A note on Hölder's inequality

by

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Abstract. Let $F$ be the indefinite integral of a nonnegative function $f$ on $(0, \infty)$ whose $L^p$ norm is 1. Let $p$ be the Hölder conjugate of $q$, and assume $1 < q < \infty$. Jodeit has shown that the measure of the set on which $x - F < s$ is bounded by a constant multiple of $s$. A new proof of this inequality is given, and during the proof it is observed that the set is covered by two intervals whose lengths are proportional to $s$.

Trudinger [3] has obtained some embeddings of Sobolev type function spaces. Moser [2] later improved these results using a reduction to one dimension and a certain inequality for functions of one variable. Subsequently Jodeit [1] gave a new, simpler proof of this inequality and extended its applicability. The essential step in Jodeit's proof was still rather difficult. In the present paper we present a very simple proof of the Moser–Jodeit inequality. This proof also sheds some light on the structure of a certain set which figures in the proof.

1. A lemma of Jodeit. Throughout the discussion we shall assume that $p$ and $q$ are Hölder conjugates ($p^{-1} + q^{-1} = 1$) and that $1 < q < \infty$. We shall assume that $f \geq 0$ is a function in $L^p(0, \infty)$, satisfying

$$\|f\|_p = \left(\int_0^\infty f^p \, dx\right)^{1/p} < 1.$$ 

Let

$$F(x) = \int_0^x f(y) \, dy, \quad 0 \leq x < \infty.$$ 

Hölder's inequality implies

$$F(x) \leq \left(\int_0^x f^p \, dy\right)^{1/p} \leq 1,$$

and therefore, in particular,

$$x - F(x) \geq 0.$$ 

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The results of Moser and Jodeit assert that in some sense the last relation must actually be a rather strong inequality. There are two equivalent ways of stating this:

**Theorem 1.** Let \( \Phi \) be any nonnegative nonincreasing function in \( L^1(0, \infty) \). There exists a constant \( C_k \) depending only on \( q \) such that

\[
\int_0^\infty \Phi(s - F(x)^p) \, dx \leq C_k \| \Phi \|_1.
\]

**Lemma.** Let \( m \) denote Lebesgue measure on the line. Then for \( 0 \leq s < \infty \)

\[
m \{ x \mid x - F(x)^p \leq s \} \leq C_k s.
\]

The equivalence of these two inequalities is almost immediate. By choosing \( \Phi \) to be the characteristic function of the interval \([0, \varepsilon]\) we see that the theorem implies the lemma. To obtain the converse we introduce an "inverse" of \( \Phi \), the function

\[
\Psi(y) = \sup \{ s \mid \Phi(s) \geq y \}.
\]

This function \( \Psi \) is nonincreasing and

\[
a < \Psi(y) \Rightarrow y \leq \Phi(a) \Rightarrow a \leq \Psi(y).
\]

Therefore

\[
\int_0^a \Psi(y) \, dy = \int_0^{\Psi(a)} \Phi(s) \, ds.
\]

If the lemma holds, then we calculate

\[
\int_0^m \Phi(s - F(x)^p) \, ds = \int_0^\infty \{ s \mid \Phi(s - F(x)^p) \geq \Psi(y) \} \, dy
\]

\[
\leq \int_0^\infty \{ s \mid x - F(x)^p \leq \Psi(y) \} \, dy
\]

\[
\leq \int_0^\infty C_k \Psi(y) \, dy
\]

\[
= C_k \| \Phi \|_1,
\]

and we conclude that the theorem also holds.

In his paper Jodeit uses the function \( \Phi(x) = e^{-x} \).

2. Proof of the lemma. Let \( E \) denote the set \( \{ x \mid x - F(x)^p \leq s \} \). We first observe that \( [0, \varepsilon] \subset E \), trivially. Suppose \( a > 2s \) and \( x \in E \). Then \( 1 \)

\[
\varepsilon - s \leq F(x)^p \leq \left( \int_0^s f^p \, dy \right)^{p-1} \varepsilon.
\]

Therefore, since \( \| f \|_p \leq 1 \),

\[
\left( 1 - \frac{s}{a} \right)^{\frac{p}{p-1}} \leq \int_0^s f^p \, dy \leq 1 - \frac{s}{a}.
\]

Therefore,

\[
\int_0^s f^p \, dy \leq 1 - \left( 1 - \frac{s}{a} \right)^{\frac{p}{p-1}} \leq \frac{as}{s},
\]

where \( a \) is a constant depending only on \( q \). (This constant can be chosen to be \( 2/(p-1) \).)

