-n_k)$ to get operators A and T satisfying (a), (b) and (c) of Proposition 1.2.

LEMMA 2.12. *If T is the operator constructed above, then for every $f\in H_\infty$ we have $\|Tf\|\leqslant\|f|\Delta\|$.*

Proof. It is known that the topology of almost uniform convergence restricted to the unit ball of H_∞ coincides with the $\sigma(H_\infty, L_1)$ topology restricted to the same ball. Fix a function $f\in H_\infty$, $\|f\|\leqslant 1$ and a subsequence (l_r) of the sequence (n_k) such that $T(f) = \lim_{r\to\infty} f\circ L^{-l_r}$.

For every $h\in L_1$ with $\|h\| = 1$ we have

$$\Big|\int T(f)h\,d\lambda\Big| = \Big|\lim_{r\to\infty}\int(f\circ L^{-l_r})h\,d\lambda\Big| \leqslant \lim_{r\to\infty}\Big|\int(f\circ L^{-l_r})h\,d\lambda\Big|$$
$$\leqslant \lim_{r\to\infty}\Big\{\Big|\int_{L^{l_r}(\Delta)}(f\circ L^{-l_r})h\,d\lambda\Big| + \Big|\int_{S\setminus L^{l_r}(\Delta)}(f\circ L^{-l_r})h\,d\lambda\Big|\Big\}$$
$$\leqslant \lim_{r\to\infty}\Big\{\|f|\Delta\|\,\|h\| + \int_{S\setminus L^{l_r}(\Delta)}|h|\,d\lambda\Big\} = \|f|\Delta\|.$$

Since $\|T(f)\| = \sup\{|\int T(f)h\,d\lambda|,\ h\in L_1 \text{ and } \|h\| = 1\}$, we infer that $\|T(f)\|\leqslant\|f|\Delta\|$. This completes the proof.

PROPOSITION 2.13. *The operator $R_\Delta\colon H_\infty\to L_\infty(\Delta)$ defined by $R_\Delta f = f|\Delta$ is an isometry on the image of AT.*

Proof. Since A is an isometry, $\|ATf\| = \|Tf\|\leqslant\|f|\Delta\|$ for an arbitrary $f\in H_\infty$. In particular, for $f = ATf$ we have $\|AT(f)\|\leqslant\|AT(f)|\Delta\|$ $\leqslant\|AT(f)\|$.

If we take into account the results of Section 1 we can state the results obtained in the following form:

THEOREM 2.14. *Let Δ be a set of positive measure in S and let t be a point of density of Δ. Then there exists a retraction $r\colon \mathfrak{M}(H_\infty)\to\mathfrak{M}(H_\infty)$ such that $\operatorname{Im} r\subset\mathfrak{M}_t(H_\infty)$ and $\hat{H}_\infty|\operatorname{Im} r$ is isometric to H_∞ and the restriction operator $R_\Delta\colon H_\infty\to L_\infty(\Delta)$ is an isometry on the image of every elementary projection given by the retraction r and some function g.*

Now let X and Y be Banach spaces isometric to H_∞. Let us identify X with $H_\infty(S_1)$ where S_1 is a unit circle, and Y with $H_\infty(S_2)$ where S_2 is some other unit circle. Under these assumptions we have

PROPOSITION 2.15. *Let $T\colon X\to Y$, $\Delta\subset S_2$ with $\lambda(\Delta)>0$ and $\varepsilon>0$. Let V_1 and V_2 be two disjoint, closed intervals in S_1. Then there exist an elementary projection P in X and a set $\Delta_1\subset\Delta$ with $\lambda(\Delta_1)>0$ such that*

(10) $$\|R_{\Delta_1}TP\|<\varepsilon,$$

(11) *P is given by a retraction r and a function g such that $\|g|V_1\|<\varepsilon$ and $\operatorname{Im} r\subset\mathfrak{M}_a\big(H_\infty(S_1)\big)$ for a certain $a\in V_2$.*

Proof. We may assume $\|T\| = 1$. Suppose our claim is false, i.e. there exists a $\Delta\subset S_2$ with $\lambda(\Delta)>0$ such that for every elementary projection in X satisfying (11) and every $\Delta_1\subset\Delta$ with $\lambda(\Delta_1)>0$ we have $\|R_{\Delta_1}TP\|>\varepsilon$. Now let us take k different points in V_2, $t_1, t_2, \ldots, t_k$ and let us find $g_1, g_2, \ldots, g_k\in A$ such that

(12) $$\|g_i\| = 1 = |g_i(t_i)| \quad \text{for} \quad i = 1, 2, \ldots, k,$$

(13) $$|g_i(t)|<\varepsilon \quad \text{for all } t\in V_1 \text{ and } i = 1, 2, \ldots, k,$$

(14) $$g_i(t_j) = 0 \quad \text{if} \quad i\neq j,$$

(15) $$\sum_{i=1}^{k}|g_i|\leqslant 1+\varepsilon.$$

Let r_i be a retraction given by Corollary 1.3 for the point t_i. Let P_i be the elementary projection given by r_i and g_i. Then we have

$$P_iP_j = \begin{cases} P_i & \text{if} \quad i = j,\\ 0 & \text{if} \quad i\neq j\end{cases}$$

and

$$\sup\{\|f_i\|\colon i = 1, 2, \ldots, k\}\leqslant\Big\|\sum_{i=1}^{k} f_i\Big\|$$
$$\leqslant(1+\varepsilon)\sup\{\|f_i\|\colon i = 1, 2, \ldots, k\} \quad \text{for all } f_i\in\operatorname{Im} P_i.$$

