On joint spectra

by

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Abstract. The commutation properties of some \(N\)-tuples of multiplication operators are used to show that for \(N \geq 2\) there exist \(N\)-tuples \(T = (T_1, \ldots, T_N)\) of commuting linear operators on a Banach space \(X\) with the property that there are two maximal commutative subalgebras \(\mathcal{A}_1\) and \(\mathcal{A}_2\) of \(L(X)\) containing \(T_1, \ldots, T_N\) such that the joint spectrum of \(T\) in \(\mathcal{A}_1\) is different from the joint spectrum of \(T\) in \(\mathcal{A}_2\). The example is closely related to an example of J. L. Taylor in [8], [9]. It is shown that, in general, Taylor’s functional calculus ([9]) is richer than the analytic functional calculus (in the sense of [4]) in any closed commutative subalgebra of \(L(X)\) containing \(T_1, \ldots, T_N\).

1. Introduction. Let \(X\) be a complex Banach space and denote by \(L(X)\) the Banach algebra of all continuous linear operators on \(X\). For an \(N\)-tuple \(T = (T_1, \ldots, T_N)\) of commuting operators in \(L(X)\) we may consider the following joint spectra:

(a) \(\sigma(T, X)\), the joint spectrum of \(T\) with respect to \(X\) in the sense of J. L. Taylor ([8]).

(b) If \(\mathcal{A}\) is a closed subalgebra of \(L(X)\) containing \(I, T_1, \ldots, T_N\) in its center, then we denote by \(\sigma_{\mathcal{A}}(T)\) the joint spectrum of \(T\) in \(\mathcal{A}\):

\[
\sigma_{\mathcal{A}}(T) := \{ z \in \mathbb{C}^N : \sum_{i=1}^N (z_i I - T_i) \mathcal{A} \neq \mathcal{A} \}.
\]

By [8], Lemma 1.1, the Taylor spectrum of \(T\) is always contained in \(\sigma_{\mathcal{A}_1}(T)\), where \(\mathcal{A}_1\) is the algebra of all continuous linear operators commuting with \(T_1, \ldots, T_N\). Moreover, J. L. Taylor showed in [8] by an example of five operators \(T_1, \ldots, T_5\) that the inclusion can be strict. We modify this example in order to show that the inclusion can be proper for every \(N \geq 2\).

For this example the algebra \(H(\sigma(T, X))\) of germs of locally analytic functions on \(\sigma(T, X)\) is strictly larger than \(H(\sigma_{\mathcal{A}_1}(T))\). Hence, Taylor’s analytic functional calculus ([9]) is, in general, richer than the analytic functional calculus (in the sense of [4]) in any commutative closed subalgebra of \(L(X)\) containing \(T_1, \ldots, T_N\). For a second (closely related example) we show that there exist maximal commutative subalgebras \(\mathcal{A}_1\) and \(\mathcal{A}_2\) of \(L(X)\) containing \(T_1, \ldots, T_N\) such that \(\sigma_{\mathcal{A}_1}(T) \neq \sigma_{\mathcal{A}_2}(T)\). This
gives an answer to a question raised by W. Żelazko at the Oberwolfach meeting on functional analysis in 1976.

2. The examples. We put \( G = G_1 \cup G_2 \) where
\[
G_1 = \{ \xi \in \mathbb{C}^N : |\xi| < 1/4 \text{ for } j = 1, \ldots, N \}, \quad G_2 = \{ \xi \in \mathbb{C}^N : 1/2 < \max|\xi_j| : j = 1, \ldots, N < 1 \}.
\]

