$h+n^{-1}$ and letting $n\to\infty$. The continuity of Φ and an application of Fatou's lemma give the result for h.

Now assuming $h>0$ and $\|g\|_q=1$, we introduce a change of variable

$$\xi=\varphi(x)\equiv\int_0^x h^p\,dy.$$

The new variable ξ has range $0<\xi<a=\int_0^a h^p\,dy$, and φ is locally absolutely continuous, strictly increasing, and has a locally absolutely continuous inverse. Define a function $f(\xi)$ by the formula

$$f(\varphi(x))=g(x)h(x)^{1-p}.$$

Note that

$$\begin{aligned}\int_0^a f(\xi)^p\,d\xi &= \int_0^a f(\varphi(x))^q\varphi'(x)\,dx\\ &= \int_0^a g^q h^{(1-p)q}h^p\,dx\\ &= \int_0^a g^q\,dx\\ &= 1.\end{aligned}$$

Thus, Theorem 1 implies that

$$\int_0^a \Phi\Big\{\xi-\Big(\int_0^\xi f(\eta)\,d\eta\Big)^p\Big\}\,d\xi\leqslant C_q\|\Phi\|_1.$$

By the changes of variable $\xi=\varphi(x)$ and $\eta=\varphi(y)$, we obtain the theorem.

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A function space $C(X)$ which is weakly Lindelöf but not weakly compactly generated

by

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Abstract. We give an example of a Banach function space $C(X)$, which is weakly Lindelöf but not weakly compactly generated. This solves in the negative an old problem of Corson and a problem of Benyamini, Rudin and Wage.

1. In this paper we construct the following example (the terminology will be explain later).

EXAMPLE. *There is a Banach function space $C(X)$ which is Lindelöf under the weak topology but not weakly compactly generated.*(1)

The example solves in the negative a problem of Corson [2] (see also Lindenstrauss [9], Problems 6 and 6′) and Problem 7 of Benyamini, Rudin and Wage [1] (cf. [1], Corollary 2-2). The reader is referred for the related topics to [2], [9], [13], and [1]. Note that Talagrand [17] showed that a weakly compactly generated Banach space is weakly Lindelöf.

Our topological terminology is taken from [3] and the terminology related of functional analysis follows [14] and [9].

The symbol $C(X)$ stands for the Banach space of all continuous real-valued functions on a compact space S with the sup-norm [14]. The space $C(X)$ is said to be *weakly Lindelöf* if it is Lindelöf under the weak topology. A Banach space E is *weakly compactly generated* if there exist a weakly compact set K in E such that E is the closed linear span of K [9]; if $E=C(S)$, then this is equivalent ([9], Theorem 3.2) to the condition that S is an *Eberlein compact*, i.e., S is homeomorphic to a weakly compact subset of a Banach space. Recall that for a compact scattered space S the weak topology of $C(S)$ coincide in the unit ball with the topology

(1) K. Kunen constructed under the Continuum Hypothesis (preprint 1975) a compact scattered space K of cardinality $\aleph_1$ such that every finite product of K is hereditarily separable (Kunen showed that the existence of such K is in fact independent on the usual axioms for set theory); one can verify that $C(K)$ is weakly Lindelöf (even hereditarily), but not weakly compactly generated (see [10], Remark 2).

of pointwise convergence ([14], Corollary 19.7.7); in particular, in this case the space $C(S)$ is weakly Lindelöf iff it is Lindelöf in the topology of pointwise convergence.

We shall use the following notation. The symbol $\boldsymbol{R}$ stands for the real line, P denote the space of irrationals and N — the space of natural numbers. $D = \{0, 1\}$ stands for the two-point discrete group. Given two topological spaces S and T we denote by $C(S, T)$ the space of all continuous functions from S to T endowed with the topology of pointwise convergence; the space $C(S, D)$ will be considered as the abelian group with the pointwise group operations. For an abelian group G and A, $B \subset G$ we put $A+B = \{a+b\colon a \in A, b \in B\}$ and we write $A+\{b\} = A+b$. Finally, we write $S \underset{\text{cl}}{\subset} T$ when S can be embedded in T as a closed subspace and $|A|$ stands for the cardinality of a set A.

