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On the inversion of pseudo-differential operators

by

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Abstract. Let A be a properly supported pseudo-differential operator on a manifold X. An important problem is to know when A has a pseudo-differential inverse. If this inverse exists, one deduces immediately that A is elliptic and injective in $C_0^\infty(X)$. When X is a compact manifold, these conditions are nearly sufficient. In fact: Let A be a self-adjoint elliptic pseudo-differential operator on a compact manifold X. Suppose that A is injective in $C^\infty(X)$. Then A has a two-sided self-adjoint pseudo-differential inverse (see [1]).

Here we consider the case of a non-compact manifold. We have a function σ with certain properties, and we ask for the existence of properly supported pseudo-differential operators, A and B, such that $A\circ B$ is the identity operator I and σ is a principal symbol of A. We state the main result in 2. The rest of the paper is devoted to prove it.

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1. Notation and basic definitions. Let X be a C^∞ paracompact manifold of dimension N. If ξ is an element in the cotangent bundle $T^*(X)$, the length $|\xi|$ of ξ can be defined in terms of a riemannian metric on X. We shall also assume that there is given a volume element on X, which in any local coordinate system can be expressed as $f dx_1 \ldots dx_n$, with $f \in C^\infty$ and $f > 0$.

A function $a = a(x, \xi) \in C^\infty(T^*(X))$, *is in the class* $S^m_{\varrho,\delta}$, or *is a symbol of order* $m \in \mathbf{R}$ *and type* ϱ, δ, $0 \leqslant 1-\varrho \leqslant \delta < \varrho \leqslant 1$ if, in local coordinates, it satisfies

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta a(x, \xi)\right| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|}$$

for all α, β.

This class is well defined on $T^*(X)$.

A linear and continuous operator $A\colon C_0^\infty(X) \to C^\infty(X)$ *belongs to the class* $I^m_{\varrho,\delta}$, or *is a properly supported pseudo-differential operator of order* m *and type* ϱ, δ if for a given local coordinate system defined in an open set U there exists $a(x, \xi) \in S^m_{\varrho,\delta}$ such that for f with support in U and $x \in U$

$$Af(x) = \int e^{ix\cdot\xi} a(x, \xi) \hat{f}(\xi)\, d\xi$$

and the distribution kernel K_A of A vanishes outside a neighborhood of the diagonal. Therefore, we can write

$$Af(x) = \lim_{\varepsilon\to 0}\int e^{i(x-y)\cdot\xi} a(x, y, \xi)\,\eta(\varepsilon\xi)\, f(y)\, dy,$$

where $a(x, y, \xi)\in S^m_{\varrho,\delta}$, $a(x, y, \xi) = 0$ for $|x-y| > M$ and $\eta\in C_0^\infty$ and equals 1 for $|\xi|\leqslant 1$.

$a(x, y, \xi)$ is called an *amplitude*. The class of $a(x, \xi)$ modulo $S^{m+(\delta-\varrho)}_{\varrho,\delta}$ is well defined on the cotangent bundle $T^*(X)$. It will be called the *principal symbol of* A, $\sigma_p(A)$. If a, b belong to $\sigma_p(A)$, we shall write $a\sim b$. A function on $T^*(X)$ which belongs to $\sigma_p(A)$ will also be called a *principal symbol of* A. We shall say that an operator A in $I^m_{\varrho,\delta}$ is *elliptic* if it has a principal symbol a such that $|a(x, \xi)|\geqslant C\,|\xi|^m$, for $|\xi|\geqslant R$ and $(x, \xi)\in T^*(X)$. The operator A *belongs to* $I^{-\infty}$ if it is in $I^m_{\varrho,\delta}$, for all $m\in \boldsymbol{R}$. For the properties of pseudo-differential operators that we shall use, see [1].

