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F-spaces with a basis which is shrinking but not hyper-shrinking

by

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Abstract. We provide examples of non-locally convex F-spaces E enjoying the following three properties: (1) E has an absolute basis. (2) The weak topology of E is metrizable. (3) Each basis of E is boundedly complete and shrinking, but no basis of E is hyper-shrinking. The existence of F-spaces satisfying (3) has been conjectured by N. J. Kalton and J. H. Shapiro in a recent work.

1. Introduction. Following Kalton and Shapiro [7], we call a basis for an F-space *shrinking* (resp., *hyper-shrinking*) if each of its bounded block bases tends weakly to zero (resp., tends to zero *in the weak topology of its closed linear span*). It was conjectured in [7] that a shrinking basis need not be hyper-shrinking (while the converse implication holds trivially).

The main purpose of this note is to establish this conjecture. In Theorem 3.3, which is our main result, we indicate a class of (generalized) Orlicz sequence spaces $l(\varphi, a)$ in which every basis, in particular the absolute basis of the unit vectors, is boundedly complete and shrinking but is not hyper-shrinking. We should like to mention that our first example of a space with such properties was the F-space H used by Bessaga, Pełczyński and Rolewicz in [3], pp. 50–51 for a different purpose; the spaces $l(\varphi, a)$ considered in Section 3 are a slight generalization of their example. These spaces are interesting also in that their weak topology is metrizable (although they are *not* locally convex). We give a few rather simple results about such spaces in Section 2. Finally, in Section 4, we collect some comments and open questions.

As regards terminology, we follow, in general, that used by Kalton and Shapiro. Throughout the paper all topological linear spaces are assumed to be Hausdorff and infinite-dimensional. If any topologies we are dealing with are not explicitly specified, we always mean the original topologies of the spaces under consideration. If (x_n) and (y_n) are bases of topological linear spaces X and Y, respectively, then we call them *equivalent* if there is a (topological) isomorphism of $\tilde{X}$ onto $\tilde{Y}$ sending x_n to y_n for all n, where $\tilde{X}$ and $\tilde{Y}$ are the completions of X and Y. If X and Y are F-spaces, then this definition is well known to be equivalent to the following:

For any scalar sequence (t_n), the series $\sum t_n x_n$ converges iff the series $\sum t_n y_n$ converges.

We shall say that a sequence (x_n) in a topological linear space is *irregular* if $t_n x_n \to 0$ for each scalar sequence (t_n).

We denote by ω the locally convex F-space of all scalar sequences with the topology of coordinatewise convergence, and by (e_n) the unit vector basis of the spaces ω, l_p, etc. It is well known that

(A) *All the bases of ω are equivalent to its unit vector basis.* (See [2], Theorem 7, [3], Theorem 5.)

Let E be a metrizable topological linear space whose (topological) dual E' is point-separating. Then the weak topology $\sigma(E) = \sigma(E, E')$ of E is Hausdorff, and the Mackey topology $\tau(E) = \tau(E, E')$ of E is easily seen to be the finest locally convex topology on E weaker than the original topology. The convex neighbourhoods of zero in E form a base at zero for $\tau(E)$, and hence if the original topology of E is metrizable, then so is $\tau(E)$.

We shall make an essential use of the following, somewhat simplified, version of Proposition 3.2 in [6] (see also [9], proof of Theorem 1).

PROPOSITION 1.1. *Let (x_n) be a basis of an F-space E. Then (x_n) is also a basis of E for the Mackey topology. Moreover, for any scalar sequence (t_n), $t_n x_n \to 0$ iff $t_n x_n \to 0$ for the Mackey topology. Hence (x_n) is bounded (resp., regular, irregular) iff it is bounded (resp., regular, irregular) for the Mackey topology.*

The proof is easy and uses the following two facts: (1) The continuity and equicontinuity of linear maps are preserved when the original topologies are replaced by the corresponding Mackey topologies. (2) If a subsequence of a basis of an F-space is regular, then the coefficient functionals corresponding to that subsequence form an equicontinuous collection.

Note that, in Proposition 1.1, the basis (x_n) of $(E, \tau(E))$ is equicontinuous, i.e., the associated sequence of partial sum operators is equicontinuous for the Mackey topology. This is clear from the proof, and is also a consequence of the fact that $\tau(E)$ is barrelled (see, e.g., the proof of Proposition 5.6 in [5]). It follows that (x_n) is also a basis for the completion of $(E, \tau(E))$.