Now suppose that \( 2s < x \leq 2a \) and that \( x_1 \) and \( x_2 \) are both in \( E \). Then

\[
x_2 - x_1 \leq x_1 - F(x_1)^p \quad (\text{by (3)})
\]

\[
\leq s + F(x_1)^p - F(x_2)^p \quad (\text{since } x_2 \in E)
\]

\[
= s + p \frac{x_1^{p-1}}{F(x_1) - F(x_2)},
\]

where the mean value theorem places \( \xi \) between \( F(x_1) \) and \( F(x_2) \).

Therefore,

\[
x_2 - x_1 \leq s + p F(x_2)^{p-1} \int_{x_1}^{x_2} f^p \, dy
\]

\[
\leq s + p \frac{x_2^{p-1}}{x_1^{p-1}} \int_{x_1}^{x_2} f^p \, dy \quad (\text{by (2)})
\]

\[
\leq s + p \frac{x_2^p}{x_1^p} \left( \int_{x_1}^{x_2} f^p \, dy \right)^{\frac{1}{p}}
\]

\[
\leq s + p \frac{x_2^p}{x_1^p} \left( \int_{x_1}^{x_2} f^p \, dy \right)^{\frac{1}{p}} (x_2 - x_1)^{1/p}
\]

\[
\leq s + p \frac{x_2^p}{x_1^p} \left( \frac{ax_1}{x_2} \right)^{1/p} (x_2 - x_1)^{1/p}.
\]

The last inequality used (3) in the case \( x = x_1 \in E \). So we obtain

\[
x_2 - x_1 \leq s + p \left( \frac{ax_1^p}{x_2^p} \right)^{\frac{1}{p}} (x_2 - x_1)^{1/p}.
\]

If we assume in addition that \( x_1 \geq 2s^p/a \), then

\[
x_2 - x_1 \leq s + \left( \frac{x_1}{2} \right)^{1/p} (x_2 - x_1)^{1/p}
\]

\[
\leq s + \frac{x_1}{2q} + \frac{x_1 - x}{p}.
\]
Therefore,
\[ a_{2} - a_{1} \leq qa + \frac{ca^{d}}{2} \]
where \( b \) is a constant depending only on \( q \). Now (4) implies
\[ a_{2} - a_{1} \leq s + p(ab)^{\frac{q}{d}}(s_{2} - s_{1})^{\frac{d}{p}}. \]
Therefore,
\[ \frac{a_{2} - a_{1}}{s} \leq 1 + c(s_{2} - s_{1})^{\frac{d}{p}}, \]
where \( c \) is a constant depending only on \( q \). Finally, the last inequality implies that
\[ \frac{a_{2} - a_{1}}{s} \leq q + c = d, \]
another constant depending only on \( q \).

The lemma now follows immediately. If \( E \subset \{ 0, 2pqsa \} \), then there is nothing more to be done. If this inclusion does not hold, we let \( s_{1} \) be the smallest number in \( E \cap \{ 2pqsa \}, \infty \). We have then shown that if \( s_{1} \) is in \( E \) and is larger than \( s_{2} \), then \( s_{2} - s_{1} < ds_{2} \). Thus, in any case \( E \) is covered by at most two intervals, each of which has length proportional to \( s \). Therefore, \( m[E] \leq (2pqsa + ds_{2}). \)

**Corollary.** Under the hypothesis of the theorem,
\[ \lim_{s \to \infty} \mu[E]^{p} = \infty. \]

The proof of the corollary follows simply by noticing that the set \( E \) is bounded.

3. **Remarks.** We remark that a routine estimation of the constants shows that \( C_{q} \leq 30g \).

Also we remark that the covering of \( E \) by two intervals can be seen in what is perhaps the simplest example. Namely, let \( f = \lambda^{1-q} \) on \([0, \lambda] \) and zero on \([\lambda, \infty) \). Then
\[ s - E^{p} = \begin{cases} \lambda^{1-q} - p^{rac{q}{d}} & \text{on} \ [0, \lambda], \\ \lambda & \text{on} \ [\lambda, \infty]. \end{cases} \]
The maximum of \( s - E^{p} \) on \([0, \lambda] \) occurs at \( s = p^{1/(d-q)} \) and equals \( p^{d} \left( \frac{1}{d-q} \right)^{1/(d-q)} = \beta_{1} \). If \( s \geq \beta_{1} \), then \( E = [0, \lambda + s] \). But if \( s < \beta_{1} \), then \( E \) equals the union of two intervals \([0, r_{1}], [r_{1}, \lambda + s] \), and a direct estimation yields \( r_{1} < 2q \) and \( \lambda - r_{1} \leq 2pq \left( \frac{1}{d-q} - 1 \right). \)