Since $\|R_\Delta TP_1\|\geqslant\varepsilon$, there exists an $f_1\in\operatorname{Im} P_1$, $\|f_1\| = 1$ such that $\|R_\Delta TP_1(f_1)\|>2\varepsilon/3$. Multiplying f_1 by a complex number of absolute value 1, we can get a positive number $\alpha_1 \geqslant \varepsilon/2$ and a set $\Delta_1 \subset \Delta$, $\lambda(\Delta_1) > 0$ such that

$$|(TP_1 f_1)(s) - \alpha_1| < \varepsilon/4 \quad \text{for} \quad s \in \Delta_1.$$

In this manner we can inductively construct a sequence of sets $\Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_k$ with $\lambda(\Delta_k) > 0$ and a sequence of functions $f_i \in \operatorname{Im} P_i$ with $\|f_i\| = 1$ for $i = 1, 2, \ldots, k$, and a sequence of positive numbers $\alpha_i \geqslant \varepsilon/2$ such that

$$|(TP_i f_i)(s) - \alpha_i| < \varepsilon/4 \quad \text{for} \quad s \in \Delta_i.$$

Hence

$$\Big\| T\Big(\sum_{i=1}^{k} f_i\Big)\Big\| \geqslant \Big\| R_{\Delta_k} T\Big(\sum_{i=1}^{k} f_i\Big)\Big\| \geqslant \sum_{i=1}^{k} \alpha_i - k\varepsilon/4 \geqslant k\varepsilon/4.$$

Since $\|\sum_{i=1}^{k} f_i\| \leqslant 1+\varepsilon$, for big k we get a contradiction of our assumption that $\|T\| \leqslant 1$. This contradiction proves our proposition.

PROPOSITION 2.16. *The space H_∞ has the property* SOB′.

Proof. Let us consider an arbitrary compact $\mathscr{K} \subset L(H_\infty, H_\infty)$. We may assume that $\sup\{\|B\|: B \in \mathscr{K}\} \leqslant 1$. Given $\varepsilon > 0$, we can find an ε-net $T_1, T_2, \ldots, T_n$ for $\mathscr{K}$. Let us fix two disjoint intervals V_1 and V_2 in the circle S. Let us apply Proposition 2.15 with $\Delta = V_1$. We get a set $\Delta_1 \subset V_1$, $\lambda(\Delta_1) > 0$ and an elementary projection P_1 over some α such that (10) and (11) are satisfied. Now we apply Proposition 2.15 with $T = T_2|\operatorname{Im} P_1$, $X = \operatorname{Im} P_1$, $Y = H_\infty$, $\Delta = \Delta_1$ and V_1 and V_2 chosen arbitrarily. We get an elementary projection P_2 in $\operatorname{Im} P_1$ and a set $\Delta_2 \subset \Delta_1$, $\lambda(\Delta_2) > 0$ such that $\|R_{\Delta_2} T_2 P_2 P_1\| < \varepsilon$. Obviously we also have $\|R_{\Delta_2} T_1 P_2 P_1\| < \varepsilon$. Observe that $P_2 P_1$ is an elementary projection over α satisfying (11). Repeating this procedure n times, we get a set Δ, $\lambda(\Delta) > 0$, $\Delta \subset V_1$ and an elementary projection Q_2 over α satisfying (11). Let us pick an s, a point of condensation of Δ and a function $f_0 \in A$ with $\|f_0\| = f_0(s) = 1$ and $f_0(\alpha) = 0$. Let us take a retraction constructed in Theorem 2.14 and let us denote by Q_1 the elementary projection given by this retraction and the function f_0. Elementary projections Q_1 and Q_2 obviously satisfy (4) and (5). Since Q_2 satisfies (11) and $f_0(\alpha) = 0$, (6) is satisfied. Proposition 2.13 implies that for arbitrary $f \in H_\infty$ we have $\|Q_1 f\| \leqslant \|R_\Delta f\|$; so $\|Q_1 T_i Q_2\| \leqslant \varepsilon$ for $i = 1, 2, \ldots, n$. Since $\|Q_1\| = \|Q_2\| = 1$ and $\{T_i\}_{i=1}^{n}$ is an ε-net for $\mathscr{K}$, we infer that $\sup\{\|Q_1 T Q_2\|: T \in \mathscr{K}\} < 2\varepsilon$. This completes the proof of the proposition.

THEOREM 2.17. *The linear group of H_∞ is contractible.*

This theorem follows from the remarks made at the beginning of this section, Lemma 2.3 and Proposition 2.16.

3. In this section we want to describe certain generalizations of our results to the spaces of bounded analytic functions of several complex variables. We will consider only the polydisc U^n and the ball B_n. Let us recall that

$$U^n = \{(z_1, z_2, \ldots, z_n) \in C^n: |z_i| < 1,\ i = 1, 2, \ldots, n\},$$

$$B_n = \Big\{(z_1, z_2, \ldots, z_n) \in C^n: \sum_{i=1}^{n} |z_i|^2 < 1\Big\}.$$

We will consider the spaces $H_\infty(U^n)$ and $H_\infty(B_n)$ of bounded analytic functions endowed with the supremum norm.