2. The main result.

THEOREM 1. *Let* $\sigma = \sigma(x, \xi)$ *be a real function* $\in C^\infty(T^*(X))$, *positively homogeneous of degree* $m\in\boldsymbol{R}$ *in* ξ *for* $|\xi|\geqslant 1$ *and such that* $\sigma(x, \xi) > 0$ *for all* $(x, \xi)\in T^*(X)$. *Then*

(i) *If* $m = 0$, *there exist operators* A, $B\in I^0_{1,0}$ *with*

$$A\circ B = I,\qquad \sigma_p(A)\sim\sigma.$$

(ii) *If* $m\neq 0$, *there exist operators* A, B *satisfying*

$$A\in I^m_{1,\delta},\ B\in I^{-m}_{1,\delta},\qquad \text{for all } 0<\delta<1,$$

$$A\circ B = I,\qquad \sigma_p(A)\sim\sigma.$$

3. The local version of the main result. We shall first show the existence of an adequate covering of the manifold X.

LEMMA 1. *There exists a covering* $\{U^a_n\}^{1\leqslant a\leqslant k}_{n\geqslant 1}$ *of* X *with relatively compact open sets such that, for each fixed* a, $U^a_n\cap U^a_h = \emptyset$ *if* $n\neq h$.

Proof. When X is a Euclidean space, there exists a family of cubes $\{Q^a_n\}$ satisfying those conditions.

Then, let us consider the general case. The Whitney immersion theorem asserts that there exists a differentiable mapping $f\colon X\to \boldsymbol{R}^{2N+1}$ such that X is homeomorphic to $f(X)$, with the topology induced by $\boldsymbol{R}^{2N+1}$ and $f(X)$ is closed in $\boldsymbol{R}^{2N+1}$ (see [2]).

If $\{Q^a_n\}$ is the covering of $\boldsymbol{R}^{2N+1}$ with cubes as above, the sets $U^a_n = f^{-1}[f(X)\cap Q^a_n]$ yield the desired covering of X.

Let $\{\varphi^a_n\}$ now be a partition of unity subordinated to the covering $\{U^a_n\}$. Consider, for each a, n,

$$\sigma^a_n = e^{\varphi^a_n\log\sigma} = \sigma^{\varphi^a_n}.$$

It is easy to see that:

(i) If $m = 0$, $\sigma^a_n\in S^0_{1,0}$.

(ii) If $m > 0$, $\sigma^a_n\in S^m_{1,\delta}$ for all $0<\delta<1$.

(iii) If $m < 0$, $\sigma^a_n\in S^0_{1,\delta}$ for all $0<\delta<1$.

In all three cases, $\sigma^a_n(x, \xi) = 1$, when $x\in X\setminus K^a_n$, for a compact set $K^a_n\subset U^a_n$.

We shall suppose, without loss of generality, that m is non-negative. In this case, $1/\sigma^a_n\in S^0_{1,0}$ when $m = 0$, and $1/\sigma^a_n\in S^0_{1,\delta}$ for all $0<\delta<1$ when $m>0$.

We are now in position to give the localized version of Theorem 1:

THEOREM 2. *There are operators* $A = A^a_n$, $B = B^a_n$ *such that*

(i) *If* $m = 0$, then $A, B\in I^0_{1,0}$, *and*

$$A\circ B = I,\qquad \sigma_p(A)\sim\sigma^a_n,$$

$A = I+\tilde{A}$, $B = I+\tilde{B}$, *with* $\tilde{A}$, $\tilde{B}\in I^0_{1,0}$, *and the corresponding distribution kernels have compact support contained in* $U^a_n\times U^a_n$.

(ii) *If* $m>0$, then $A\in I^m_{1,\delta}$, $B\in I^0_{1,\delta}$ *for all* $0<\delta<1$.

Furthermore, these operators satisfy the conditions in (i), *except that now* $\tilde{A}\in I^m_{1,\delta}$ *and* $\tilde{B}\in I^0_{1,\delta}$, *for all* $0<\delta<1$, *instead of* $I^0_{1,0}$.

We shall prove this theorem in 5. If we accept it for a moment, we derive from it Theorem 1 in the following way:

For $1\leqslant a\leqslant k$ fixed, the compositions

$$A^a = \ldots\circ A^a_n\circ\ldots\circ A^a_2\circ A^a_1,$$
$$B^a = B^a_1\circ B^a_2\circ\ldots\circ B^a_n\circ\ldots$$

have a meaning as operators on $C_0^\infty(X)$ and they define properly supported pseudo-differential operators. In fact, given a compact subset K of X and f with support in K, since the U^a_n are disjoint, there exists $N(K)$ such that

$$A^a_n\circ\ldots\circ A^a_1(f) = A^a_N\circ\ldots\circ A^a_1(f)\quad\text{for}\quad n\geqslant N(K).$$

Furthermore, there exists also $N_1(K)$ such that

$$B^a_n(f) = f\quad\text{for}\quad n\geqslant N_1(K).$$

Then, the operators

$$A = A^k\circ\ldots\circ A^1\qquad B = B^1\circ\ldots\circ B^k$$

satisfy Theorem 1.