Let us denote by \( \mathcal{B}(G) \) the space of all continuous functions on \( G \) and by \( \mathcal{B}(G) \) the space of all functions \( f \in \mathcal{B}(G) \) such that \( \frac{\partial f}{\partial \xi_j} \) (in the sense of distributions) is continuous on \( G \) for \( j = 1, \ldots, N \) (see [9], [10]). Let \( X_0 \) be the Banach algebra of all continuous functions on \( G \), endowed with the supremum norm \( \| \cdot \|_0 \), and let \( X \) be the space of all functions \( f \in X_0 \) such that the restriction of \( f \) to \( G \) belongs to \( \mathcal{B}(G) \) and such that for \( j = 1, \ldots, N \) the functions \( \frac{\partial f}{\partial \xi_j} \) have continuous extensions to \( \hat{G} \) (again denoted by \( \frac{\partial f}{\partial \xi_j} \)). Endowed with the norm \( \| \cdot \|_1 \),
\[
\| f \|_1 = \| f \|_0 + \sum_{j=1}^N \left\| \frac{\partial f}{\partial \xi_j} \right\|_0 \quad (f \in X),
\]
\( X_0 \) is a Banach algebra. Spaces of this type have also been used in [1] and [3]. We consider the following multiplication operators on \( X_k (k = 0, 1) \):
\[
(T_kf)(z) = z_k f(z) \quad (f \in X_k \text{ and } z = (z_1, \ldots, z_k) \in \hat{G} (j = 1, \ldots, N)).
\]

1. Lemma. For \( k = 0, 1 \) the \( N \)-tuples \( T_k = (T_{k_1}, \ldots, T_{k_N}) \) are \( X_0 \)-toral (in the sense of [3]) and hence decomposable in the sense of [6].

The (unique) spectral capacity for \( T_k \) is given by
\[
\sigma(T_k) = \{ \lambda \in X : \text{supp}(\lambda f) = P \} \quad (P = P \subseteq \mathbb{C}^N).
\]

Moreover,
\[
\sigma(T_0) = \sigma(0) \cap \sigma(T_1) = \sigma(T_0) \cap \sigma(T_1).
\]

Proof. \( \Phi_0(X_0) \to L(X_0) \) with \( \Phi(g)f = g \cdot f \) is obviously an algebra homomorphism with \( \Phi(1) = I \) and \( \Phi(\xi_j) = X_j \), where \( \xi_j : G \to \mathbb{C} \) are the coordinate functions with \( \xi_j(x) = \xi_j \) for \( x = (x_1, \ldots, x_N) \in G \). Hence \( \Phi(T_k) \) is decomposable by [3]. Theorem 4. Obviously, \( \Phi(\xi_j) \) as defined by (1) is a spectral capacity for \( T_k \) (which is unique by [9]).

(3) follows by Theorem 6 in [3].

The proof of the following proposition is similar to the proof of Theorem 4.4.6 in [5] and of Proposition 2.4 in [1].

2. Proposition. For \( A \in L(X_n, X_m) \) (\( n, m = 0, 1 \)) the following statements are equivalent:
(a) \( A \Phi(\xi_j) \in \Phi(\xi_j) \) for every closed \( \xi_j \subseteq \mathbb{C}^N \).
(b) \( A \Phi(\xi_j) = 0 \) in the case \( \xi_j = 0 \), \( n = m = 1 \). In the case \( n = m, A \) is a multiplication operator
\[
A f = af \quad (f \in X_n)
\]
with \( a \in X_n \), and in the case \( n = 1, m = 0 \), \( A \) is a differential operator of the type
\[
A f = a f + \sum_{j=1}^N b_j \frac{\partial f}{\partial \xi_j} \quad (f \in X_1)
\]
with \( a, b_1, \ldots, b_N \in X_1 \).

Proof. As obviously every operator of the type (b) fulfills condition (a), we have only to show that (a) implies (b). Condition (a) can also be written in the form
\[
\supp(Af) = \supp(f) \quad \text{for all } f \in X_n.
\]
Therefore, for every \( w \in G \) the map \( u_w : \Phi(\mathbb{C}^N) \to C \) defined by \( u_w(f) = \langle A(f), w \rangle \) is a continuous linear functional with support contained in \( \{w \} \). Hence, we have in the case \( n = 0 \)
\[
\{ A(f), w \} = \{ a(w) f(w) \} \quad \text{for every } f \in \Phi(\mathbb{C}^N)
\]
with \( a(w) \in C \), and in the case \( n = 1 \)
\[
\{ A(f), w \} = \{ a(w) f(w) + \sum_{j=1}^N b_j(w) \frac{\partial f}{\partial \xi_j}(w) + c_j(w) \frac{\partial f}{\partial \xi_j}(w) \}
\]
for every \( f \in \Phi(\mathbb{C}^N) \) with \( b_j(w), c_j(w) \in C \) (\( j = 1, \ldots, N \)). Applying \( A \) successively to the polynomials \( 1, \xi_1, \ldots, \xi_j, \ldots, \xi_N \), we obtain that the functions \( w \to a(w), w \to b_j(w), w \to c_j(w) (j = 1, \ldots, N) \) are elements of \( X_m \).