We start with the easier case, namely $H_\infty(U^n)$. In this case we will use the results of Sections 1 and 2 and we will work coordinatewise. Let $\{\nu_k^1\}_{k=1}^{\infty}, \ldots, \{\nu_k^n\}_{k=1}^{\infty}$ be n lacunary sequences. Using Lemma 1.1, we can construct n functions $h_1(z_1), h_2(z_2), \ldots, h_n(z_n)$, and using the conformal maps

$$\Phi_k(z_1, z_2, \ldots, z_n) = \big(L^{\nu_k^1}(z_1), \ldots, L^{\nu_k^n}(z_n)\big),$$

we can construct an operator $T: H_\infty(U^n) \to H_\infty(U^n)$ such that for every $f \in H_\infty(U^n)$ Tf is the almost uniform limit of a subsequence of the sequence $\{f \circ \Phi_k\}_{k=1}^{\infty}$. Exactly as in the proof of Proposition 1.2, we get the following

PROPOSITION 3.1. *For every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha_i| = 1$ for $i = 1, 2, \ldots, n$ there exists an analytical retraction r_α: $\mathfrak{M}(H_\infty(U^n)) \to \mathfrak{M}(H_\infty(U^n))$ such that* $\operatorname{Im} r_\alpha \subset \mathfrak{M}_\alpha(H_\infty(U^n))$ *and* $\operatorname{Im} r_\alpha$ *is homeomorphic to $\mathfrak{M}(H_\infty(U^n))$.*

Now it is clear (cf. [20]) that each point $t \in S^n$ is a peak point for the algebra $A(U^n)$. Moreover, one can find a sequence of points $(\alpha_k) \in S^n$ and functions $f_k \in A(U^n)$ satisfying (2), (3), (4) of Section 1; hence we can repeat the proof of Corollary 1.6 to get

THEOREM 3.2. *The space $H_\infty(U^n)$ is isomorphic to $\big(\sum H_\infty(U^n)\big)_\infty$.*

The existence of a Bočkariov basis for the disc algebra implies the existence of an orthonormal system $\{b_s^n\}_{s=1}^{\infty}$ which is a basis for $A(U^n)$. In fact, (b_s^n) is certain ordering of the functions $b_{i_1}(z_1)\cdot b_{i_2}(z_2)\cdot \ldots \cdot b_{i_n}(z_n)$ (cf. [3]).

This observation allows us to get

THEOREM 3.3. *The space $H_\infty(U^n)$ is isomorphic to $\big(\sum H_\infty^s(U^n)\big)_\infty$ where* $\dim H_\infty^s(U^n) = s$.

Our goal now is to show

THEOREM 3.4. *The linear group of the space $H_\infty(U^n)$ is contractible.*

The main step in the proof is the following

PROPOSITION 3.5. *Let Δ be a subset of S^n such that* $(1, 1, 1, \ldots, 1)$ *is a point of density of Δ. Then there exist lacunary sequences $\{\nu_k^1\}_{k=1}^{\infty}, \ldots, \{\nu_k^n\}_{k=1}^{\infty}$ such that* $\lim_{k\to\infty} \lambda_n(\Phi_k(\Delta)) = 1$.

Theorem 3.4 follows from Proposition 3.5 exactly as Theorem 2.17 follows from Proposition 2.9: one can repeat the proof verbatim.

Proof of Proposition 3.5. If our claim is false, then there exist an $\eta > 0$ and a natural number N such that $\lambda_n(\Phi_{k_1,k_2,\ldots,k_n}(\Delta)) < 1-\eta$ for $k_i \geqslant N$, $i = 1, 2, \ldots, n$ where

$$\Phi_{k_1,k_2,\ldots,k_n}(z_1, z_2, \ldots, z_n) = (L^{k_1}(z_1), L^{k_2}(z_2), \ldots, L^{k_n}(z_n)).$$

By Lemma 2.7 we know that there exists a natural number k such that $I_s = (\gamma_{sk}-\varepsilon_{sk}, \gamma_{sk}+\varepsilon_{sk})$ are disjoint intervals and $\lambda_1(L^{sk}(I_s)) > 1-\delta$ where δ satisfies $(1-\delta)^n > 1-\eta/2$. Let us denote $V_{s_1,s_2,\ldots,s_n} = I_{s_1}\times \times I_{s_2}\times \ldots \times I_{s_n}$. We have

$$\lambda_n(\Phi_{ks_1,ks_2,\ldots,ks_n}(V_{s_1,s_2,\ldots,s_n})) \geqslant (1-\delta)^n > 1-\eta/2;$$

hence

$$\lambda_n(\Phi_{ks_1,ks_2,\ldots,ks_n}(V_{s_1,s_2,\ldots,s_n}\setminus\Delta)) \geqslant \eta/2$$

for all $s_1, s_2, \ldots, s_n$ greater than or equal to a certain natural number $M \geqslant N$. Corollary 2.5 implies that

$$\lambda_n(V_{s_1,s_2,\ldots,s_n}\setminus\Delta) \geqslant (9^n 2k^{2n} s_1^2\cdot s_2^2\cdot \ldots \cdot s_n^2)^{-1}\eta;$$

hence for every $s > M$ we have

$$\frac{\lambda_n((0, \gamma_{sk}+\varepsilon_{sk})^n\cap\Delta)}{\lambda_n((0, \gamma_{sk}+\varepsilon_{sk})^n)} \leqslant 1-\frac{\sum\limits_{s_1,s_2,\ldots,s_n\geqslant s}\lambda_n(V_{s_1,\ldots,s_n}\setminus\Delta)}{\lambda_n((0, \gamma_{sk}+\varepsilon_{sk})^n)}$$

$$\leqslant 1-\frac{\sum\limits_{s_1,\ldots,s_n\geqslant s}(9^n 2k^{2n}s_1^2\cdot\ldots\cdot s_n^2)^{-1}\eta}{(\gamma_{sk}+\varepsilon_{sk})^n}$$

$$\leqslant 1-\frac{\eta}{9^n 2k^{2n}C} < 1.$$

This estimate contradicts the fact that $(1, 1, \ldots, 1)$ is a point of density of Δ. This contradiction proves the proposition.