Remark. In general, given two operators $A\in I^{m_1}_{\varrho,\delta}$, $B\in I^{m_2}_{\varrho,\delta}$, $C = A\circ B\in I^{m_1+m_2}_{\varrho,\delta}$. But in certain cases the order of C can be less than m_1+m_2 as happens, for example, in this case.

4. Preliminary results.

LEMMA 2. *Let U be a relatively compact open subset of X and let $\sigma = \sigma(x, \xi) \in S^m_{\varrho,\delta}$, $m \geqslant 0$, a real symbol such that, for $0 < c_1 < c_2$ and a compact set $K \subset U$,*

$$\sigma(x, \xi) > c_1 \quad \text{for all } (x, \xi) \in T^*(X),$$

$$\sigma(x, \xi) = c_2 \quad \text{if} \quad x \in X \setminus K,$$

$$\sqrt{\sigma - c_1} \in S^{m/2}_{\varrho,\delta}.$$

Then there exists a self-adjoint operator $A \in I^m_{\varrho,\delta}$ such that

$$\sigma_p(A) \sim \sigma, \quad A \text{ is injective in } C_0^\infty(X),$$

$A = c_2 I + C$, $C \in I^m_{\varrho,\delta}$ and the support of the distribution kernel K_C of C is a compact subset of $U \times U$.

Proof. It suffices to define

$$A = c_1 I + (\tilde{A} + \sqrt{c_2 - c_1}\, I)^* \circ (\tilde{A} + \sqrt{c_2 - c_1}\, I)$$

where the principal symbol of $\tilde{A}$ coincides with $\sqrt{\sigma - c_1} - \sqrt{c_2 - c_1}$ and its kernel $K_{\tilde{A}}$ vanishes outside an adequate neighborhood of the diagonal in $X \times X$.

LEMMA 3. *Let U be a relatively compact open subset of X and $\sigma = \sigma(x, \xi) \in S^0_{\varrho,\delta}$ a real symbol such that*

$$\sigma(x, \xi) > 0 \quad \text{for all } (x, \xi) \in T^*(X), \quad \sqrt{\sigma} \in S^0_{\varrho,\delta},$$

$1/\sigma \in S^{m'}_{\varrho,\delta}$, for $m' \geqslant 0$ and satisfies the hypothesis of Lemma 2.

Then there exists a self-adjoint operator $B \in I^0_{\varrho,\delta}$ with

$$\sigma_p(B) \sim \sigma, \quad B \text{ is injective in } C_0^\infty(X),$$

$B = cI + C$, for some $c > 0$ and $C \in I^0_{\varrho,\delta}$, such that the support of the distribution kernel K_C is a compact subset of $U \times U$.

Proof. If $\sigma(x, \xi) = c$, $c > 0$, in the complement of a compact subset K of U, there exists $\tilde{B} \in I^0_{\varrho,\delta}$ such that the principal symbol of $\tilde{B}$ coincides with $\sqrt{\sigma} - \sqrt{c}$ and its kernel $K_{\tilde{B}}$ vanishes outside a neighborhood of the diagonal in $X \times X$. Then the operator $B_1 = (\tilde{B} + \sqrt{c}I)^* \circ (\tilde{B} + \sqrt{c}I)$ satisfies all conditions except, possibly, the injectivity. We shall modify B_1, in order to obtain also this condition.

There is an operator $A \in I^{m'}_{\varrho,\sigma}$ satisfying Lemma 2 with respect to $1/\sigma$. Since $\sigma_p(A \circ B_1) \sim 1$, we can write $A \circ B_1 = I + D$, where D belongs to $I^{-(\varrho-\delta)}_{\varrho,\delta}$ and the distribution K_D has compact support in $U \times U$.

Now, D is a compact operator in $L^2(X)$ and so the null space of $I + D$ is finite dimensional. Furthermore, it consists of C_0^∞ functions (see [1], p. 36). Then, the null space M of B_1 as a bounded operator in $L^2(X)$ is also finite dimensional and consists of functions in $C_0^\infty(U)$.

