STUDIA MATHEMATICA, T. LXIV. (1979)

A note on Hölder's inequality

by

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Abstract. Let F be the indefinite integral of a nonnegative function f on $(0, \infty)$ whose L^q norm is 1. Let p be the Hölder conjugate of q, and assume $1 < q < \infty$. Jodeit has shown that the measure of the set on which $x - F^p \leqslant s$ is bounded by a constant multiple of s. A new proof of this inequality is given, and during the proof it is observed that the set is covered by two intervals whose lengths are proportional to s.

Trudinger [3] has obtained some embeddings of Sobolev type function spaces. Moser [2] later improved these results using a reduction to one dimension and a certain inequality for functions of one variable. Subsequently Jodeit [1] gave a new, simpler proof of this inequality and extended its applicability. The essential step in Jodeit's proof was still rather difficult. In the present paper we present a very simple proof of the Moser–Jodeit inequality. This proof also sheds some light on the structure of a certain set which figures in the proof.

1. A lemma of Jodeit. Throughout the discussion we shall assume that p and q are Hölder conjugates ($p^{-1}+q^{-1} = 1$) and that $1 < q < \infty$. We shall assume that $f \geqslant 0$ is a function in $L^q(0, \infty)$, satisfying

$$\|f\|_q = \Big[\int_0^\infty f^q\,dx\Big]^{1/q} \leqslant 1.$$

Let

$$F(x) = \int_0^x f(y)\,dy, \qquad 0 \leqslant x < \infty.$$

Hölder's inequality implies

$$F(x)^p \leqslant \Big(\int_0^x f^q\,dy\Big)^{p-1} x, \tag{1}$$

and therefore, in particular,

$$x - F(x)^p \geqslant 0. \tag{2}$$

* Partially supported by the National Science Foundation under NSF grant GP MPS75-08010.

The results of Moser and Jodeit assert that in some sense the last relation must actually be a rather strong inequality. There are two equivalent ways of stating this:

THEOREM 1. *Let Φ be any nonnegative nonincreasing function in $L^1(0,\infty)$. There exists a constant C_q depending only on q such that*

$$\int_0^\infty \Phi\big(x-F(x)^p\big)\,dx \leqslant C_q\|\Phi\|_1 .$$

LEMMA. *Let m denote Lebesgue measure on the line. Then for $0\leqslant s<\infty$*

$$m\big(\{x \mid x-F(x)^p\leqslant s\}\big)\leqslant C_q s .$$

The equivalence of these two inequalities is almost immediate. By choosing Φ to be the characteristic function of the interval $[0,s]$ we see that the theorem implies the lemma. To obtain the converse we introduce an "inverse" of Φ, the function

$$\Psi(y)=\sup\{x \mid \Phi(x)\geqslant y\}.$$

This function Ψ is nonicreasing and

$$x<\Psi(y)\Rightarrow y\leqslant\Phi(x)\Rightarrow x\leqslant\Psi(y).$$

Therefore

$$\int_0^\infty \Psi(y)\,dy=\int_0^\infty \Phi(x)\,dx .$$

If the lemma holds, then we calculate

$$\begin{aligned}\int_0^\infty \Phi\big(x-F(x)^p\big)\,dx &= \int_0^\infty m\Big(\big\{x \mid \Phi\big(x-F(x)^p\big)\geqslant y\big\}\Big)\,dy\\ &\leqslant \int_0^\infty m\big(\{x \mid x-F(x)^p\leqslant \Psi(y)\}\big)\,dy\\ &\leqslant \int_0^\infty C_q\Psi(y)\,dy\\ &= C_q\|\Phi\|_1,\end{aligned}$$

and we conclude that the theorem also holds.

In his paper Jodeit uses the function $\Phi(x)=e^{-x}$.

2. Proof of the lemma. Let E denote the set $\{x \mid x-F(x)^p\leqslant s\}$. We first observe that $[0,s]\subset E$, trivially. Suppose $x>2s$ and $x\in E$. Then (1) implies

$$x-s\leqslant F(x)^p\leqslant\Big(\int_0^x f^q dy\Big)^{p-1}x .$$

Therefore, since $\|f\|_q\leqslant 1$,

$$\Big(1-\frac{s}{x}\Big)^{1/(p-1)}\leqslant\int_0^x f^q dy\leqslant 1-\int_x^\infty f^q dy .$$

Therefore,

$$\int_x^\infty f^q dy\leqslant 1-\Big(1-\frac{s}{x}\Big)^{1/(p-1)}\leqslant\frac{as}{x}, \tag{3}$$

where a is a constant depending only on q. (This constant can be chosen to be $2/(p-1)$.)

