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On M-hyponormal operators

by

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Abstract. Direct integral decompositions of dominant (or M-hyponormal) operators and spectral operators which are quasi-affine transforms of M-hyponormal operators are considered.

According to Stampfli and Wadhwa [6], a (bounded) operator T on a Hilbert space H is said to be *dominant* if range$(T-z) \subseteq$ range$(T-z)^*$ for all $z \in \boldsymbol{C}$, and T is said to be *M-hyponormal* if

$$\|(T-z)^* x\| \leqslant M\|(T-z)x\|$$

for all $z \in \boldsymbol{C}$ and $x \in H$. It is not hard to see that the following statements are each equivalent to each other:

1. T is dominant.
2. For each $z \in \boldsymbol{C}$, there is an operator A_z such that

$$T-z = (T-z)^* A_z.$$

3. For each $z \in \boldsymbol{C}$, there is a positive number M_z such that

$$\|(T-z)^* x\| \leqslant M_z\|(T-z)x\| \qquad (x \in H),$$

i.e.,

$$(T-z)(T-z)^* \leqslant M_z^2 (T-z)^*(T-z).$$

This follows from [1]. Also [1] implies that T is M-hyponormal if and only if, for each $z \in \boldsymbol{C}$, there is an operator A_z such that $\|A_z\| \leqslant M$ and $T-z = (T-z)^* A_z$.

In this paper we present some variants of the results in [6]. First we record a lemma which appears in [3].

LEMMA 1. *Let T be a spectral operator on a Hilbert space H with the resolution of the identity E. Let C be a closed set in $\boldsymbol{C}$ and $x \in H$. If there exists a bounded function $g\colon \boldsymbol{C}-C \to H$ such that $(T-z)g(z) = x$ for all z, then $E(C)x = x$.*

The next lemma is the basis of the subsequent results. The proof is a modification of [6].

LEMMA 2. *Let T be an M-hyponormal operator. Suppose there exists an operator W one-one with dense range and a spectral operator S such that $TW = WS$. Then there exist a positive operator P, a normal operator N and a quasi-nilpotent operator Q such that $(T-N)P = PQ$ and $TN = NT$.*

Proof. By the polar decomposition of W and the fact that S is spectral, we may replace W by a positive operator P and assume that the scalar part N of S is normal. Let $N = \int z dE_z$ be the spectral decomposition of N. Since T is M-hyponormal, for each z in C, there is an operator A_z such that $\|A_z\| \leqslant M$ and $T-z = (T-z)^* A_z$. Let K be a closed set in C and $x \in E(K)H$. Then there is an analytic function $f\colon C-K \to H$ such that $(S-z)f(z) = x$. Thus, for $z \notin K$,

$$(T-z)^* A_z Pf(z) = (T-z)Pf(z) = P(S-z)f(z) = Px.$$

Hence

$$(S-z)^* PA_z Pf(z) = P(T-z)^* A_z Pf(z) = P^2 x.$$

Let C be an arbitrary closed set in C containing K^* $(= \{z \in C\colon \bar{z} \in K\})$ and a neighborhood of the infinity. Then $g(z) = PA_{\bar{z}} Pf(\bar{z})$ is bounded on $C-C$ and $(S^*-z)g(z) = P^2 x$. By Lemma 1, $P^2 x \in E(C^*)x$. (Note that $X \to E(X^*)$ is the spectral measure of N^* which is the scalar part of S^*.) Therefore $P^2 x \in E(K)H$. We have shown that $E(K)H$ is an invariant subspace of P^2 for every closed set K in C. Regularity of the spectral measure E thus implies that N commutes P.

Now the identity $TP = PS$ can be written $(T-N)P = PQ$. Furthermore,

$$NTP = N(PS) = PNS = PSN = TPN = TNP.$$

Since the range of P is dense, we have $TN = NT$. ∎

COROLLARY 3. *If a spectral operator is M-hyponormal, then it has a normal scalar part.*

Proof. From the proof of Lemma 2, we see that if W is invertible, then so is P. Hence there is a normal operator N such that $TN = NT$ and $T-N$ is quasi-nilpotent. The conclusion follows from the uniqueness of the canonical reduction of a spectral operator (see Dunford and Schwartz [2], Theorem XV, 4.5). ∎

The following corollary is a special case of [3]; ([3] is based on a result of Putnam [4]).

COROLLARY 4. *If $TW = WS$, where S is spectral, T is hyponormal and W has a dense range, then T is normal, S is a scalar operator and S is similar to T.*

Proof. From Lemma 2, we have $TN = NT$ and $(T-N)P = PQ$ where N is a normal operator, P is a positive operator with a dense range and Q is similar to the radical part of S. Now it suffices to show that $T-N = 0$.

Since N is normal and $TN = NT$, Fuglede's theorem yields $T^*N = NT^*$. Furthermore, since T is hyponormal, we have, for each $x \in H$ and $z \in C$,

$$\|(T-N-z)^* x\|^2 = \|(T-z)^* x\|^2 - 2\,\mathrm{Re}\big((T-z)^* x \,|\, N^* x\big) + |z|^2 \|x\|^2$$
$$= \|(T-z)x\|^2 - 2\,\mathrm{Re}\big(Nx \,|\, (T-z)x\big) + |z|^2\|x\|^2 = \|(T-N-z)x\|^2.$$

Therefore $T-N$ is hyponormal.