Let $\{f_1, \ldots, f_h\}$ be an orthonormal basis for M and let

$$Bf = B_1 f + \sum_{j=1}^{h} (f, f_j)_{L^2} f_j, \quad f \in L^2(X).$$

The operator $\sum_{j=1}^{h} (f, f_j)_{L^2} f_j$, is the orthogonal projection on M. Since the f_j are in $C_0^\infty(U)$, it is an integral operator with C_0^∞ kernel, of compact support contained in $U \times U$. Thus, B satisfies the same conditions as B_1. Furthermore, B is injective in $L^2(X)$. For if $Bf = 0$, since B_1 is self-adjoint, we have

$$0 = (B_1 f, f_k)_{L^2} = -\sum_{j=1}^{h} (f, f_j)_{L^2} (f_j, f_k)_{L^2} = -(f, f_k)_{L^2}.$$

Then, $B_1 f = Bf = 0$. Therefore, f is in M and since f is orthogonal to M, we conclude that $f = 0$.

It will be necessary to know, in Theorem 2, whether the L^2-inverse of a certain pseudo-differential operator is also pseudo-differential (see [1]):

THEOREM 3. *Let $C \in I^m_{\varrho,\delta}$, $m < 0$, and suppose that the distribution kernel K_C of C has compact support $\subset K \times K$. Suppose further that $I + C$ is injective in $C_0^\infty(X)$. Then $I + C$ has a two-sided inverse of the form $I + C'$, $C' \in I^m_{\varrho,\delta}$.*

To prove this, we need the following

LEMMA 4. *Let $\Phi\colon X \to L^2(X)$ be a strong measurable mapping with compact support $K_1 \subset X$. Then there exists a measurable function $\varphi\colon X \times X \to \mathbf{C}$, $\mathbf{C}$ the complex field, such that*

$$\Phi(x)(y) = \varphi(x, y) \quad \text{a.e. in } X.$$

Proof of Lemma 4. For each $x \in X$, $\Phi(x)$ is a class of square integrable functions such that any two of them coincide almost everywhere. We want to show that it is possible to select an element h_x in each class such that $\varphi(x, y) = h_x(y)$ is a measurable function on $X \times X$. The strong measurability of Φ, implies that there exists a sequence $\{\Phi_n\}_{n \geqslant 1}$ such that

(i) $\Phi_n(x, y) = \sum_{j=1}^{H(n)} h_j^{(n)}(y)\, \chi_j^{(n)}(x)$, where $h_j^{(n)} \in L^2(X)$, and for each n, $\chi_j^{(n)}$ are the characteristic functions of measurable disjoint subsets of X.

(ii) $\lim_{n \to \infty} \int_X |\Phi_n(x, y) - \Phi(x)(y)|^2\, dy = 0$ a.e. in X.

Let $C_n = \{x \in X \mid \|\Phi_n(x, \quad)\|_{L^2} \leqslant 2\|\Phi(x)\|_{L^2} \leqslant 2n\}$. Writing $\Phi_n =$

$= \Phi_n(x,y)\cdot\chi_{C_n}(y)$, χ_{C_n} the characteristic function of C_n, we have

$$\|\tilde{\Phi}_n(x,\ )\|_{L^2} \leqslant 2\|\Phi(x)\|_{L^2}, \quad x\in X,\ n\geqslant 1.$$

Let $K_1^{(m)} = \{x\in K_1|\ \|\Phi(x)\|_{L^2}\leqslant m\}$, $m\geqslant 1$. According to the dominated convergence theorem we have

$$\lim_{n,k\to\infty}\int\limits_{K_1^{(m)}\times X} |\tilde{\Phi}_n(x,y)-\tilde{\Phi}_k(x,y)|^2\,dx\,dy = 0.$$

Since $K_1^{(m)}\subset K_1^{(m+1)}$, we can extract a subsequence $\{\tilde{\Phi}_j(x,y)\}$ converging a.e. in $K_1\times X$; its limit is the desired function $\varphi(x,y)$.