Let now \( w \) be an arbitrary point in \( G \). There exists a function \( h \in \Phi(\mathbb{C}^N) \) with compact support contained in \( G \), such that \( 0 < h < 1 \) and such that \( h = 1 \) in a neighbourhood \( U \) of \( w \). Then we have (by (5)) for every \( f \in X_n \)
\[
\{ A(f), w \} = \{ A(f), h \} + \{ A(1 - h), f \} \quad \text{for all } f \in X_n
g \text{ can be approximated in the norm of } X_n \text{ by functions in } \Phi(\mathbb{C}^N) \text{ (by the proof of Lemma 2.5 in [9]). As } A \text{ is continuous and } h = 1 \text{ in } U, \text{ we obtain that (6), resp. (7), are valid for all } f \in X_n \text{ and } w \in G. \]

In the case \( n = 0 \), (6) holds (by continuity of \( a \) and \( f \)) for all \( w \in G \).

This proves (3) in the case \( n = m = 0 \).

Let us now consider the case \( n = 1 \). For an arbitrary \( w \in G \) we can find \( h \in \Phi(\mathbb{C}^N) \) as above, with the additional property that the diameter
of \( \text{supp}(h) \) is smaller than 1. For \( j = 1, \ldots, N \) the functions \( g_j \), with

\[
g_j(z) = \begin{cases} h(z) & \text{for } z \in \text{supp}(h), \\ 0 & \text{for } z \notin \text{supp}(h), \end{cases}
\]

belong to \( X_j \). Then we have for \( z \in U \)

\[
(Ag_j)(z) = a(z)g_j(z) + \sum_{j=1}^{N} \left( b_j(z) \frac{\partial g_j}{\partial z} (z) + c_j(z) \frac{\partial g_j}{\partial z_j} (z) \right)
\]

By the continuity of the functions \( Ag_j, a, b, \frac{\partial g_j}{\partial z}, \frac{\partial g_j}{\partial z_j} \), we obtain the continuity of \( \frac{\partial g_j}{\partial z_j} \) at the point \( w \). For \( z \in U, z \neq w \)

\[
\frac{\partial g_j}{\partial z_j} (z) = \ln \left( \ln |z_j - w_j| \right) - (\ln |z_j - w_j|) - 1,
\]

this is only possible if \( c_j(w) = 0 \). As \( w \) was an arbitrary point in \( G \) and as \( c \) is continuous on \( G \), we have \( c_j = 0 \) on \( G \) and (4) is proved.

In the case \( n = m = 1 \), we obtain for \( z \in U \)

\[
(Ag_j)(z) = a(z)g_j(z) + b_j(z) \frac{\partial g_j}{\partial z} (z)
\]

with \( Ag_j, a, b_j \in X_1 \). Hence,

\[
\frac{\partial}{\partial z} \left( b_j(z) \frac{\partial g_j}{\partial z} \right) = \frac{\partial b_j}{\partial z} \frac{\partial g_j}{\partial z} + b_j \frac{\partial^2 g_j}{\partial z^2}
\]

has to be continuous at \( w \). As \( \frac{\partial^2 g_j}{\partial z^2} \) is not continuous at \( w \), this is only possible if \( b_j(w) = 0 \) \( (k = 1, \ldots, N) \). Consequently, \( b_j = 0 \) \( (k = 1, \ldots, N) \) and (3) holds for \( n = m = 1 \).

For \( n = 0, m = 1, w \in G \), consider \( \frac{\partial g_j}{\partial z} \) (which belongs to \( X_0 \)) because of \( g_j \in X_1 \). As \( A \left( \frac{\partial g_j}{\partial z} \right) = a \frac{\partial g_j}{\partial z} \in X_1 \), the function \( \frac{\partial}{\partial z} \left( a \frac{\partial g_j}{\partial z} \right) \) has to be continuous at \( w \). As above, we obtain that this is only possible if \( a(w) = 0 \). Hence, \( a = 0 \) on \( G \), i.e., \( A = 0 \), and the proof is complete.