2. The space $C(X)$. The following simple construction seems belong to the topological folklor (cf. [12], the proof of Theorem 1, or [5], the proof of Theorem 4.1, or else [14], 8.5.10 (G)).

Let Ω be the set of all countable ordinals, let $\Gamma \subset \Omega$ be the set off all non-limit ordinals and $\Lambda = \Omega \setminus \Gamma$ be the set of all limit ordinals from Ω. Attach to each $\lambda \in \Lambda$ a sequence $s_\lambda\colon N \to \Gamma$ such that $s_\lambda(n) < \lambda$ for $n \in N$ and $\lim s_\lambda(n) = \lambda$. We give the set Ω a topology as follows: the points from Γ are isolated and the basic neighbourhoods of a point $\lambda \in \Lambda$ are of the form $\{\lambda\} \cup \{s_\lambda(n)\colon n \geqslant m\}$. We let $X = \Omega \cup \{\omega_1\}$ to be the one-point compactification of the locally compact space Ω, where ω_1 is the "point at infinity".

M. Wage ([16], Example p. 20) observed that the space X is not an Eberlein compact([2]) and thus the space $C(X)$ is not weakly compactly generated (see Section 1).

3. Auxiliary lemmas. Our goal is to prove that $C(X)$ is weakly Lindelöf, or — equivalently — that $C(X, \boldsymbol{R})$ is a Lindelöf space (see Section 1). It is reasonable to reduce this problem first to a simpler one (Lemma 1) and then to give a simple sufficient condition (Lemma 4) which we shall verify in the next section. Our approach is somewhat more general that it is realy needed, however we think that this shed a proper light on our situation.

LEMMA 1. *Let S be a compact zero-dimensional space. The space $C(S, \boldsymbol{R})$ is Lindelöf if and only if the product $C(S, D)^N$ is Lindelöf. Moreover, given a point $p \in S$, the space $C(S, D)$ can be replaced in this equivalence by the space $G_p = \{f \in C(S, D)\colon f(p) = 0\}$.*

([2]) This follows easily from Rosenthal's [13] characterization of Eberlein compacts and the well-known theorem on regressive functions on normal sets of ordinals ([8], Theorem 8, p. 347).

Proof. One can restrict ourselves to infinite S; in this case the space $C(S, D)$ is not countably compact, i.e., $N \underset{\text{cl}}{\subset} C(S, D)$. By the exponential law [3] we have

$$(1) \qquad C(S, D)^N \underset{\text{top}}{=} C(S, D^N) \underset{\text{cl}}{\subset} C(S, \boldsymbol{R})$$

which proves the necessity. To prove sufficiency let us take a countable dense set $Q \subset D^N$ and let for every $q \in Q$ the family $\{V_{qn}\}$ be an open base at q. Put $L_q = \{f \in C(S, D^N)\colon q \notin f(S)\}$; we verify that

$$(2) \qquad L = \bigcap_{q \in Q} L_q \text{ is Lindelöf.}$$

Indeed, since $L_q = \bigcup_{n \in N} \{f \in C(S, D^N)\colon f(S) \cap V_{qn} = \emptyset\}$ is an F_σ-set in $C(S, D^N)$ and hence it is a continuous image of a closed subset of the product $C(S, D^N) \times N$, the product $\underset{q \in Q}{P} L_q$ is a continuous image of a closed subset of the product $C(S, D^N)^N \times N^N \underset{\text{cl}}{\subset} C(S, D^N)$. Now, (2) follows from the observation that L is homeomorphic to the "diagonal" of the space $\underset{q \in Q}{P} L_q$.