PROPOSITION 1.2. *Let (x_n) be an unconditional basis of an F-space E. Then the following are equivalent:*

(a) *(x_n) is not hyper-shrinking.*

(b) *There is a block basis (y_n) for (x_n) such that (y_n), considered as a basis of $(Y, \tau(Y))$, where $Y = \overline{\text{lin}}(y_n)$, is equivalent to the unit vector basis of l_1.*

Proof. Suppose (x_n) is not hyper-shrinking. Then there is a bounded block basis (y_n) for (x_n) and a continuous linear functional f on $Y = \overline{\text{lin}}(y_n)$ such that $f(y_n) = 1$ for all n ([7], Lemma 4.4). By Proposition 1.1, (y_n) is a bounded unconditional basis of the completion $\tilde{Y}$ of $(Y, \tau(Y))$. It is now easy to see that $\sum t_n y_n$ converges in $\tilde{Y}$ iff $(t_n) \in l_1$. This proves that (a) $\Rightarrow$ (b). The converse implication is obvious.

2. F-spaces with irregular bases. If E is a locally convex space, then the following conditions are equivalent: (1) E' is countably-dimensional; (2) $\sigma(E)$ is metrizable (and hence equals $\tau(E)$); (3) E is isomorphic to a subspace of ω; (4) E has a basis equivalent to the unit vector basis of ω; (5) E has an irregular basis; (6) E contains an irregular sequence with a dense linear span.

The equivalence of conditions (1) through (6) is well known from the theory of locally convex spaces, or at least easy to verify. We indicate only that (3) $\Rightarrow$ (4) holds by a result of Bessaga and Pełczyński [1], Theorem 2 (see also [3], Proposition 2.2), according to which every subspace of ω has an equicontinuous basis, and this basis must be equivalent to the unit vector basis of ω by (A).

Now, if E is an arbitrary topological linear space with a separating dual and such that the locally convex space $(E, \tau(E))$ satisfies the equivalent conditions listed above, we shall say that E is an *ω-space*. If, in addition, E is an F-space, we shall call it an *$F\omega$-space*. We shall see in Section 3 that non-locally convex $F\omega$-spaces do exist. Let us note, however, that if E is an F-space such that each closed subspace of E is an $F\omega$-space (in its relative topology), then $E \cong \omega$, i.e., E is minimal ([5], [7]). In fact, if E is non-minimal, then E contains a regular basic sequence. Then the closed linear span of this sequence admits a continuous norm (see [7], proof of the implication (iii) $\Rightarrow$ (iv) of Theorem 3.2) and hence its weak topology cannot be metrizable. It is obvious that if a topological linear space E with a separating dual contains an irregular sequence whose linear span is dense, then E is an ω-space. Note, however, that the existence of such a sequence alone does not imply the existence of non-trivial continuous linear functionals. (Consider, for instance, the Haar system in the F-space of all measurable functions on $[0, 1]$.) Nevertheless, we have the following

PROPOSITION 2.1. *A topological linear space E is isomorphic to ω iff it contains a sequence (x_n) with a dense linear span and such that the series $\sum t_n x_n$ converges for each $(t_n) \in \omega$.*

Proof. By the Banach–Steinhaus theorem, the linear mapping $T\colon \omega \to 0$ defined by $T((t_n)) = \sum t_n x_n$ is continuous. Since $\omega / T^{-1}(0) \cong \omega$ (by [3], Theorem 4) and ω is a minimal space (see [5], Proposition 4.1), the associated map $\hat{T}\colon \omega/T^{-1}(E)\to E$ is an isomorphism, and its range is evidently all of E. Thus $E\cong\omega$. The converse is obvious.

The next result is an easy consequence of the preceding considerations combined with 1.1 and (A).