Proof of Theorem 3. First, observe that $I+C$ is elliptic and is a bounded operator in $L^2(X)$. If f is in the null space of $I+C$, then $f\in C^\infty(X)$ and f has support in K, since $f = -Cf$ and Cf has support in K. Thus, $f\in C_0^\infty(X)$ and our hypotheses imply that $f=0$. Therefore, $I+C$ is injective in $L^2(X)$ and since C is a compact operator in $L^2(X)$, it follows that $I+C$ has a two-sided inverse B_1 as an operator in $L^2(X)$. Since

$$B_1\circ(I+C)f = f, \quad (I+C)\circ B_1f = f, \quad f\in L^2(X),$$

given the assumed properties of C, it follows that $B_1f=f$ if $\operatorname{supp}(f)$ is disjoint from K, and $B_1f=f$ outside K, for all f. Since $I+C$ is elliptic, it has a two-sided inverse B, modulo $I^{-\infty}$. This inverse has the form $B = I+C'$, with $C'\in I^m_{\varrho,\delta}$. Now, B can actually be taken so that if K_1 is a compact set containing K in its interior, then $Bf=f$ for $\operatorname{supp}(f)$ disjoint from K_1 and $Bf=f$ outside K_1, for all f. For, let φ vanish outside K_1 and let $\varphi=1$ on K. Then $I+\varphi C'\varphi$ has the desired property. (See [1], p. 37.) Furthermore, it is also a two-sided inverse of $I+C$ modulo $I^{-\infty}$.

Now, let

$$B\circ(I+C) = I+R_1, \quad (I+C)\circ B = I+R_2,$$

then R_1 and R_2 belong to $I^{-\infty}$. Setting $S = B-B_1$, we will show that $S\in I^{-\infty}$ which will imply that $B_1\in I^0_{\varrho,\delta}$ and our assertion will be established. Now, from the preceding identities and the fact that

$$B_1\circ(I+C) = (I+C)\circ B_1 = I,$$

we obtain

$$S\circ(I+C) = R_1, \quad (I+C)\circ S = R_2$$

and multiplying on the right and on the left by B, we obtain, respectively,

$$S+S\circ R_2 = R_1\circ B, \quad S+R_1\circ S = B\circ R_2,$$

and multiplying the first equation on the left by R_1 and subtracting from the second we obtain

$$S = R_1\circ S\circ R_2 - R_1^2\circ B + B\circ R_2.$$

Since $Bf=f$ and $B_1f=f$ if $\operatorname{supp}(f)$ is disjoint from K_1, then $Sf=0$ if $\operatorname{supp}(f)$ is disjoint from K_1. Furthermore, since $Bf=f$ and $B_1f=f$ outside K_1, it follows that $\operatorname{supp}(Sf)\subset K_1$.

Now, suppose we show that $R_1\circ S\circ R_2$ is an integral operator with square integrable kernel. Then, since $R_1^2\circ B$ and $B\circ R_2$ are in $I^{-\infty}$, they are also integral operators with square integrable kernels, and S itself will be an integral operator with a square integrable kernel. Thus, on account of the above properties of S, this kernel will have support in $K_1\times K_1$. But then, as is readily seen, $R_1\circ S\circ R_2$ is an integral operator with a compactly supported $C^\infty(X\times X)$ kernel and therefore $R_1\circ S\circ R_2\in I^{-\infty}$. Thus, we will have that $S\in I^{-\infty}$, and our theorem will be established.

Let $x\in X$. The linear mapping

$$L^2(X)\to C,$$
$$f\to R_1\circ S\circ R_2f(x)$$

is continuous and, therefore, there exists $h_x\in L^2(X)$ such that

$$R_1\circ S\circ R_2f(x) = (f, h_x)_{L^2}.$$

If we show that the mapping

$$K_1\overset{\Phi}{\to} L^2(X),$$
$$x\to h_x$$

is strongly measurable, on account of Lemma 4, we will deduce that there exists φ such that $\varphi(x,y) = \Phi(x)(y)$ a.e. in X.

It is sufficient to prove that Φ is weakly measurable. This is clear, for, given $f\in L^2(X)$, the function

$$K_1\to C,$$
$$x\to R_1\circ S\circ R_2f(x) = \int\limits_X h_x(y)f(y)\,dy$$

is continuous.