Let now \( X \) be the Banach space \( X = X_0 \oplus X_1 \). We consider the following operators in \( L(X) \):

\[
T_j := T_j^0 \oplus T_j^1 \quad (j = 1, \ldots, N),
\]

\[
\Phi(h) := \Phi^0(h) \oplus \Phi^1(h) \quad h \in X_1,
\]

\[
D_j \text{ with } D_j(f, g) := \left( \frac{\partial g_j}{\partial z} (0), 0 \right) \quad (j = 1, \ldots, N) \text{ for } (f, g) \in X,
\]

\[
S_j := T_j + D_j \quad (j = 1, \ldots, N),
\]

\[
\Psi(k) := \Phi^k(0) \oplus 0 \quad k \in X_0.
\]

3.Lemma. (a) \( T = (T_1, \ldots, T_N) \) is a \( X \)-scalar \( N \)-tuple and hence decomposable. The spectral capacity for \( T \) is given by

\[
\sigma(T) = \sigma(T^0) \oplus \sigma(T^1),
\]

\[
= \{ (f, g) \in X : \text{supp}(f) \cup \text{supp}(g) \subseteq F \}
\]

for \( F = F \subset C^N \). Moreover,

\[
\sigma(T, X) = \sigma(T, X).
\]

(b) \( S = (S_1, \ldots, S_N) \) is decomposable, the spectral capacity for \( S \) coincides with that of \( T \), and

\[
\sigma(S, X) = \sigma(T, X).
\]

Proof. (a) is proved in the same way as Lemma 1.

(b) The operators \( S_1, \ldots, S_N, T_1, \ldots, T_N \) commute and \( S_j - T_j = D_j \) is nilpotent for \( j = 1, \ldots, N \). Therefore, \( T \) and \( S \) are quasi-nilpotent equivalent in the sense of [6], Definition 4.1, by Remark 4.3 in [6]. Theorem 4.1 in [6] implies that \( \sigma(T, X) = \sigma(S, X) \). By Proposition 4.1 in [6], the \( N \)-tuple \( S \) is decomposable and the spectral capacities for \( T \) and \( S \) coincide.

4. Proposition. For \( A \in L(X) \) the following two conditions are equivalent:

(a) \( A \sigma(F) = \sigma(F) \) for every closed \( F \subset C^N \).

(b) There are functions \( h \in X_1 \), \( h \in X_0 \), and \( k_1, k_2, \ldots, k_N \), \( k \in X_0 \) such that

\[
A = \Phi(h) + \Psi(k) + \sum_{j=1}^{N} \Psi(k_j) D_j.
\]

Proof. Obviously, (b) implies (a), so that we have to prove only the converse implication. Denote for \( n = 0, 1 \) by \( J_n : X_n \to X \) the canonical injection and by \( P_n : X \to X_n \) the canonical projection. Then, \( A \) can be written in the form

\[
A = \begin{bmatrix} A_{20} & A_{21} \\ A_{10} & A_{11} \end{bmatrix}
\]

where \( A_{ij} = P_j A J_i \) for \( i, j = 0, 1 \).
As $\mathcal{A}_j \subseteq \mathcal{A}(\mathcal{F})$, for $j = 0, 1$ and every closed $\mathcal{F} \subseteq C^0$, condition (a) implies $A_j \subseteq \mathcal{A}(\mathcal{F})$ for every closed $\mathcal{F} \subseteq C^0$ and $j = 0, 1$. By Proposition 2 we obtain

$$A_i = \mathcal{A}(\mathcal{F}_i) \quad \text{with} \quad h \in \mathcal{F}_i \quad (i = 0, 1),$$

$$A_0 = 0,$$

$$A_1 = \mathcal{A}(\mathcal{F}_1) + \sum_{k=1}^n \mathcal{A}(\mathcal{F}_{k_1}) \quad \text{with} \quad k_1, k_2, \ldots, k_N \in \mathcal{F}_1.$$

Hence, $A$ is of the type (8) (with $k := k_1$ and $k := k_2$).

We are now able to compute the commutant algebra for $\mathcal{T}$ and $S$.

**Proposition.**

(a) $(\mathcal{T})' = \{ \mathcal{A}(\mathcal{F}) + \mathcal{F}(\mathcal{F}_i) + \sum_{k=1}^n \mathcal{A}(\mathcal{F}_{k_1}) \mathcal{D}_j : k, k_1, \ldots, k_N \in \mathcal{F}_1, h \in \mathcal{F}_1 \}$.

