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A formula for the eigenvalues of a compact operator

by

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Abstract. Let T be a compact operator in a Banach space X. Let $\lambda_n(T)$ denote the eigenvalues of T ordered in non-increasing absolute value and counted according to their multiplicity. We prove that

$$|\lambda_n(T)| = \lim_{m\to\infty} a_n(T^m)^{1/m},$$

where a_n denotes the approximation numbers or any other s-number sequence in the sense of [5]. For $n = 1$ this is the well-known formula for the spectral radius of T. As a corollary one has Weyl's inequality in Hilbert spaces. We also give an estimate for $|\lambda_n(T)|$ by the approximation numbers of T with respect to an equivalent norm on X.

1. Introduction. Let X and Y be complex Banach spaces. We denote the continuous linear operators from X into Y by $\mathscr{L}(X, Y)$, the compact operators by $\mathscr{K}(X, Y)$. Let $\mathscr{L}(X) = \mathscr{L}(X, X)$ and $\mathscr{K}(X) = \mathscr{K}(X, X)$. The approximation numbers of $T \in \mathscr{L}(X, Y)$ are defined by

$$a_n(T) = \inf\{\|T-T_n\|: T_n \in \mathscr{L}(X, Y),\ \text{rank}\ T_n < n\},$$

for any positive integer $n \in \mathbf{N}$, with $a_1(T) = \|T\|$. The isomorphism numbers of $T \in \mathscr{L}(X, Y)$ are defined as follows: If rank $T < n$, $i_n(T) = 0$. If rank $T \geqslant n$, there exists a Banach space Z of dimension $\geqslant n$ and operators $A \in \mathscr{L}(Z, X)$ and $B \in \mathscr{L}(Y, Z)$ such that BTA is the identity I_Z on Z. Let

$$i_n(T) = \sup\{\|A\|^{-1}\|B\|^{-1}\},$$

the supremum taken over all Z, A and B with the above property. The approximation numbers and isomorphism numbers are examples of s-number sequences of continuous linear operators in the sense of Pietsch [5]. If s_n is any s-number sequence, we have by [5] for any $T \in \mathscr{L}(X, Y)$

$$i_n(T) \leqslant s_n(T) \leqslant a_n(T).$$

Let $T \in \mathscr{L}(X)$ be an operator the spectrum of which consists of eigenvalues of finite multiplicity only. We assume always that the eigenvalues are ordered in non-increasing absolute value and counted according to their multiplicity. We denote them by $\lambda_n(T)$, $n \in \mathbf{N}$.

The s-numbers in Hilbert spaces H were studied by Gohberg–Krein [2]. For $T \in \mathcal{K}(H)$ and any s-number sequence s_n

$$s_n(T) = \lambda_n(|T|),$$

where $T = U|T|$ is the polar decomposition of T, cf. [5], [6].

2. Estimates for the eigenvalues. C. R. Loesener [4] has shown in his thèse that for any matrix operator T in the complex euclidean N-space C^N,

$$|\lambda_n(T)| = \lim_{m\to\infty} \lambda_n(|T^m|)^{1/m}, \quad n = 1, \ldots, N.$$

We generalize this finite-rank Hilbert space result to compact operators on a Banach space.

THEOREM 1. *Let X be a complex Banach space and $T \in \mathcal{L}(X)$ be an operator the spectrum of which consists of eigenvalues of finite multiplicity only. Then for any $n \in N$ and any s-number sequence s_n,*

(2.1) $$|\lambda_n(T)| = \lim_{m\to\infty} s_n(T^m)^{1/m}.$$

Remarks. (a) This contains Loesener's result, since in Hilbert spaces $s_n(T) = \lambda_n(|T|)$. Of course, any operator $T \in \mathcal{L}(X)$ with $T^N \in \mathcal{K}(X)$ for some $N \in N$ fulfills the assumption of Theorem 1. Recall $s_1(S) = \|S\|$, so for $n = 1$ Theorem 1 gives nothing but the formula for the spectral radius of T. (b) Theorem 1 is also valid for Riesz operators, i.e. operators with $\dim\ker(I-\lambda T) < \infty$, $\operatorname{codim}\operatorname{im}(I-\lambda T) < \infty$ for any complex number λ.