We have $L = \{f \in C(S, D^N)\colon f(S) \cap Q = \emptyset\} \underset{\text{top}}{=} C(S, P)$, as $D^N \setminus Q \underset{\text{top}}{=} P$, and it remains to prove that the space $C(S, \boldsymbol{R})$ is a continuous image of the space $C(S, P)$. To this end let us choose an open mapping $u\colon P \xrightarrow[\text{onto}]{} \boldsymbol{R}$ and put $F(f) = u \circ f$. Then $F\colon C(S, P) \to C(S, \boldsymbol{R})$ is continuous. For a $g \in C(S, \boldsymbol{R})$ the set-valued function $G(s) = u^{-1} \circ g(s)$ has a continuous selection $f \in C(S, P)$ (i.e., $f(s) \in G(s)$ for $s \in S$); indeed, if $U \subset P$ is open, then the set $\{s\colon G(s) \cap U \neq \emptyset\} = g^{-1} \circ u(U)$ is an open F_σ-set in zero-dimensional space S and we can use for example [8], Theorem 1, p. 458. We have $g = F(f)$ and thus F is onto.

The remark about G_p follows from the equality $C(S, D) \underset{\text{top}}{=} G_p \times D$.

A space E is said to have the *strong condensation property* provided that for every uncountable subset A of E there exists an uncountable subset C of A which is *concentrated* around a point $c \in E$ (cf. [7], § 40, VII), i.e., the set $C \setminus V$ is at most countable whenever V is a neighbourhood of c.

LEMMA 2. *Let E be a regular space of weight $\leqslant \aleph_1$ with the strong condensation property. Then the countable product E^N is Lindelöf.*

Proof. Since the weight of $E^N \leqslant \aleph_1$, it is enough to verify that each uncountable subset A of E^N has a point of condensation in E^N, i.e. for some $x \in E^N$ any neighbourhood of the point x contains uncountably many points of A.

Let $A \subset E^N$ be a set of cardinality $\aleph_1$. Let p_n assign to each $x = (x_n) \in E^N$ the nth coordinate x_n of x. We shall choose succesively uncountable sets $A_0 = A \supset A_1 \supset \ldots$ and points $c_1, c_2, \ldots \in E$ such that for an $n \in N$ either the projection p_n restricted to the set A_n is one-to-one and the set $p_n(A_n)$ is concentrated around the point c_n, or else $p_n(A_n) = \{c_n\}$.

Assume that the set A_n is chosen and consider the set $B = p_{n+1}(A_n)$. If B is uncountable, then there exist an uncountable set $C \subset B$ and a point $c_{n+1} \in E$ such that C is concentrated around c_{n+1}; in this case we choose A_{n+1} taking one point from each set $p_{n+1}^{-1}(c) \cap A_n$ where c runs over C. If the set B is countable, then there exists a point $c_{n+1} \in B$ such that the set $A_{n+1} = p_{n+1}^{-1}(c_{n+1}) \cap A_n$ is uncountable.

We shall verify that the point $c = (c_n)$ is a point of condensation of the set A. For let $V = V_1 \times \ldots \times V_k \times E \times \ldots$ be a basic neighbourhood of c. By our choice, for every $n \leqslant k$ the set $p_n(A_k) \setminus V_n = B_n$ is either non-empty, at most countable and the projection p_n restricted to the set A_k is one-to-one, or else B_n is empty. Therefore the set $\bigcup_{n \leqslant k} p_n^{-1}(B_n) \cap A_k = H$ is at most countable and the uncountable set $A_k \setminus H$ is contained in $V \cap A$.

LEMMA 3. *Let G be an abelian topological group and let E be a subset of G such that the product E^N is Lindelöf and for every $a \in G$ there exist $a_1, \ldots, a_m \in E$ such that $a = a_1 + \ldots + a_m$. Then the product G^N is Lindelöf.*

Proof. The space $E^N \times N^N$ is Lindelöf (because either E is compact, or $N \subseteq_{\mathrm{cl}} E$). Let us define a function $f\colon E^N \times N^N \to G^N$ by the formula $f\colon \big((a_1, a_2, \ldots), (m_1, m_2, \ldots)\big) \to (a_1 + \ldots + a_{m_1}, a_{m_1+1} + \ldots + a_{m_1+m_2}, \ldots)$. The function f is continuous and onto.