(b) $(\mathcal{S})' = \{ \mathcal{A}(\mathcal{F}) + \sum_{k=1}^n \mathcal{A}(\mathcal{F}_{k_1}) \mathcal{D}_j : k, k_1, \ldots, k_N \in \mathcal{F}_1, h \in \mathcal{F}_1 \cap \mathcal{H}(\mathcal{G}) \}$,

where $\mathcal{H}(\mathcal{G})$ is the algebra of locally analytic functions on $\mathcal{G}$.

**Proof.**

(a) Obviously, every operator of the type (8) commutes with $T_1, \ldots, T_N$. On the other hand, every operator commuting with $T_1, \ldots, T_N$ fulfills condition (a) in Proposition 4 by Corollary 4.5 in [8] and is therefore of the type (8) by Proposition 4.

(b) Let $A$ be an operator in $\mathcal{L}(\mathcal{F})$ commuting with $S_1, \ldots, S_N$. As in the proof of (a), $A$ must be of the type (8). I.e. there are functions $k, k_1, k_2, \ldots, k_N \in \mathcal{F}_1$ and $h \in \mathcal{F}_1$ such that

$$A = \mathcal{A}(\mathcal{F}) + \mathcal{F}(\mathcal{F}_1) + \sum_{k=1}^n \mathcal{A}(\mathcal{F}_{k_1}) \mathcal{D}_j.$$

Consequently, for $j = 1, \ldots, N$,

$$0 = S_j A - A S_j = (T_j + D_j) A - A (T_j + D_j) = D_j A - A D_j = C_j,$$

because of (a). Now, $D_j D_j = D_j \mathcal{F}(\mathcal{F}_1) = 0$ for $i, j = 0, 1, \ldots, N$. Hence

$$0 = C_j = D_j \mathcal{A}(\mathcal{F}) - \mathcal{A}(\mathcal{F}) \mathcal{A}(\mathcal{F}) + \mathcal{F}(\mathcal{F}_1) \mathcal{D}_j \quad \text{for} \quad j = 1, \ldots, N.$$

Therefore,

$$0 = C_j(\mathcal{F}_1) = \left( \frac{\partial \mathcal{F}_1}{\partial \mathcal{F}_1} \right) \quad \text{for} \quad j = 1, \ldots, N,$$

and so $h \in \mathcal{H}(\mathcal{G})$. This implies $\mathcal{A}(\mathcal{F}) = D_j \mathcal{A}(\mathcal{F})$ and therefore

$$0 = C_j(\mathcal{F}_1) = \left( \frac{\partial \mathcal{F}_1}{\partial \mathcal{F}_1} \right) \quad \text{i.e.} \quad h = 0,$$

and we have shown that $A$ is of the desired type. On the other hand, if $A = \mathcal{A}(\mathcal{F}) + \sum_{k=1}^n \mathcal{A}(\mathcal{F}_{k_1}) \mathcal{D}_j$ with $k_1, k_2, \ldots, k_N \in \mathcal{F}_1$ and $h \in \mathcal{F}_1 \cap \mathcal{H}(\mathcal{G})$, then clearly $A S_j = S_j A$ for $j = 1, \ldots, N$.

6. **Theorem.**

(a) $\mathcal{A}_1 := \{ \mathcal{A}(\mathcal{F}) + \mathcal{F}(\mathcal{F}_1) D_j : h \in \mathcal{F}_1, k \in \mathcal{F}_1 \}$ and $\mathcal{A}_2 := \{ \mathcal{A}(\mathcal{F}) + \mathcal{F}(\mathcal{F}_1) D_j : h \in \mathcal{F}_1, k \in \mathcal{F}_1 \}$ are maximal commutative subalgebras of $\mathcal{L}(\mathcal{F})$ containing $T_1, \ldots, T_N$. If $N \geq 2$, then

$$\sigma_{\mathcal{A}_1}(T) = \sigma_{\mathcal{A}_2}(T) = \sigma_{\mathcal{S}}(T) = \{ \mathcal{F}(\mathcal{F}_1) : h \in \mathcal{F}_1, k \in \mathcal{F}_1 \}.$$

(b) $\sigma_{\mathcal{A}_1}(T) = \sigma_{\mathcal{S}}(T) = \mathcal{F}(\mathcal{F}_1) \cap \mathcal{H}(\mathcal{G})$.