Now we will concentrate our attention on the space $H_\infty(B_n)$. Our aim is to prove the following

THEOREM 3.6. *The space $H_\infty(B_n)$ is isomorphic to $(\sum H_\infty(B_n))_\infty$.*

Let us define the following map on B_n:

$$\Phi(z_1, z_2, \ldots, z_n) = (\varphi_1(z_1, \ldots, z_n), \ldots, \varphi_n(z_1, \ldots, z_n))$$
$$= \left(\frac{z_1+i(z_1-1)}{1+i(z_1-1)}, \frac{z_2}{1+i(z_1-1)}, \ldots, \frac{z_n}{1+i(z_1-1)}\right).$$

The properties of this map are summarised in

LEMMA 3.7. (a) $\Phi(1, 0, \ldots, 0) = (1, 0, \ldots, 0)$,

(b) *Φ is a biholomorphical map of B_n onto B_n,*

(c) *for every integer k the k-th iteration of Φ is given by*

$$(1)\quad \Phi^k(z_1, z_2, \ldots, z_n) = (\varphi_1^k(z_1, z_2, \ldots, z_n), \ldots, \varphi_n^k(z_1, z_2, \ldots, z_n))$$
$$= \left(\frac{z_1+ki(z_1-1)}{1+ki(z_1-1)}, \frac{z_2}{1+ki(z_1-1)}, \ldots, \frac{z_n}{1+ki(z_1-1)}\right),$$

(d) *for every compact set $K\subset B_n$ there exists a constant C such that*

$$\|\Phi^k(z_1, z_2, \ldots, z_n)-(1, 0, \ldots, 0)\| \leqslant C/|k|$$

for all $(z_1, z_2, \ldots, z_n)\in K$.

Proof. Part (a) is immediate and part (d) easily follows from Lemma 1.0. The proof of part (c) is a straightforward, but somewhat tedious calculation; it is left to the reader. In view of (c), to prove (b) it is enough to show that $\Phi(B_n) = B_n$, or, what is obviously equivalent, that Φ transforms the unit sphere in C^n onto the unit sphere. This last fact is an easy calculation and is left to the reader.

Now let us define

$$h_1(z_1, z_2, \ldots, z_n) = z_1\prod_{k=0}^{\infty}\varphi_1^{2^k}(z_1, z_2, \ldots, z_n),$$

$$h_s(z_1, z_2, \ldots, z_n) = 1-\prod_{k=0}^{\infty}(1-\varphi_s^{2^k}(z_1, z_2, \ldots, z_n)) \quad \text{for} \quad s = 2, 3, \ldots, n.$$

The above products converge almost uniformly on B_n and define functions from $H_\infty(B_n)$. For the function h_1 this fact is contained in Lemma 1.1; for the other functions the convergence follows directly from Lemma 3.7(d).

LEMMA 3.8. *For every $s = 1, 2, \ldots, n$, $h_s(\Phi^{-2^k}(z_1, \ldots, z_n))$ converges almost uniformly on B_n to z_s.*

Proof. For h_1, see Lemma 1.1. For $s > 1$ we have

$$|h_s(\Phi^{-2^k}(z_1, \ldots, z_n))-z_s|$$
$$= \left|1-\prod_{j=0}^{\infty}(1-\varphi_s^{2^j}(\Phi^{-2^k}(z_1, \ldots, z_n)))-z_s\right|$$
$$= \left|\prod_{j=0}^{k-1}(1-\varphi_s^{2^j}(\Phi^{-2^k}(z_1, \ldots, z_n)))(1-z_s)\prod_{j=k+1}^{\infty}(1-\varphi_s^{2^j}(\Phi^{-2^k}(z_1, \ldots, z_n)))-(1-z_s)\right|$$
$$\leqslant |1-z_s|\left(\sum_{j=0}^{k-1}|\varphi_s^{2^j}(\Phi^{-2^k}(z_1, \ldots, z_n))|+\sum_{j=k+1}^{\infty}|\varphi_s^{2^j}(\Phi^{-2^k}(z_1, \ldots, z_n))|\right)$$

If $(z_1, \ldots, z_n)$ is in a fixed compact $K\subset B_n$, the last expression is less than or equal to $|1-z_s|C(k\cdot 2^{-k+1}+2^{-k})$. This completes the proof of the lemma.

Now we will consider the n-tuple of functions $(h_1, h_2, \ldots, h_n)$ as a map $H\colon \mathfrak{M}(H_\infty(B_n)) \to C^n$ defined by

$$H(\varphi) = (\hat{h}_1(\varphi), \hat{h}_2(\varphi), \ldots, \hat{h}_n(\varphi)).$$

It is a continuous map.