Let us summarize the result of this section in the form convenient for the application to $C(X, \boldsymbol{R})$ (observe, that the weight of $C(X, \boldsymbol{R})$ is $\aleph_1$).

LEMMA 4. *Let $G = \{f \in C(X, D)\colon f(\omega_1) = 0\}$. If there exists a set $E \subset G$ such that*

(a) *for every $f \in G$ there exist $f_1, \ldots, f_m \in E$ such that $f = f_1 + \ldots + f_m$,*

(b) *the space E has the strong condensation property,*

then the space $C(X, \boldsymbol{R})$ is Lindelöf.

4. The space $C(X)$ is weakly Lindelöf. As was noticed, we have to prove that $C(X, \boldsymbol{R})$ is Lindelöf. Let us put

$$E = \{f \in C(X, D)\colon |f^{-1}(1) \cap \Lambda| \leqslant 1\}. \tag{3}$$

By virtue of Lemma 4 it is enough to prove that $E \subset G$ satisfies the conditions (a) and (b) of this lemma.

Let $f \in G$ and let $f^{-1}(1) \cap \Lambda = \{\lambda_1, \ldots, \lambda_k\}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. For $i \leqslant k$ let $f_i \in E$ satisfy $f_i(\lambda_i) = 1$; then $f_{k+1} = f - (f_1 + \ldots + f_k) \in E$ and we have $f = f_1 + \ldots + f_{k+1}$. This proves (a).

We pass to the proof of (b). Let

$$E_0 = \{f \in C(X, D)\colon f^{-1}(1) \subset \Gamma\} \subset E. \tag{4}$$

Observe, that E_0 is a soubgroup of G consisting exactly of the functions with finite support contained in Γ and $E_0 + E \subset E$. The reasononings given in the proof of (5) below are taken from Engelking's paper [4], proof of Lemma 2.(³)

(5) *The space E_0 has the strong condensation property.*

Let $A \subset E_0$ be of cardinality $\aleph_1$ and let $S_f = f^{-1}(1)$ for $f \in A$. Since the sets S_f are finite, there exist an uncountable set $C \subset A$ and a finite set $S \subset \Gamma$ such that $S_f \cap S_g = S$ for distinct $f, g \in C$ (cf. [4], [1]). Let $c \in E_0$ be the characteristic function of the set S; then for every neighbourhood V of c the set $C \setminus V$ is finite.

Let $A \subset E$ be a set of cardinality $\aleph_1$ and let

$$\Sigma = \{\lambda \in \Lambda\colon \text{there exists an } f \in A \text{ with } f(\lambda) = 1\}. \tag{6}$$

Case 1. *The set Σ is bounded in Ω, i.e., $\Sigma \subset [0, \alpha]$ where $\alpha < \omega_1$.* If $|A \cap E_0| = \aleph_1$ then one can use (5) to choose an uncountable concentrated subset of A; if this is not the case, then there exist a $\lambda \in [0, \alpha]$ and an uncountable set $B \subset A$ such that $f(\lambda) = 1$ for $f \in B$. Let $u \in E$ be a function satisfying $u(\lambda) = 1$. Then $B - u \subset E_0$ and by (5) there exists an uncountable set $C_1 \subset B - u$ concentrated around a point $c_1 \in E_0$. The set $C = C_1 + u \subset A$ is concentrated around the point $c = c_1 + u \in E$.