Now suppose that $2s<x_1<x_2$ and that x_1 and x_2 are both in E. Then

$$\begin{aligned}x_2-x_1&\leqslant x_2-F(x_1)^p \qquad \text{(by (2))}\\ &\leqslant s+F(x_2)^p-F(x_1)^p \qquad (\text{since } x_2\in E)\\ &= s+p\,\xi^{p-1}\big(F(x_2)-F(x_1)\big),\end{aligned}$$

where the mean value theorem places ξ between $F(x_1)$ and $F(x_2)$. Therefore,

$$\begin{aligned}x_2-x_1&\leqslant s+pF(x_2)^{p-1}\int_{x_1}^{x_2} f\,dy\\ &\leqslant s+px_2^{(p-1)/p}\int_{x_1}^{x_2} f\,dy \qquad \text{(by (2))}\\ &\leqslant s+px_2^{1/q}\Big(\int_{x_1}^{x_2} f^q dy\Big)^{1/q}(x_2-x_1)^{1/p}\\ &\leqslant s+px_2^{1/q}\Big(\int_{x_1}^{\infty} f^q dy\Big)^{1/q}(x_2-x_1)^{1/p}\\ &\leqslant s+px_2^{1/q}\Big(\frac{as}{x_1}\Big)^{1/q}(x_2-x_1)^{1/p}.\end{aligned}$$

The last inequality used (3) in the case $x=x_1\in E$. So we obtain

$$x_2-x_1\leqslant s+p\Big(\frac{asx_2}{x_1}\Big)^{1/q}(x_2-x_1)^{1/p}. \tag{4}$$

If we assume in addition that $x_1\geqslant 2p^q as$, then

$$\begin{aligned}x_2-x_1&\leqslant s+\Big(\frac{x_2}{2}\Big)^{1/q}(x_2-x_1)^{1/p}\\ &\leqslant s+\frac{x_2}{2q}+\frac{x_2-x_1}{p}.\end{aligned}$$

Therefore,

$$x_2 - x_1 \leqslant qs + \frac{x_2}{2};$$

$$x_2 \leqslant 2x_1 + 2qs \leqslant 2x_1 + qp^{-q}a^{-1}x_1 = bx_1,$$

where b is a constant depending only on q. Now (4) implies

$$x_2 - x_1 \leqslant s + p(asb)^{1/q}(x_2-x_1)^{1/p}.$$

Therefore,

$$\frac{x_2-x_1}{s} \leqslant 1 + c^{1/q}\left(\frac{x_2-x_1}{s}\right)^{1/p},$$

where c is a constant depending only on q. Finally, the last inequality implies that

$$\frac{x_2-x_1}{s} \leqslant q + c = d,$$

another constant depending only on q.

The lemma now follows immediately. If $E \subset [0, 2p^q as]$, then there is nothing more to be done. If this inclusion does not hold, we let x_1 be the smallest number in $E \cap [2p^q as, \infty)$. We have then shown that if x_2 is in E and is larger than x_1, then $x_2 - x_1 \leqslant ds$. Thus, in any case E is covered by at most two intervals, each of which has length proportional to s. Therefore, $m(E) \leqslant (2p^q a + d)s$.

COROLLARY. *Under the hypothesis of the theorem,*

$$\lim_{x\to\infty}\left(x - F(x)^p\right) = \infty.$$

The proof of the corollary follows simply by noticing that the set E is bounded.

3. Remarks. We remark that a routine estimation of the constants shows that $C_q \leqslant 30q$.

Also we remark that the covering of E by two intervals can be seen in what is perhaps the simplest example. Namely, let $f = \lambda^{-1/q}$ on $[0, \lambda]$ and zero on (λ, ∞). Then

$$x - F^p = \begin{cases} x - \lambda^{1-p}x^p & \text{on} \quad [0, \lambda], \\ x - \lambda & \text{on} \quad [\lambda, \infty]. \end{cases}$$

The maximum of $x - F^p$ on $[0, \lambda]$ occurs at $x = p^{-1/(p-1)}\lambda$ and equals $q^{-1}p^{-1/p-1} = \beta\lambda$. If $s \geqslant \beta\lambda$, then $E = [0, \lambda + s]$. But if $s < \beta\lambda$, then E equals the union of two intervals $[0, r_1] \cup [r_2, \lambda + s]$, and a direct estimation yields $r_1 \leqslant qs$ and $\lambda - r_2 \leqslant qs(p^{1/p-1} - 1)$.

Notice that this example shows that we cannot expect to be able to cover E with a single interval whose length is proportional to s. Also, if we choose $s = \beta\lambda$, then $m(E) = \lambda + s = (\beta^{-1}+1)s$. Thus,

$$C_q \geqslant qp^{1/(p-1)} + 1 > q.$$

Thus, the estimate $C_q \leqslant 30q$ exhibits the proper qualitative size of the constant.

The example can also be used to show that if we allow Φ to be an arbitrary nonnegative L^1 function, then the left side of the inequality of the theorem may be infinite.

4. Generalization. It is very easy to extend these results to include the case of a Hölder inequality involving two "arbitrary" functions, rather than the pair f and 1 which were treated before. Thus, suppose g and h are nonnegative functions on an interval $(0, a) \subset (0, \infty)$, and suppose that $g \in L^q$ and $h \in L^p(0, b)$ for all $b < a$. Of course, p and q are Hölder conjugates and $1 < q < \infty$.