Proof. We have to show that the limit in (2.1) exists and is equal to $|\lambda_n(T)|$. Since

$$i_n(S) \leqslant s_n(S) \leqslant a_n(S),$$

it is enough to show

(2.2) $$\overline{\lim_{m\to\infty}}\, a_n(T^m)^{1/m} \leqslant |\lambda_n(T)|$$

and

(2.3) $$\underline{\lim_{m\to\infty}}\, i_n(T^m)^{1/m} \geqslant |\lambda_n(T)|.$$

(a) Both inequalities are true for $n = 1$. To show the induction step for (2.2), we may assume that $\lambda_{n-1}(T) \neq 0$, for otherwise by the induction assumption and ordering of the eigenvalues $\lambda_n(T) = 0$ and $a_n(T^m)^{1/m} \to 0$. Choose $k \in N$ minimal with the property $\lambda_{n-k}(T) \neq \lambda_n(T)$. So $k = 1$, if $\lambda_{n-1}(T)$ has multiplicity one. Let $\Lambda = \{\lambda_1(T), \ldots, \lambda_{n-k}(T)\}$ and let $P = P(\Lambda, T)$ be the spectral projection of T with respect to the spectral set Λ, defined by the Dunford integral (cf. [1], Chapter 7) for the properties of spectral projections. Since the multiplicity of an eigenvalue λ

is the dimension of the range of the spectral projection associated to λ, we have $n-k = \dim P(X)$. Therefore rank $PT^m \leqslant n-k$ for all $m \in N$ and

$$a_n(T^m) \leqslant a_{n-k+1}(T^m) \leqslant \|T^m - PT^m\| = \|(I-P)T^m\| = \|((I-P))T^m\|.$$

We used here that P commutes with T and that $(I-P)$ is a projection too. The spectrum of $(I-P)T$ is equal to the spectrum of T minus Λ, so consists of the eigenvalues $\{\lambda_{n-k+i}(T)\colon i \in N\}$. Hence by the classical formula for the spectral radius r,

$$\overline{\lim_{m\to\infty}}\, a_n(T^m)^{1/m} \leqslant \overline{\lim_{m\to\infty}}\, \|(I-P)T^m\|^{1/m}$$
$$= r((I-P)T) = |\lambda_{n-k+1}(T)| = |\lambda_n(T)|.$$

(b) We prove next the induction step for inequality (2.3). Again we may assume $\lambda_n(T) \neq 0$. Let $\Lambda = \{\lambda_1(T), \ldots, \lambda_n(T)\}$ and $P = P(\Lambda, T)$ be the spectral projection relative to Λ. Then $k := \dim P(X) \geqslant n$ and $\lambda_k(T) = \lambda_n(T)$. So $k = n$, if $\lambda_n(T)$ has multiplicity one.

T maps $P(X)$ into itself. Let $\overline{T}\colon P(X) \to P(X)$ be the restriction and astriction of T to $P(X)$. But $0 \notin \sigma(\overline{T}) = \Lambda$ and hence $\overline{T}$ is injective and an isomorphism because of $\dim P(X) < \infty$. Let $A\colon P(X) \to X$ be the natural injection and $B_m\colon X \to P(X)$ be defined as $B_m = (\overline{T}^{-1})^m P$. Then $B_m T^m A$ is the identity on $P(X)$, and by definition of the isomorphism numbers of T^m,

$$i_n(T^m) \geqslant i_k(T^m) \geqslant \|A\|^{-1}\|B_m\|^{-1} \geqslant \|P\|^{-1}\|(\overline{T}^{-1})^m\|^{-1}.$$

In absolute value, the largest eigenvalue of $\overline{T}^{-1}$ is $\lambda_k(T)^{-1} = \lambda_n(T)^{-1}$, hence again by the spectral radius formula, this time for $\overline{T}^{-1}$,

$$\underline{\lim_{m\to\infty}}\, i_n(T^m)^{1/m} \geqslant \underline{\lim_{m\to\infty}}\, \|P\|^{-1/m}\|(\overline{T}^{-1})^m\|^{-1/m} = r(\overline{T}^{-1})^{-1} = |\lambda_n(T)|^{-1}.$$

Of course, $\lim_{m\to\infty} \|P\|^{-1/m} = 1$, since P does not depend on $m \in N$.