LEMMA 3.9. *Let $\psi\colon \mathfrak{M}(H_\infty(B_n)) \to C^n$ be an analytical map, i.e. the coordinates of ψ are functions from $H_\infty(B_n)$. Let $a \in \bar{B}_n \setminus B_n$ be given. Suppose that $\psi(\mathfrak{M}_a(H_\infty(B_n))) \subset \bar{B}_n$. Then there exists an analytical map $\tilde{\psi}\colon \mathfrak{M}(H_\infty(B_n)) \to \bar{B}_n$ such that*

$$\tilde{\psi} \mid \mathfrak{M}_a(H_\infty(B_n)) = \psi \mid \mathfrak{M}_a(H_\infty(B_n)).$$

Proof. In this proof we will consider the space C^n with the natural Euclidean norm. Let us pick a function $f \in A(B_n)$ such that $f(a) = 1 = \|f\|$ and $|f(t)| < 1$ for all $t \in \bar{B}_n$, $t \neq a$. We inductively construct an increasing sequence of indices (n_k) and a sequence of open sets U_k in $\mathfrak{M}(H_\infty(B_n))$ containing $\mathfrak{M}_a(H_\infty(B_n))$ such that

$$\sup_{\varphi\in\mathfrak{M}(H_\infty(B_n))} \|\hat{f}^{n_k}(\varphi)\psi(\varphi)\| \leqslant 1+2^{-k-1}, \tag{2}$$

$$U_k = \{\varphi \in \mathfrak{M}(H_\infty(B_n))\colon \|\hat{f}^{n_1}(\varphi)\psi(\varphi)\| < 1+2^{-k-1}\}, \tag{3}$$

$$\|\hat{f}^{n_k}(\varphi)\psi(\varphi)\| < 1/2 \quad \text{for} \quad \varphi \notin U_k. \tag{4}$$

We put

$$\tilde{\psi} = \sum_{k=1}^{\infty} 2^{-k}\hat{f}^{n_k}\psi.$$

It is clear that $\tilde{\psi}$ is an analytical map. Since $\hat{f}|\mathfrak{M}_a(H_\infty(B_n)) = 1$, we obtain

$$\tilde{\psi}|\mathfrak{M}_a(H_\infty(B_n)) = \psi|\mathfrak{M}_a(H_\infty(B_n)).$$

We have to check that $\|\tilde{\psi}(\varphi)\| \leqslant 1$ for all $\varphi \in \mathfrak{M}(H_\infty(B_n))$. If $\varphi \in U_k$ for all k, this is clear, and so suppose that, for a certain k_0, $\varphi \in U_{k_0} \setminus U_{k_0+1}$. Then we have

$$\begin{aligned}\|\tilde{\psi}(\varphi)\| &\leqslant \sum_{k=1}^{\infty} 2^{-k}\|\hat{f}^{n_k}(\varphi)\psi(\varphi)\| \\ &= \sum_{k=1}^{k_0} 2^{-k}\|\hat{f}^{n_1}(\varphi)\psi(\varphi)\| + \sum_{k=k_0+1}^{\infty} 2^{-k}\|\hat{f}^{n_k}(\varphi)\psi(\varphi)\| \\ &\leqslant (1+2^{-k_0-1})\sum_{k=1}^{k_0} 2^{-k} + 1/2 \sum_{k=k_0+1}^{\infty} 2^{-k} = 1-2^{-2k_0-1}.\end{aligned}$$

Remark 3.10. The proof of this lemma is a trivial modification of the proof of Theorem II.12.5 of [5].

Now we can apply Lemma 3.9 to the map H to get the map $\tilde{H}\colon \mathfrak{M}(H_\infty(B_n)) \to \bar{B}_n$. This allows us to repeat the proof of Proposition 1.2 with the use of the map $\tilde{H}$ instead of the function h and the conformal map Φ instead of the conformal map L. We get

PROPOSITION 3.11. *For every $a \in C^n$, $\|a\| = 1$ there exists an analytical retraction $r_a\colon \mathfrak{M}(H_\infty(B_n)) \to \mathfrak{M}(H_\infty(B_n))$ such that $\operatorname{Im} r_a \subset \mathfrak{M}_a(H_\infty(B_n))$, $\operatorname{Im} r_a$ is homeomorphic to $\mathfrak{M}(H_\infty(B_n))$ and $H_\infty(B_n)|\operatorname{Im} r_a$ is isometric to $H_\infty(B_n)$.*

Theorem 3.6 follows from this proposition exactly as, in Section 1, Corollary 1.6 follows from Proposition 1.2.

Remark 3.12. The author would like to express his gratitude to Dr E. Ligocka for several useful conversations concerning this section.

4. In the present section we investigate the finite-dimensional norm-one projections in spaces A and H_∞. The main result of this section states that the image of the adjoint projection is orthogonal to the Lebesgue measure. As an application we find that the disc algebra A is not a π_1-space. Earlier examples of this phenomenon were given by Gurarii [7] and a stronger example was given by Enflo [4]. Their examples were rather artificial while the disc algebra is a "natural" space. Our next application is the result stating that the space $L_1(\lambda)/H_1^0(\lambda)$ does not have finite-dimensional norm-one-projections other than one-dimensional ones.

If $f \in L_\infty(\lambda)$, we denote by $M(f)$ the set $\{t \in S\colon |f(t)| = \|f\|\}$. This set is defined modulo sets of Lebesgue measure zero, but we will always be interested in the measure of this set, so that this will not lead to misunderstandings.