Next, for a bounded operator A and $k > 0$, we write $M(A;k)$ for the spectral manifold

$$\{x \in H\colon \text{there is an analytic function } f\colon \{z;\,|z| > k\} \to H \text{ such that } (A-z)f(z) = x \text{ for all } z\}.$$

It follows from the Laurant expansion that this set is equal to

$$\{x \in H\colon \limsup_n \|(A-z)^n x\|^{1/n} \leqslant k\}.$$

From $(T-N)P = PQ$, we have $P\big(M(Q;k)\big) \subseteq M(T-N;k)$. Note that $M(Q;k) = H$ for all $k > 0$ and $M(T-N;k)$ is always closed. (In fact, $M(T-N;k) = \{x \in H\colon \|(T-N)^n x\| \leqslant k^n \|x\| \text{ for all } n \geqslant 1\}$, since a hyponormal operator is paranormal.) Hence $M(T-N;k) = H$ for all $k > 0$. By Baire's category theorem, it is easy to show that $\mathrm{Sp}(T-N) = \{0\}$. Now $T-N$ is a quasi-nilpotent hyponormal operator. Hence $T-N = 0$. ∎

Next we consider direct integral decompositions of M-hyponormal operators.

LEMMA 5. *Let $T = \int_X^{\oplus} T(t)\,dm(t)$ be a direct integral decomposition of T.*

(a) *If T is dominant, then $T(t)$ is dominant a.e. (t).*

(b) *T is M-hyponormal if and only if $T(t)$ is M-hyponormal a.e. (t).*

Proof. Since the proof of (b) is similar to and easier than (a), we only prove part (a). By hypothesis, for each $z \in C$, there exits a positive number M_z such that the operator

$$D_z = M_z (T-z)^*(T-z) - (T-z)(T-z)^*$$

is positive. (For definiteness, we assume that M_z is the smallest positive number making $D_z \geqslant 0$.) Hence $D_z(t) \geqslant 0$ a.e. (t) for each z. Let $P_n = \{z \in C;\ M_z \leqslant n\}$. Then $\bigcup_{n=1}^{\infty} P_n = C$. Let Q_n be a countable dense subset of P_n. Let $Y = \{t \in X\colon D_z(t) \geqslant 0 \text{ for } z \in \bigcup_{n=1}^{\infty} Q_n\}$. Then $m(X-Y) = 0$. Now it is easy to check that $T(t)$ is dominant for $t \in Y$. ∎

LEMMA 6. *Let T be a dominant operator. Then*

(a) $\ker(T-z)^2 = \ker(T-z) \subseteq \ker(T-z)^*$ *for each* $z \in C$, *and*

$$\ker(T-z) \perp \ker(T-z') \quad \text{if} \quad z \neq z',$$

(b) *if T is algebraic or of finite rank, then T is normal.*

Proof. Straightforward. ■

The following theorem follows immediately from the above two lemmas.

THEOREM 7 (see [6]). *If T is dominant and either T is n-normal or there is a nonconstant polynomial p such that $p(T)$ is normal, then T is normal.*

As a result of Lemma 5, we obtain:

COROLLARY 8. *If T is M-hyponormal, N is normal and $TN = NT$, then $T+N$ is M-hyponormal.*

Remark 1. The above corollary fails if "M-hyponormal" is replaced by "dominant". Take any dominant operator S which is not M-hyponormal for every $M > 0$. (Such operator exists, see e.g. [6].) Let T be a direct sum of countably many copies of S, say $T = \sum_{k=1}^{\infty}{}^{\oplus} S_k$, with S_k is unitarily equivalent to S for each k. We can choose $z_k \in C$ such that $\lim_{k\to\infty} M_k = \infty$, where

$$M_k = \inf\{M > 0: \|(S-z_k)^* x\| \leqslant M\|(S-z_k)x\| \text{ for all } x\}.$$

Obviously $\{z_k\}_k$ must be bounded. Let $N = \sum^{\oplus} z_k I_k$. Then there is no positive number M such that $\|(T+N)^* x\| \leqslant M\|(T+N)x\|$ for each x.

Remark 2. We give an alternative proof of Corollary 8, without using the direct integral technique as follows: Let $N = \int_{\mathrm{Sp}(N)} z\,dE_z$ be the spectral decomposition of N. Take a partition $B = \{B_1, \ldots, B_n\}$ of $\mathrm{Sp}(N)$ into Borel sets of small diameter. Take some z_k in B_k for each k. Put $N_B = \sum_{k=1}^{n} z_k E(B_k)$. Now each $E(B_k)H$ reduces T. Let $T_k = T | E(B_k)H$. Then obviously $T = \sum^{\oplus} T_k$ and each T_k is M-hyponormal. Hence, for each k, there exists an operator A_k on $E(B_k)H$ such that $\|A_k\| \leqslant M$ and $(T_k+z_k)^* = (T_k+z_k)A_k$. Let $A_B = \sum^{\oplus} A_k$. Then $(T+N_B)^* = (T+N_B)A_B$ and $\|A_B\| \leqslant M$. Note that the net $\{N-N_B: B\}$ tends to zero. Choose a subnet of $\{A_B: B\}$ which converges in the weak operator topology to some A. Then $(T+N)^* = (T+N)A$ and $\|A\| \leqslant M$. Now it is clear that $T+N$ is M-hyponormal.

Combining Corollary 3 and Corollary 8, we obtain:

THEOREM 9. *A spectral operator is M-hyponormal if and only if its scalar part is normal and its radical part is M-hyponormal.*

References

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