I am indebted to the referee for pointing out that Weyl's inequality in Hilbert spaces is a consequence of Theorem 1: Let $S_p(H)$ denote the class of compact operators $T \in \mathcal{K}(H)$ in the Hilbert space H for which

$$\sigma_p(T) = \Big(\sum_{n\in N} s_n(T)^p\Big)^{1/p} < \infty.$$

Then for $0 < p, q < \infty$ with $1/r = 1/p + 1/q$ (cf. [6])

(2.4) $$\sigma_r(ST) \leqslant \sigma_p(S)\sigma_q(T), \quad S \in S_p(H),\ T \in S_q(H).$$

WEYL'S INEQUALITY. *Suppose $T \in S_p(H)$ with $0 < p < \infty$. Then the eigenvalues of T are absolutely p-summable with*

$$\Big(\sum_{n\in N} |\lambda_n(T)|^p\Big)^{1/p} \leqslant \Big(\sum_{n\in N} s_n(T)^p\Big)^{1/p}.$$

Proof. Given $\varepsilon > 0$ and $N \in \mathbf{N}$, there is by Theorem 1 a positive integer m such that

$$|\lambda_n(T)| \leqslant (1+\varepsilon) s_n(T^m)^{1/m} \quad \text{for all } n = 1, \ldots, N.$$

Hence

$$\Big(\sum_{n\leqslant N} |\lambda_n(T)|^p\Big)^{1/p} \leqslant (1+\varepsilon)\Big(\sum_{n\leqslant N} s_n(T^m)^{p/m}\Big)^{1/p} = (1+\varepsilon)\,\sigma_{p/m}(T^m)^{1/m}$$

and by (2.4),

$$\leqslant (1+\varepsilon)\,\sigma_p(T).$$

Another consequence is the

Corollary. *Let* $\dim X = l < \infty$ *and* $T \in \mathscr{L}(X)$ *be invertible. Then for any s-number sequence* s_n *and* $1 \leqslant k \leqslant l$

$$|\lambda_k(T)/\lambda_{l-k+1}(T)| = \lim_{m\to\infty} \big(s_k(T^m)s_k(T^{-m})\big)^{1/m}.$$

Proof. The inequality

$$i_{l-k+1}(S)\,a_k(S^{-1}) \leqslant 1$$

holds by [5]. Hence by Theorem 1,

$$|\lambda_{l-k+1}(T)| = \lim_{m\to\infty} i_{l-k+1}(T^m)^{1/m} \leqslant \lim_{m\to\infty} a_k(T^{-m})^{-1/m} = \big(|\lambda_k(T^{-1})|\big)^{-1} = |\lambda_{l-k+1}(T)|.$$

Therefore $|\lambda_{l-k+1}(T)| = \lim_{m\to\infty} a_k(T^{-m})^{-1/m}$. A similar argument shows this for the isomorphism numbers and thus for any s-number sequence.

Let $\|\cdot\|_1$ be an equivalent norm on X. We write

$$\|\cdot\|_1 \sim \|\cdot\| \quad \text{and} \quad \|T\|_1 := \big\|T\colon (X, \|\cdot\|_1) \to (X, \|\cdot\|_1)\big\|.$$

Then by Holmes [3] there is another formula for the spectral radius of an operator $T \in \mathscr{L}(X)$,

$$r(T) = \inf\{\|T\|_1\colon \|\cdot\|_1 \sim \|\cdot\|\}.$$

This formula has only a weak generalization to s-number sequences. Let $s_n(T)_1$ denote the s-numbers of T with respect to an equivalent norm $\|\cdot\|_1$ on X.

Theorem 2. *Let* $T \in \mathscr{L}(X)$ *be as in Theorem 1. Then for any s-number sequence* s_n *and any* $n \in \mathbf{N}$,

$$\inf\{s_n(T)_1\colon \|\cdot\|_1 \sim \|\cdot\|\} \leqslant |\lambda_n(T)| \leqslant \sup\{s_n(T)_1\colon \|\cdot\|_1 \sim \|\cdot\|\}. \tag{2.5}$$

Both inequalities may be strict in general.

Proof. (a) To show the first inequality for the approximation numbers a_n, let $\Lambda = \{\lambda_1(T), \ldots, \lambda_{n-k}(T)\}$ and P be as in part (a) of the proof of Theorem 1. Then by the mentioned formula for the spectral radius

$$\begin{aligned}\inf\{a_n(T)_1\colon \|\cdot\|_1 \sim \|\cdot\|\} &\leqslant \inf\{a_{n-k+1}(T)_1\colon \|\cdot\|_1 \sim \|\cdot\|\} \\ &\leqslant \inf\{\|(I-P)T\|_1\colon \|\cdot\|_1 \sim \|\cdot\|\} \\ &= r\big((I-P)T\big) = |\lambda_n(T)|.\end{aligned}$$