LEMMA 4.1. *Let E be a subspace of H_∞, $\dim E \geqslant 2$. Then the set of $f \in E$ such that $\lambda(M(f)) = 0$ is dense in E.*

Proof. Let $f \in E$, $f \neq 0$, and let $1 > \varepsilon > 0$ be arbitrary but fixed. Let a functional f^* support f, and pick $g \in \ker f^* \cap E$ with $\|g\| = 1$. Let us consider $f+ag$ where $|a| < \varepsilon$. If there exists an a such that $\lambda(M(f+ag)) = 0$, we have our proof. Suppose to the contrary that for every a with $|a| < \varepsilon$ we have $\lambda(M(f+ag)) > 0$. Then for every $\eta > 0$ we can find a_1 and a_2, $a_1 \neq a_2$ such that

$$\lambda(M(f+a_1g)\cap M(f+a_2g)) > 0 \quad \text{and} \quad |a_1| < \eta \text{ and } |a_2| < \eta. \tag{1}$$

Let us denote

$$h_1 = \frac{f+a_1g}{\|f+a_1g\|}, \quad h_2 = \frac{f+a_2g}{\|f+a_2g\|}$$

and put $u = h_1+\zeta h_2$ where $|\zeta| = 1$ and ζ is chosen in such a way that $\|u\| = 2$. It is possible because of (1). For $t \in M(u)$ we have $h_1(t) = \zeta h_2(t)$, and so by the uniqueness theorem for H_∞ functions and the fact that h_1 is not equal to ζh_2 we infer that $\lambda(M(u)) = 0$. Now consider

$$v = \left(\frac{1}{\|f+a_1 g\|} + \frac{\zeta}{\|f+a_2 g\|}\right)^{-1} u.$$

Obviously $\lambda(M(v)) = 0$ and

$$v = f + \left(\frac{1}{\|f+a_1 g\|} + \frac{\zeta}{\|f+a_2 g\|}\right)^{-1}\left(\frac{a_1}{\|f+a_1 g\|} + \frac{\zeta a_2}{\|f+a_2 g\|}\right) g.$$

From this formula it easily follows that if η is close to 0, the coefficient at g is close to 0, so we have a contradiction of our assumption that for all a, with $|a| < \varepsilon$, $\lambda(M(f+ag)) > 0$. This contradiction proves the lemma.

Now we recall some facts about functionals on A and H_∞. We will consider A as a subspace of $C(S)$ and H_∞ as a subspace of $L_\infty(\lambda)$. From F. and M. Riesz's theorem it follows that A^* is equal to $\{\mu\colon\ \mu \perp \lambda\} \oplus_1 L_1(\lambda)/H_1^0(\lambda)$. This, in particular, means that if we have $(f_1^*, f_2^*, \ldots, f_n^*) \subset A^*$ and each f_i^* has a Hahn–Banach extension to a measure μ_i singular with respect to the Lebesgue measure, then $\operatorname{span}\{f_i^*\}_{i=1}^n$ is isometric to $\operatorname{span}\{\mu_i\}_{i=1}^n$. In the case of the space H_∞ it follows from the Gleason–Whitney theorem [6] (cf. also Havin's [10]) that the space H_∞^* admits a direct sum decomposition $H_\infty^* = X \oplus_1 R$ where X is the space of functionals on H_∞ such that every norm-preserving extension to $L_\infty(\lambda)$ is singular with respect to the Lebesgue measure and R is the space of functionals such that every norm-preserving extension to L_∞ is absolutely continuous with respect to the Lebesgue measure. Moreover, the space R can be isometrically identified with $L_1(\lambda)/H_1^0(\lambda)$.

Remark 4.2. The facts stated above can be proved also by using the Havin Lemma [9].

THEOREM 4.3. (a) *If $P\colon A \to A$ is a norm-one finite-dimensional projection with* $\dim \operatorname{Im} P > 1$, *then* $\operatorname{Im} P^* \subset \{\mu\colon\ \mu \perp \lambda\}$.

(b) *If $P\colon H_\infty \to H_\infty$ is a norm-one finite-dimensional projection with* $\dim \operatorname{Im} P > 1$, *then* $\operatorname{Im} P^* \subset X$.

Proof. We will only prove part (a). The proof of (b) is almost the same.

Let $E = \operatorname{Im} P$, $\dim E = n$. We construct two sequences, $(e_1, e_2, \ldots, e_n) \subset E$ and $(e_1^*, e_2^*, \ldots, e_n^*) \subset E^*$, such that

(2) $\|e_i\| = \|e_i^*\| = e_i^*(e_i) = 1$ for $i = 1, 2, \ldots, n$,

(3) $\lambda(M(e_i)) = 0$ for $i = 1, 2, \ldots, n$,

(4) The matrix $(e_i^*(e_j))_{i,j=1}^n$ is non-singular.

We begin the construction with arbitrary e_1, $\|e_1\| = 1$ and $\lambda(M(e_1)^*_s) = 0$. Lemma 4.1 says that we can do it. The next e_k's we choose in $\bigcap_{s<k} \ker e_s^*$ with $\|e_k\| = 1$ and $\lambda(M(e_k)) = 0$; e_k^* is a supporting functional to e_k. Lemma 4.1 makes this possible if $k < n$. To find e_n we take $g \in \bigcap_{s<n} \ker e_s^*$, $\|g\| = 1$ and take e_n such that $\lambda(M(e_n)) = 0$ and e_n is very close to g; e_n^* is, as usual, the supporting functional.

If we take e_n close enough to g, the matrix $(e_i^*(e_j))_{i,j=1}^n$ will be non-singular because $e_i^*(e_j) = 0$ for $i < j < n$, $e_n^*(e_n) = 1$ and for $i < n$ the numbers $e_i^*(e_n)$ can be made as small as we wish.