Furthermore, φ is a square integrable function. For, the mapping

$$L^2(X)\to C^0(K_1),$$
$$f\to R_1\circ S\circ R_2f$$

is continuous, where $C^0(K_1)$ indicates the class of continuous functions $f\colon X\to C$ with compact support in K_1, with the topology of uniform convergence. Thus, there exists $M > 0$ such that

$$\|f\|_{L^2} \leqslant 1 \quad \text{implies} \quad \sup_{x \in K_1} \Big| \int_X \varphi(x, y) f(y)\, dy \Big| \leqslant 1/M.$$

For a fixed $x \in X$,

$$\Big(\int_X |\varphi(x, y)|^2 dy \Big)^{1/2} = \sup_{\|f\|_{L^2} \leqslant 1} |R_1 \circ S \circ R_2 f(x)| = \sup_{\|f\|_{L^2} \leqslant 1} \Big| \int_X \varphi(x, y) f(y)\, dy \Big| \leqslant 1/M.$$

Therefore

$$\int_{X \times X} |\varphi(x, y)|^2 dy\, dx \leqslant 1/M^2 \operatorname{meas}(K_1).$$

5. Proof of Theorem 2. In order to simplify the notation, U will be a fixed open set of the covering $\{U_n^a\}$ and φ will be the corresponding function in the subordinate partition of unity. Let also $\sigma_1 = \sigma^\varphi$.

It is clear that σ_1 satisfies the assumptions of Lemma 2, with $c_2 = \varrho = 1$, and $\delta = 0$ if $m = 0$ or $0 < \delta < 1$ if $m > 0$.

On the other hand, $1/\sigma_1$ satisfies the assumptions of Lemma 3 with $\varrho = 1$, $m' = m$ and $\delta = 0$ if $m = 0$ or $0 < \delta < 1$ if $m > 0$. Thus, we can obtain operators A and B_1 as in Lemmas 2 and 3, respectively. Therefore $A \circ B_1$ has the properties of Theorem 3 and then its L^2-inverse $\tilde{B}$ is a pseudo-differential operator.

Therefore the operators A and $B_1 \circ \tilde{B}$ verify Theorem 2 with respect to U.

Remark. When $m = 0$, both symbols σ_1 and $1/\sigma_1$ satisfy the assumptions of Lemma 2.

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Extensions of a Fourier multiplier theorem of Paley, II*

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Abstract. Let $A(U^N)$ be the algebra of functions that are analytic in the interior of the unit polydisc U^N and continuous on the closure of U^N. Denote the positive cone in the integer lattice Z^N by Z_+^N; then, for each function f in $A(U^N)$, denote the Taylor coefficients of f by $\{\hat{f}(a)\}_{a \in Z_+^N}$. Call a function p on Z_+^N a *Paley multiplier* if $\sum_{a \in Z_+^N} |p(a)\hat{f}(a)| < \infty$ for all f in $A(U^N)$. Call a region W in Z_+^N a *proper cone* if the ratios $(\min_n a_n)/|a|$, remain bounded away from 0 as a runs through W. Every element of $l^2(Z_+^N)$ is a Paley multiplier; it is shown in this paper that, if p is a Paley multiplier, then $\sum_{a \in W} |p(a)|^2 < \infty$ for every proper cone W. This is a considerable improvement on previous results, but it remains unknown, when $1 < N < \infty$, whether every Paley multiplier belong to $l^2(Z_+^N)$.

The proof is based on a simple construction that also yields partial solutions to some problems about homogenous expansions of functions in $A(U^N)$. Other applications of the constructions are also discussed.

1. Introduction. We use the notation and terminology of Rudin's book [29], except that we denote the Taylor coefficients of a function f in $A(U^N)$ by $\hat{f}(a)$ rather than $c(a)$. Such a function is completely determined by its restriction to the distinguished boundary T^N of U^N, and its Taylor coefficients are just the Fourier coefficients of its restriction to T^N.

Paley's theorem [26] is that, when $N = 1$, every Paley multiplier belongs to $l^2(Z^+)$. Helson [15] found a second proof of Paley's theorem, and generalized it to several variables in the following way. Choose a half-space S in Z^N, and let A be the set of continuous functions on T^N whose Fourier coefficients vanish off S; then a function p, on the set S, has the property that $\sum_{a \in S} |p(a)\hat{f}(a)| < \infty$, for all f in A, if and only if $p \in l^2(S)$. Rudin ([28], p. 222) extended this result to the context of compact abelian groups with totally-ordered dual groups.

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