Case 2. *The set Σ is unbounded in Ω.* One can define in this case a transfinite sequence $\{\lambda_\xi\colon \xi < \omega_1\} \subset \Sigma$ such that for every $\xi < \omega_1$ we have $\mu_\xi = \sup\{\lambda_\alpha\colon \alpha < \xi\} < \lambda_\xi$. Let us choose for every $\xi < \omega_1$ a point $f_\xi \in A$ with $f_\xi(\lambda_\xi) = 1$ (thus $f_\xi \neq f_\eta$ for $\xi \neq \eta$). For every $\xi < \omega_1$ put $J_\xi = (\mu_\xi, \lambda_\xi]$; then $J_\xi \cap J_\eta = \emptyset$ for $\xi \neq \eta$. Let

$$S_\xi = f_\xi^{-1}(1) \cap J_\xi, \qquad T_\xi = f_\xi^{-1}(1) \setminus J_\xi.$$

The sets T_ξ are finite and $S_\xi \cap S_\eta = \emptyset$ for distinct ξ and η. Let g_ξ be the characteristic function of S_ξ and let h_ξ be the characteristic function of T_ξ; we have $g_\xi \in E$, $h_\xi \in E_0$ and $f_\xi = g_\xi + h_\xi$. Observe, that the set $\{g_\xi\colon \xi < \omega_1\}$ is concentrated around the function identically equal to 0. Since $\{h_\xi\colon \xi < \omega_1\} \subset E_0$, there exists by (5) an uncountable set Θ of ordinals less than ω_1 and a point $c \in E_0$ such that for every neighbourhood V of c the relation $h_\xi \notin V$ holds for at most countably many $\xi \in \Theta$. It follows that the uncountable set $C = \{f_\xi\colon \xi \in \Theta\}$ is concentrated around the point $c \in E_0$.

This completes the proof that $C(X)$ is weakly Lindelöf.

(³) One can also prove (5) as follows: the space $A = \{x \in E_0\colon |x^{-1}(1)| \leqslant 1\}$ is homeomorphic to the one-point compactification of the discrete space of cardinality $\aleph_1$ thus the space $T = \bigoplus_{n \in N} A^n$ has the strong condensation property and E_0 is a continuous image of T.

5. Remarks.

Remark 1. One can prove ([10], Theorem) that *if S is a compact separable space with the ω_1-th derived set empty and the space $C(S)$ is weakly Lindelöf, then S is countable.* Thus the space $C(X)$ is, in some sense, the simplest example of a function space with the properties mentioned in the title.

Remark 2. The Banach space $B = C(X)$ can be described also as follows. Let $m(\Gamma)$ be the Banach space of all bounded real functions on the set of all countable non-limit ordinals Γ endowed with the sup-norm and let E (cf. (3)) be the set consisting of all characteristic functions of subsets of Γ which are finite or equal to $s_\lambda(N) = N_\lambda$ for some $\lambda < \omega_1$ (see Section 2). Then B is the Banach space generated by E in $m(\Gamma)$; one can say that B is the Banach space associated with the quasi-disjoint family(4) $\{N_\lambda: \lambda < \omega_1\}$ of subsets of the set Γ; cf. Johnson and Lindenstrauss [6], where some examples of Banach spaces were constructed by means of a quasi-disjoint family of subsets of N.

Remark 3. The following observation seems to be worth whil noticing. Given a set of sequences of ordinals $\mathscr{S} = \{s_\lambda: \lambda \in \Lambda\} \subset \Gamma^N$ chosen as in Section 2 one can topologize this choice in a few natural ways obtaining spaces interesting from quite different points of view. First, as was done by Stone [15], one can consider $\mathscr{S}$ with the "first difference" metric, which yields a striking example in non-separable Borel theory; next, one can enrich this topology by new open sets $\{s_\xi: \xi \leqslant \lambda\}$ where λ runs over Λ, obtaining an example in general topology [11]; finally, the Banach space generated by the characteristic functions of the sets $s_\lambda(N)$ and the finite subsets of Γ (see Remark 2) provides an example in functional analysis.

Added in proof. A solution of the problem of Corson was obtained independently by M. Talagrand, *Espaces de Banach faiblement K-analytiques*, C.R.A.S. 284 (1977), pp. 745–748.

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