Now we extend e_i^* to the functionals $f_i^* \in A^*$ by the formula $f_i^*(x) = e_i^*(Px)$ for $x \in A$. Clearly $f_i^* \in \operatorname{Im} P^*$ for $i = 1, 2, \ldots, n$. We still have $\|f_i^*\| = f_i^*(e_i) = 1$. Conditions (2) and (3) imply that $f_i^* \in \{\mu\colon\ \mu \perp \lambda\}$ for $i = 1, 2, \ldots, n$. Condition (4) ensures that $\operatorname{span}\{f_i^*\}_{i=1}^n = \operatorname{Im} P^*$. This completes the proof.

COROLLARY 4.4. *The space $L_1(\lambda)/H_1^0(\lambda)$ does not admit any norm-one projection P with* $1 < \dim \operatorname{Im} P < \infty$.

Proof. The corollary follows from part (b) of Theorem 4.3 and the fact that $(L_1(\lambda)/H_1^0(\lambda))^* = H_\infty$ and in the decomposition $H_\infty^* = X \oplus_1 R$, R is the image of $L_1(\lambda)/H_1^0(\lambda)$ when canonically embedded into $(L_1(\lambda)/H_1^0(\lambda))^{**}$.

LEMMA 4.5. *Let $P_n\colon A \to A$ be a sequence of projections such that for every $x \in A$ we have $P_n(x) \to x$. Then $\overline{\operatorname{span}} \bigcup_{n=1}^\infty P_n^*(A^*)$ is not isomorphic to a subspace of an $L_1(\nu)$-space.*

Proof. Let us take a finite-dimensional subspace $F \subset A^*$. Given $\varepsilon > 0$, there exists a finite-dimensional subspace $E \subset A$ such that for $f \in F$

$$\|f\| \leqslant (1+\varepsilon)\sup\{|f(e)|\colon\ e \in E \text{ and } \|e\| = 1\}.$$

Hence there exists an n such that for $f \in F$

$$\|f\| \leqslant (1+\varepsilon)\sup\{|f(x)|\colon\ x \in \operatorname{Im} P_n,\ \|x\| = 1\}.$$

But this means that for $f \in F$ we have

$$\begin{aligned}\|P_n^*(f)\| &= \sup\{|P_n^* f(x)|\colon\ x \in A,\ \|x\| \leqslant 1\} = \sup\{|f(P_n x)|\colon\ x \in A,\ \|x\| \leqslant 1\} \\ &\geqslant \|P_n\|^{-1}\sup\{|f(x)|\colon\ x \in \operatorname{Im} P_n,\ \|x\| \leqslant 1\} \geqslant M^{-1}(1+\varepsilon)^{-1}\|f\|\end{aligned}$$

where $M = \sup \|P_n\|$.

This shows that F can be isomorphically embedded into $\overline{\operatorname{span}} \bigcup_{n=1}^\infty P_n^*(A^*)$ with constants independent of F. By Proposition 7.1 of [15] this shows that if $\overline{\operatorname{span}} \bigcup_{n=1}^\infty P_n^*(A^*)$ is isomorphic to a subspace of $L_1(\nu)$-

space, then also A^* is isomorphic to a subspace of an $L_1(\nu)$-space. This is not true by Corollaire 1 of [18]. This completes the proof of the lemma.

THEOREM 4.6. *The disc algebra A is not a π_1-space, i.e. there does not exist a sequence of finite-dimensional, norm-one projections $P_n\colon A\to A$ such that $P_n(x)\to x$ for every $x\in A$. In particular, A does not have a monotone basis.*

Proof. The proof is immediate from part (a) of Theorem 4.3 and Lemma 4.5.

Theorem 4.3 easily implies

COROLLARY 4.7. *A finite-dimensional, norm-one complemented subspace of A is isometric to l_∞^n.*

Let us remark that it is well known that there exists a sequence of finite-dimensional, norm-one operators $T_n\colon A\to A$ such that $T_n(x)\to x$ for every $x\in A$. T_n can be chosen so as to be the Fejér means.

DEFINITION 4.8. Let (f_i) be a basic sequence in $C(K)$. (f_i) is called an *interpolating sequence* if there exists a sequence of different points $(t_i)\subset K$ such that for every $f=\sum_{i=1}^{\infty}a_if_i$ we have $f(t_k)=\sum_{i=1}^{n}a_if_i(t_k)$ for all $k=1,2,\ldots,n$ and $n=1,2,\ldots$

PROPOSITION 4.9. *$A\subset C(S)$ is not spanned by an interpolating basic sequence.*

Proof. Let (f_i) be an interpolating basic sequence spanning A and let $\delta_k(f)=f(t_k)$. It follows from the definition of an interpolating basic sequence that the coefficient functional $f_n^*=\sum_{i=1}^{n}\beta_i^n\delta_i$ for some scalars β_i^n. By F. and M. Riesz's theorem span $\{f_n^*\}_{n=1}^{\infty}$ is isomorphic to l_1, but this contradicts Lemma 4.5.

The above proposition answers the question of J. Wronicz.

5. Remarks and open problems. We start with a few remarks about the retraction r_a constructed in Section 1 and the elementary projection $P(f)=\hat{f}\circ r_a$. It is easily seen that ImP is a $\sigma(L_\infty,L_1)$ closed subalgebra of H_∞, generated by the function h. In other words, Im$P=\{f\in H_\infty\colon f=\sum_{n=0}^{\infty}a_nh^n\}$, the series being almost uniformly convergent. There is another natural projection onto ImP, namely the conditional expectation projection. The behaviour of those projections is very different. In particular, P is not $\sigma(H_\infty,L_1/H_1^0)$ continuous, while the conditional expectation projection is.