PROPOSITION 2.2. *Let E be an F-space with a basis (x_n). Then the following statements are equivalent:*

(a) *(x_n) is irregular.*

(b) *E is an $F\omega$-space.*

(c) *(x_n), as a basis of $(E,\tau(E))$, is equivalent to the unit vector basis of ω.*

(d) *The weak topology of E coincides with the weak topology defined by the sequence of coefficient functionals of (x_n).*

COROLLARY 2.3. *Suppose E is an F-space with an irregular basis. Then every basis of E is irregular and shrinking. Moreover, if E has a boundedly complete basis, then every basis of E is boundedly complete.*

Proof. The first assertion follows directly from the preceding proposition. To prove the second one, we use also Lemma 1(b) of [8], which says that a basis of an F-space is boundedly complete iff every bounded subset is relatively compact for the weak topology defined by the coefficient functionals of the basis. It is therefore enough to observe that, by the above proposition, this topology does not depend on a particular choice of a basis (in an $F\omega$-space).

Thus we have also the following

COROLLARY 2.4. *Let E be an F-space with an irregular boundedly complete basis. Then every bounded subset of E is weakly relatively compact.*

The next corollary will be immediately applicable to the spaces considered in Section 3. Recall that a basis (x_n) of an F-space $E=(E,|\cdot|)$ is said to be *absolute* (with respect to the F-norm $|\cdot|$) if $|x|=\sum|t_nx_n|$ whenever the series $\sum t_nx_n$ converges to x. It is obvious that every block basis of an absolute basis is absolute and that every absolute basis is boundedly complete.

COROLLARY 2.5. *Let $E=(E,|\cdot|)$ be an F-space with an absolute basis (x_n), and let $a_n=\sup_t|tx_n|$. Then:*

(a) *$E\cong\omega$ iff there is an m such that $\sum_{n=m}^{\infty}a_n<\infty$.*

(b) *If $a_n\to 0$ and $\sum_{n=m}^{\infty}a_n=\infty$ for all m, then E is not locally convex and every basis of E is irregular, boundedly complete and shrinking.*

We close this section with an analogue (in fact, a consequence) of a result due to Bessaga and Pełczyński ([1], Lemma 4, [2], Theorem 5); we include a proof for completeness.

PROPOSITION 2.6. *Let E be an F-space which admits no continuous norm. Then if (x_n) is a basis of E, there is a subsequence (y_n) of (x_n) such that (y_n) is irregular.*

Proof. In view of Proposition 1.1 it is enough to show that a subsequence of (x_n) is irregular for the Mackey topology. Let (p_n) be a non-decreasing sequence of semi-norms defining $\tau(E)$. Since the partial sum operators are equicontinuous for $\tau(E)$, we may assume that $p_n\big(\sum_{i=1}^{k}t_ix_i\big)\leqslant p_n\big(\sum_{i=1}^{m}t_ix_i\big)$ for all scalar sequences (t_i) and all n and k,m with $k\leqslant m$. Since none of the semi-norms p_n is a norm, there is a sequence (j_n) of natural numbers such that $p_n(x_{j_n})=0$ for each n. A simple argument shows that (j_n) may be chosen so as to be strictly increasing. It is then obvious that $(y_n)\equiv(x_{j_n})$ is as required.

Remark. Note that the weak topology of E induces on $\overline{\mathrm{lin}}(y_n)$ its proper weak topology.

3. EXAMPLE. Let $\varphi\colon[0,\infty)\to[0,\infty)$ be a non-decreasing, subadditive and continuous function vanishing only at 0, and let $a=(a_n)$ be a sequence of positive numbers. Then we denote by $l(\varphi,a)$ the linear space of all real (or complex) sequences $x=(t_n)$ such that

$$|x|=\sum_{n=1}^{\infty}a_n\varphi(|t_n|)<\infty.$$

Then $|\cdot|$ is an F-norm on $l(\varphi,a)$ and $l(\varphi,a)=\big(l(\varphi,a),|\cdot|\big)$ is an F-space. If $a_n=1$ for all n, then we denote this space simply by $l(\varphi)$. Note that, when dealing with the spaces $l(\varphi)$, we may always assume φ to be bounded. In fact, if we define ψ by $\psi(t)=\min\{1,\varphi(t)\}$, then the identity mapping of $l(\varphi)$ onto $l(\psi)$ is an isomorphism. Evidently, the spaces l_p, $0<p\leqslant 1$, are of the type $l(\varphi)$.

It is clear that the sequence (e_n) of the unit vectors forms an absolute basis for $l(\varphi,a)$. From Corollary 2.5 we infer that $l(\varphi,a)\cong\omega$ iff φ is bounded and $\sum a_n<\infty$.