THEOREM 2. *Under the above assumptions there exists a constant C_q depending only on q such that for any nonnegative nonincreasing function Φ in $L^1(0, \infty)$,*

$$\int_0^a \Phi\left\{\|g\|_q^p \int_0^x h^p\,dy - \left(\int_0^x gh\,dy\right)^p\right\} h(x)^p\,dx \leqslant C_q \|g\|_q^{-p}\|\Phi\|_1.$$

Notice that the connection with Hölder's inequality arises because

$$\int_0^x gh\,dy \leqslant \|g\|_q \left(\int_0^x h^p\,dy\right)^{1/p},$$

which shows that the argument of Φ in the theorem is indeed nonnegative.

Now we give the proof. First note that it sufficies to give the proof in case $\|g\|_q = 1$. If the theorem is known to hold in that case, then it can be applied to the functions $g/\|g\|_q$ and $\Phi(\|g\|_q^p x)$ to obtain the more general result.

Second, it will be convenient for the next remark to be able to assume Φ is continuous. This can be achieved by constructing a sequence $\Phi_1 \geqslant \Phi_2 \geqslant \ldots$ of continuous nondecreasing functions in $L^1(0, \infty)$ such that

$$\lim_{n\to\infty} \Phi_n(x) = \Phi(x-) \geqslant \Phi(x) \quad \text{for all } x,$$

and applying the theorem to each Φ_n. In the limit as $n \to \infty$, Fatou's lemma and Lebesgue's dominated convergence theorem show that the theorem holds for Φ.

It will also be convenient to assume in the proof that $h > 0$. The sufficiency of this case is seen by simply writing down the theorem for

$h+n^{-1}$ and letting $n \to \infty$. The continuity of Φ and an application of Fatou's lemma give the result for h.

Now assuming $h > 0$ and $\|g\|_q = 1$, we introduce a change of variable

$$\xi = \varphi(x) \equiv \int_0^x h^p\, dy.$$

The new variable ξ has range $0 < \xi < a = \int_0^a h^p\, dy$, and φ is locally absolutely continuous, strictly increasing, and has a locally absolutely continuous inverse. Define a function $f(\xi)$ by the formula

$$f(\varphi(x)) = g(x)\, h(x)^{1-p}.$$

Note that

$$\int_0^a f(\xi)^p\, d\xi = \int_0^a f(\varphi(x))^q \varphi'(x)\, dx = \int_0^a g^q h^{(1-p)q} h^p\, dx = \int_0^a g^q\, dx = 1.$$

Thus, Theorem 1 implies that

$$\int_0^a \Phi\left\{\xi - \left(\int_0^\xi f(\eta)\, d\eta\right)^p\right\} d\xi \leqslant C_q \|\Phi\|_1.$$

By the changes of variable $\xi = \varphi(x)$ and $\eta = \varphi(y)$, we obtain the theorem.

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Received February 7, 1977 (1260)

STUDIA MATHEMATICA, T. LXIV. (1979)

A function space $C(X)$ which is weakly Lindelöf but not weakly compactly generated

by

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Abstract. We give an example of a Banach function space $C(X)$, which is weakly Lindelöf but not weakly compactly generated. This solves in the negative an old problem of Corson and a problem of Benyamini, Rudin and Wage.

1. In this paper we construct the following example (the terminology will be explain later).

EXAMPLE. *There is a Banach function space $C(X)$ which is Lindelöf under the weak topology but not weakly compactly generated.*(1)

The example solves in the negative a problem of Corson [2] (see also Lindenstrauss [9], Problems 6 and 6′) and Problem 7 of Benyamini, Rudin and Wage [1] (cf. [1], Corollary 2-2). The reader is referred for the related topics to [2], [9], [13], and [1]. Note that Talagrand [17] showed that a weakly compactly generated Banach space is weakly Lindelöf.

Our topological terminology is taken from [3] and the terminology related of functional analysis follows [14] and [9].

The symbol $C(X)$ stands for the Banach space of all continuous real-valued functions on a compact space S with the sup-norm [14]. The space $C(X)$ is said to be *weakly Lindelöf* if it is Lindelöf under the weak topology. A Banach space E is *weakly compactly generated* if there exist a weakly compact set K in E such that E is the closed linear span of K [9]; if $E = C(S)$, then this is equivalent ([9], Theorem 3.2) to the condition that S is an *Eberlein compact*, i.e., S is homeomorphic to a weakly compact subset of a Banach space. Recall that for a compact scattered space S the weak topology of $C(S)$ coincide in the unit ball with the topology

(1) K. Kunen constructed under the Continuum Hypothesis (preprint 1975) a compact scattered space K of cardinality $\aleph_1$ such that every finite product of K is hereditarily separable (Kunen showed that the existence of such K is in fact independent on the usual axioms for set theory); one can verify that $C(K)$ is weakly Lindelöf (even hereditarily), but not weakly compactly generated (see [10], Remark 2).