The fact that the projections we are working with are not w^*-continuous is the main difficulty in applying our technigues to the solving of the following problems:

PROBLEM 5.1. *Is the space $L_1(\lambda)/H_1^0(\lambda)$ isomorphic to its sum in the sense l_1?*

PROBLEM 5.2. *Is the space A isomorphic to its sum in the sense c_0?*

The image of the retraction r_1 is in the fibre $\mathfrak{M}_1$, and so one can ask if it can happen that Im$r_1=\mathfrak{M}_1$. The answer is no. To see it, let us consider the space $A_1=\hat{H}|\mathfrak{M}_1$. It is known (cf. [11]) that A_1 is a uniform algebra and $\mathfrak{M}(A_1)=\mathfrak{M}_1$. The following proposition contains some easy observations about this space.

PROPOSITION 5.3. (a) *Let $R\colon H_\infty\to A_1$ be the restriction map. Then $R^*(A_1^*)$ is complemented in H_∞^*.*

(b) *A_1 contains a subspace isomorphic to $c_0(\Gamma)$, card$\Gamma=\mathfrak{c}$ but it does not contain $l_\infty(\Gamma)$, and so A_1 is not complemented in a conjugate Banach space.*

(c) *There is no linear extension operator from A_1 into H_∞.*

(d) *A_1 contains H_∞ as a complemented subspace.*

(e) *A_1^* is isomorphic to H_∞^*.*

Proof. (a) Since the point 1 is a peak point of the disc algebra A, the fibre $\mathfrak{M}_1$ is a peak set of the algebra H_∞. By Glicksberg's theorem ([5], Th. II.12.7) if μ is a measure on $\mathfrak{M}(H_\infty)$ annihilating H_∞, then $\mu|\mathfrak{M}_1$ also annihilates H_∞. This implies that the restriction of a measure on $\mathfrak{M}(H_\infty)$ to $\mathfrak{M}_1$ induces a projection in H_∞^* and its image is $R^*(A_1^*)$.

(b) Let f_n be the same as in the proof of Proposition 1.5. Let us take a bounded sequence of complex numbers (a_n) and consider the series $\sum a_nf_n$. This series is almost uniformly convergent and represents an H_∞ function. Using the ideas of the proof of Proposition 1.5, one can show that $\|\widehat{\sum a_nf_n}|\mathfrak{M}_1\|\geqslant\limsup|a_n|$ and that, for $(a_n)\in c_0$, $\widehat{\sum a_nf_n}|\mathfrak{M}_1=0$. These facts imply that A_1 contains a subspace isomorphic to l_∞/c_0. It is well known (cf. [19]) that l_∞/c_0 contains $c_0(\Gamma)$ where card$\Gamma=\mathfrak{c}$, and hence the same is true for A_1. A_1 does not contain $l_\infty(\Gamma)$ with card$\Gamma>\aleph_0$, because card$l_\infty(\Gamma)>\mathfrak{c}$ while card$A_1\leqslant$card$H_\infty=\mathfrak{c}$. The last claim of (b) follows from the results of Rosenthal [19].

(c) The existence of the linear extension operator from A_1 into H_∞ would imply the existence of a projection from H_∞ onto A_1. Since H_∞ is a conjugate space, it is impossible by (b).

(d) It follows immediately from Proposition 1.2.

(e) Let us begin with an observation that $H_\infty^*\sim(\sum H_\infty^*)_1$. To see this let us consider the natural embedding

$$i\colon \left(\sum H_\infty\right)_{c_0}\to\left(\sum H_\infty\right)_\infty.$$

Then

$$i^*\colon \left(\sum H_\infty\right)^*_\infty \xrightarrow{\text{onto}} \left(\sum H^*_\infty\right)_1$$

is the restriction of a functional to the subspace $(\sum H_\infty)_{c_0}$. Now define $j\colon (\sum H^*_\infty)_1 \to (\sum H_\infty)^*_\infty$ by

$$j(h^*_n)(h_n) = \sum h^*_n(h_n).$$

It is an isometrical embedding and $i^*\circ j = \mathrm{id}$ on $(\sum H^*_\infty)_1$; hence ji^* is a projection from $(\sum H^*_\infty)^*_\infty$ onto $(\sum H^*_\infty)_1$. But $(\sum H_\infty)^*_\infty$ is, by Theorem 1.7, isomorphic to H^*_∞; so, for some V,

$$H^*_\infty \sim V + \left(\sum H^*_\infty\right)_1 \sim V + \left(\sum H^*_\infty\right)_1 + \left(\sum H^*_\infty\right)_1 \sim \left(\sum H^*_\infty\right)_1,$$

This proves the observation.

Since (a) implies that $A_1^* + Z \sim H^*_\infty$ and part (d) implies that $A_1^* \sim V + H^*_\infty$ for some Banach spaces Z and V, our claim follows from the decomposition method.

The above proposition shows that the fibre algebra A_1 can be thought of as an "analytical Calkin algebra".

Many results are proved in the present paper for the space H_∞ only. We do not know if they are true for $H_\infty(U^n)$ and $H_\infty(B_n)$. It would be interesting to investigate the spaces $H_\infty(V)$ for more general regions in C^n.

Added in proof. (a) Theorem 1.11 was independently obtained by T. Ando, *On the predual of H_∞*, Comm. Math. Tomus specialis in honorem Ladislai Orlicz, Warszawa 1978, pp. 33–40. A generalisation of this theorem was obtained by J. Chaumat, C. R. Acad. Sci. Paris (1979).

(b) Positive solutions of Problems 5.1 and 5.2 have been obtained by the author, *Decompositions of H_p spaces*, Duke Math. J. (September 1979).

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