PROPOSITION 3.1. *Suppose $a=(a_n)$ and $b=(b_n)$ are sequences of positive numbers, and a satisfies*

$$(*)\qquad a_n\to 0\quad\textit{and}\quad\sum_{n=1}^{\infty}a_n=\infty.$$

Then there is a block basis (f_k) for the basis (e_n) of $l(\varphi,a)$ such that (f_k) is equivalent to the basis (e_k) of $l(\varphi,b)$.

Proof. From $(*)$ it easily follows that there exist sequences (m_k) and (n_k) such that $m_1 \leqslant n_1 < m_2 \leqslant n_2 < \ldots$ and

$$(+) \qquad b_k \leqslant A_k = \sum_{m_k \leqslant i \leqslant n_k} a_i \leqslant 2b_k$$

for all k. Define a block basis (f_k) for (e_n) by

$$f_k = \sum_{m_k \leqslant i \leqslant n_k} e_i.$$

Then, given any scalar sequence (t_k), we have

$$\Big|\sum_{k\in F} t_k f_k\Big|_a = \sum_{k\in F} A_k \varphi(|t_k|)$$

for any finite subset F of N. This and $(+)$ imply that (f_k) is equivalent to the unit vector basis of $l(\varphi, b)$.

COROLLARY 3.2. *If a satisfies $(*)$, then a block basis for the basis (e_n) of $l(\varphi, a)$ is equivalent to the basis (e_n) of $l(\varphi)$.*

We finally come to the main result of the paper.

THEOREM 3.3. *Suppose φ is bounded and a satisfies $(*)$. Then $l(\varphi, a)$ is a non-locally convex $F\omega$-space with an absolute basis, and each basis of $l(\varphi, a)$ is irregular, boundedly complete and shrinking, but is not hyper-shrinking.*

Proof. In view of Corollary 2.5 we only have to prove that no basis is hyper-shrinking. Since each basis of $l(\varphi, a)$ is boundedly complete, Theorem 4.9 of [7] implies that either all bases of $l(\varphi, a)$ are hyper-shrinking or else none of them enjoys this property. So it is enough to show that the basis (e_n) is not hyper-shrinking. That it is indeed so follows from Corollary 3.2, Proposition 1.2 and the known fact that the Mackey topology of $l(\varphi)$ is simply the relative l_1-norm topology. (For a general result identifying the Mackey topology of Orlicz spaces, see [6], Theorem 3.3. In our case, where the function φ is subadditive, the elementary inequality $\varphi(t) \geqslant (t/2)\varphi(1)$, $0 \leqslant t \leqslant 1$, (cf. [8], Lemma 4) shows that $l(\varphi)$ embeds continuously in l_1 as a dense subspace. It is then easily seen that the dual of $l(\varphi)$ may be identified with the dual of l_1.)

4. Some comments and open questions.

(1) In [9] Shapiro proves that the weak basis theorem fails in certain F-spaces with bases, and in question (b) on p. 1299 of his paper he asks if this is true for all non-locally convex F-spaces with a weak basis. An examination of the proof of his Theorem 2 reveals that whenever an F-space has a basis which is not irregular, i.e., a basis with a regular sub-basis, then there is a weak basis which is not a basis for the original topology. So his question reduces to this: Does the weak basis theorem fail in every non-locally convex F-space with an irregular basis?

(2) Does every $F\omega$-space have a basis? The author does not even know whether every F-space with a separating dual which is separable under its weak (or Mackey) topology, in particular an $F\omega$-space, must be separable under the original topology.

(3) Does there exist an $F\omega$-space with a hyper-shrinking basis? Does there exist a pseudo-reflexive [7] $F\omega$-space? (We mean of course non-locally convex spaces.)

(4) Proposition 3.1 can be viewed as a result on universality of $l(\varphi, a)$: If a satisfies $(*)$, then $l(\varphi, a)$ contains an isomorphic copy of each $l(\varphi, b)$, for any b. Is it always true that if both a and b satisfy $(*)$, then $l(\varphi, a)$ and $l(\varphi, b)$ are isomorphic?

(5) It is not difficult to see that, in an F-space with an absolute basis, every infinite-dimensional Banach subspace contains a subspace isomorphic to l_1 (cf. [8]). We do not know, however, whether or not every Banach subspace is then isomorphic to a subspace of l_1.