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Notice that this example shows that we cannot expect to be able to cover \( E \) with a single interval whose length is proportional to \( s \). Also, if we choose \( s = \beta_{1} \), then \( m[E] = \lambda + s = (\lambda^{1-q} + 1) \). Thus,
\[ C_{q} \geq gq^{d} \left( \frac{1}{d-q} + 1 \right) > q. \]

Thus, the estimate \( C_{q} \leq 30g \) exhibits the proper qualitative size of the constant.

The example can also be used to show that if we allow \( \Phi \) to be an arbitrary nonnegative \( L^{1} \) function, then the left side of the inequality of the theorem may be infinite.

**4. Generalization.** It is very easy to extend these results to include the case of a Hölder inequality involving two "arbitrary" functions, rather than the pair \( f \) and \( 1 \) which were treated before. Thus, suppose \( g \) and \( h \) are nonnegative functions on an interval \( (0, s) \subset (0, \infty) \), and suppose that \( g \in L^{q} \) and \( h \in L^{p}(0, b) \) for all \( b < s \). Of course, \( p \) and \( q \) are Hölder conjugates and \( 1 < q < \infty \).

**Theorem 2.** Under the above assumptions there exists a constant \( C_{q} \) depending only on \( q \) such that for any nonnegative nonincreasing function \( \Phi \) in \( L^{1}(0, \infty) \),
\[ \int \Phi \left( \int g \right)^{p} \left( \int h \right)^{q} \leq C_{q} \left( \int g \right)^{p} \left( \int h \right)^{q}. \]

Notice that the connection with Hölder's inequality arises because
\[ \int g \left( \int h \right)^{q} \leq \left( \int g \right)^{p} \left( \int h \right)^{q}, \]
which shows that the argument of \( \Phi \) in the theorem is indeed nonnegative.

Now we give the proof. First note that it suffices to give the proof in case \( \int g \leq 1. \) If the theorem is held to hold in that case, then it can be applied to the functions \( g/\|g\|_{1} \) and \( \Phi(\|g\|_{1}^{p}) \) to obtain the more general result.

Second, it will be convenient for the next remark to be able to assume \( \Phi \) is continuous. This can be achieved by constructing a sequence \( \Phi_{1} \succ \Phi_{2} \succ \cdots \) of continuous nondecreasing functions in \( L^{1}(0, \infty) \) such that
\[ \lim_{s \to \infty} \Phi_{s}(s) = \Phi(\infty) \]
and applying the theorem to each \( \Phi_{s} \). In the limit as \( s \to \infty \), Fatou's lemma and Lebesgue's dominated convergence theorem show that the theorem holds for \( \Phi \).

It will also be convenient to assume in the proof that \( h > 0 \). The sufficiency of this case is seen by simply writing down the theorem for
\[ h \mapsto \xi \] and letting \( h \to \infty \). The continuity of \( \Phi \) and an application of Egorov's lemma give the result for \( h \).

Now assuming \( h > 0 \) and \( |\eta_0| = 1 \), we introduce a change of variable
\[ \xi = \varphi(a) = \int_0^a \xi \, dy. \]

The new variable \( \xi \) has range \( 0 < \xi < a = \int_0^a \xi \, dy \), and \( \varphi \) is locally absolutely continuous, strictly increasing, and has a locally absolutely continuous inverse. Define a function \( f(\xi) \) by the formula
\[ f(\varphi(a)) = \varphi(a) \| h(\varphi) \|^{-\varphi}. \]

Note that
\[
\int_0^a \xi \int_0^\xi f(\xi) \, d\xi d\varphi = \int_0^a \xi \int_0^\xi f(\xi) \, d\xi d\varphi
\]
\[
= \int_0^a \xi \int_0^\xi f(\xi) \, d\xi d\varphi
\]
\[
= \int_0^a \xi \, d\varphi
\]
\[
= 1.
\]

Thus, Theorem 1 implies that
\[
\int_0^\xi \Phi \left( \xi - \int_0^\xi \left( f(\eta) \, d\eta \right) \right) \, d\xi \leq C_\| \Phi \|_1.
\]

By the changes of variable \( \xi = \varphi(a) \) and \( \eta = \varphi(y) \), we obtain the theorem.

References


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