(b) We show next $|\lambda_n(T)| \leqslant \sup i_n(T)_1$. If this supremum is infinite, nothing is to show. As in (b) of the previous proof, let $\Lambda = \{\lambda_1(T), \ldots \ldots, \lambda_n(T)\}$ and $P = P(\Lambda, T)$ with $k := \dim P(X) \geqslant n$ and $\lambda_k(T) = \lambda_n(T)$. We may assume $\lambda_n(T) \neq 0$. Then $\overline{T}\colon P(X) \to P(X)$ again has an inverse and

$$|\lambda_n(T)|^{-1} = r(I\overline{T}^{-1}P\colon X \to X)$$

is the spectral radius of $I\overline{T}^{-1}P$, where $I\colon P(X) \to X$ is the injection. Hence by the Holmes formula,

$$\begin{aligned}|\lambda_n(T)| &= r(I\overline{T}^{-1}P\colon X \to X)^{-1} \\ &= \sup\{\|I\overline{T}^{-1}P\|_1^{-1}\colon \|\cdot\|_1 \sim \|\cdot\|\} \\ &\leqslant \sup\{i_n(T)_1\colon \|\cdot\|_1 \sim \|\cdot\|\},\end{aligned}$$

where we used that $(\overline{T}^{-1}P)TI$ is a factorization of the identity on $P(X)$, $\dim P(X) = k \geqslant n$.

Parts (a) and (b) together with $i_n(T) \leqslant s_n(T) \leqslant a_n(T)$ prove Theorem 2.

We give an example in which strict inequalities occur in Theorem 2.

Let X be $\mathbf{C}^2$ in the euclidean norm, $T\colon X \to X$ be given by the matrix $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ and $s_n = a_n$, $n = 1, 2$.

Then

$$\lambda_1(T) = 2, \quad \lambda_2(T) = 1$$

and

$$a_1(T) = \lambda_1(|T|) = \sqrt{3+\sqrt{5}}, \quad a_2(T) = \lambda_2(|T|) = \sqrt{3-\sqrt{5}},$$

hence

$$\inf\{a_2(T)_1\colon \|\cdot\|_1 \sim \|\cdot\|\} \leqslant \sqrt{3-\sqrt{5}} < 1 = |\lambda_2(T)|$$

and

$$\sup\{a_1(T)_1\colon \|\cdot\|_1 \sim \|\cdot\|\} \geqslant \sqrt{3+\sqrt{5}} > 2 = |\lambda_1(T)|.$$

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On projections in spaces of bounded analytic functions with applications

by

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Abstract. Projections in spaces A and H_∞ are investigated. It is shown that H_∞ is isomorphic to its l_∞-sum and has a contractible linear group. Certain generalizations to spaces of bounded analytic functions of several complex variables are presented. Norm-one finite rank projections in A and H_∞ are described. The description shows in particular that A and L_1/H_1 are not π_1-spaces. We also investigate isometric and isomorphic preduals of H_∞.

Introduction. In the present paper we consider projections in spaces of bounded analytic functions. Our main interest lies in the space $H_\infty(U)$, the space of bounded analytic functions in the unit disc U, but generalizations to $H_\infty(U^n)$ and $H_\infty(B_n)$ (the spaces of bounded analytic functions in n-polydisc and n-dimensional ball) are also presented. First we exhibit a class of elementary projections which play the crucial role in our paper. Those are projections given by linear extension operators from certain subsets of the fibres of $\mathfrak{M}(H_\infty)$. Using those projections, we show that H_∞ is isomorphic to its direct sum in the sense of l_∞. Applying the result of Bočkariov [2], we infer that H_∞ is isomorphic to a second conjugate space, thus answering the question of Rickart, asked in [22]. We show the isomorphic character of this result, proving that isometrically H_∞ has a unique predual space (this result answers the question of Porcelli [24] problem 59) and is not isometric to the second conjugate space of any Banach space. This is done in Section 1.

Section 2 contains the proof that the group of linear isomorphisms of $H_\infty(U)$ is contractible. This is done by using the general scheme elaborated by B. S. Mitiagin [17]. We show that this scheme is applicable by a detailed analysis of certain elementary projections, used also in Section 1.

In Section 3 we consider spaces of bounded analytic functions in polydiscs $U^n \subset C^n$ and balls $B_n \subset C^n$. We are able to generalize our main results to polydiscs. The space $H_\infty(U^n)$ is isomorphic to its direct sum in the sense of l_∞ and has a contractible linear group. As regards the space $H_\infty(B_n)$, we show that it is isomorphic to its l_∞-sum.