(6) Let (φ_n) be a sequence of non-decreasing, subadditive and continuous functions defined on $[0, \infty)$ and vanishing only at 0. Writing $a_n = \sup_t \varphi_n(t)$, suppose that $a_n \to 0$ and $\sum_{n=m}^{\infty} a_n = \infty$ for each m. Then the F-space $l((\varphi_n))$, defined analogously as $l(\varphi, a)$, is an $F\omega$-space with an absolute basis composed of unit vectors. This class of sequence F-spaces may be identified with the class of F-spaces considered in Corollary 2.5(b).

Additional note. After this paper was submitted for publication, the author answered Shapiro's question in the affirmative, see *The weak basis theorem fails in non-locally convex F-spaces*, Canad. J. Math. 29 (1977), pp. 1069–1071.

References

[1] C. Bessaga and A. Pełczyński, *An extension of the Krein–Milman–Rutman theorem concerning bases to the case of B_0-spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 5 (1957), pp. 379–383.

[2] — — *Własności baz w przestrzeniach typu B_0* (*Properties of bases in spaces of type B_0* (Polish; Russian and English summaries), Prace Matematyczne 3 (1959), pp. 123–142.

[3] — — and S. Rolewicz, *Some properties of the space (s)*, Colloq. Math. 7 (1959), pp. 45–51.

[4] N. J. Kalton, *Bases in non-closed subspaces of ω*, J. London Math. Soc. (2) 3 (1971), pp. 711–716.

[5] — *Basic sequences in F-spaces and their applications*, Proc. Edinburgh Math. Soc. 19 (1974), pp. 151–167.

[6] N. J. Kalton, *Orlicz sequence spaces without local convexity*, Math. Proc. Cambridge Philos. Soc. 81 (1977), pp. 253–277.
[7] — and J. H. Shapiro, *Bases and basic sequences in F-spaces*, Studia Math. 56 (1976), pp. 47–61.
[8] J. H. Shapiro, *On convexity and compactness in F-spaces with bases*, Indiana Univ. Math. J. 21 (1972), pp. 1073–1090.
[9] — *On the weak basis theorem in F-spaces*, Canad. J. Math. 26 (1974), pp. 1294–1300.

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On weakly compact operators from some uniform algebras

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Abstract. We prove that weakly compact operators from a class of uniform algebras, containing many natural algebras of analytic functions on planar compact sets, behave exactly like weakly compact operators from $C(K)$ spaces. In particular, a non-weakly compact operator from such an algebra is an isomorphism on some subspace isomorphic to c_0.

Recently S. V. Kislakov [15] and F. Delbaen [5], generalizing the results of A. Grothendieck [9] and A. Pełczyński [17] and [18], have proved that the disc algebra has the Pełczyński property and the Dunford-Pettis property. Let us recall that the Banach space X has the *Dunford–Pettis property* if every weakly compact operator defined on X transforms weakly convergent sequences into norm convergent sequences. This is equivalent to the following: for every $(x_n) \subset X$, $x_n \xrightarrow{w} 0$ and $(x_n^*) \subset X_n^*$, $x_n^* \xrightarrow{w} 0$ we have $\lim x_n^*(x_n) = 0$.

The Banach space X *has the Pełczyński property* if every non-weakly compact operator defined on X is an isomorphism when restricted to some subspace of X, isomorphic to c_0. We say that the series $\sum f_n$, $f_n \in X$ is *weakly unconditionally convergent* (and abbreviate it to "(f_n) is w.u.c.") if for every $x^* \in X^*$ we have $\sum |x^*(f_n)| < \infty$. With the use of this concept the Pełczyński property can be equivalently defined as follows: A Banach space X has the Pełczyński property if every set $V \subset X^*$ such that $\lim_n \sup\{|x^*(f_n)| : x^* \in V\} = 0$ for every w. u. c. (f_n) in X is weakly relatively compact.

It is well known (cf. [5], [9], [17], [18], [20]) that if the Banach space X has both the Dunford–Pettis property and the Pełczyński property, then

(a) The following are equivalent for an arbitrary Banach space E:
(a_1) An operator $T: X \to E$ is weakly compact.
(a_2) An operator $T: X \to E$ is strictly singular.
(a_3) An operator $T: X \to E$ is not an isomorphism when restricted to any subspace of X isomorphic to c_0.
(a_4) An operator $T: X \to E$ is unconditionally converging.

(b) If $T: X \to X$ is a weakly compact operator, then T^2 is compact.