**Proof.**

(a) Obviously $\mathcal{A}_1$ and $\mathcal{A}_2$ are commutative subalgebras of $\mathcal{L}(\mathcal{F})$ with $T_1, \ldots, T_N \in \mathcal{A}_1$. As $\mathcal{A}_2 = \{ \mathcal{A}(\mathcal{F}) + \mathcal{F}(\mathcal{F}_1) D_j : h \in \mathcal{F}_1 \}$ is the maximal subalgebra of $\mathcal{L}(\mathcal{F})$ containing $T_1, \ldots, T_N$ in its center, it is a maximal commutative subalgebra of $\mathcal{L}(\mathcal{F})$.

Let now $A \in \mathcal{L}(\mathcal{F})$ be an operator commuting with all $B \in \mathcal{A}_1$. By Proposition 6(a) there are functions $k, k_1, k_2, \ldots, k_N \in \mathcal{F}_1$ and $h \in \mathcal{F}_1$ such that

$$A = \mathcal{A}(\mathcal{F}) + \mathcal{F}(\mathcal{F}_1) + \sum_{k=1}^n \mathcal{A}(\mathcal{F}_{k_1}) \mathcal{D}_j.$$

As $D_0 = \mathcal{F}(\mathcal{F}_1) A \in \mathcal{A}_1$, we obtain

$$0 = (A D_0 - D_0 A)(0, 1) = (h, 0),$$

and therefore $h = 0$. $\mathcal{A}(\mathcal{F}) \subseteq \mathcal{A}_1$ implies

$$0 = (A \mathcal{F}(\mathcal{F}_1) - \mathcal{F}(\mathcal{F}_1) A)(0, 1) = (k, 0),$$

i.e. $k_1 = 0$ (j = 1, ..., N). Therefore, $A \in \mathcal{A}_1$ and we have shown that $\mathcal{A}_1$ is a maximal commutative subalgebra of $\mathcal{L}(\mathcal{F})$.

As $\mathcal{F}(\mathcal{F}_1) \subseteq \mathcal{A}_1$, we have $\sigma_{\mathcal{A}_1}(T) = \mathcal{A}(\mathcal{F}_1)$ by Theorem 6 in [3].

Let us now prove that $\sigma_{\mathcal{A}_1}(T) = \mathcal{A}(\mathcal{F}_1)$. If $\mathcal{F} \neq \mathcal{A}_1$, i.e. $|\mathcal{F}_1| > 1$ for some $p \in \mathcal{A}_1$, then we obtain with $u_p(s) := (u_p - u_p)^{-1}$ and $u_p = 0$ for $j \neq p, j = 1, \ldots, N$,

$$\sum_{j=1}^N (u_p I - T_j) \mathcal{A}(\mathcal{F}_1) = \mathcal{A}(\mathcal{F}_1),$$

hence $s \notin \sigma_{\mathcal{A}_1}(T)$. If $\mathcal{F} \neq \mathcal{A}_1$, then there are $U_1, \ldots, U_N \in \mathcal{A}_1$,

$$U_j = \mathcal{A}(\mathcal{F}) + \sum_{k=1}^n \mathcal{A}(\mathcal{F}_{k_1}) \mathcal{D}_j$$

with $k_1, k_2, \ldots, k_N \in \mathcal{F}_1, u_p \in \mathcal{F}(\mathcal{F}_1) \cap \mathcal{H}(\mathcal{G})$ (j = 1, ..., N; p = 0, 1, ..., N), such that

$$\sum_{j=1}^N (u_p I - T_j) U_j = I.$$
If we apply this equation to \((1, 0) \in X\), we obtain
\[
\left( \sum_{j=1}^{N} (w_j - \pi_j)w_j, 0 \right) = (1, 0),
\]
and
\[
(9) \quad \sum_{j=1}^{N} (w_j - \pi_j)w_j(x) = 1 \quad \text{for all } x \in \overline{G}_j.
\]

As \(N \geq 2\), there are unique continuous functions \(\eta_j : K \to \mathbb{C}\) which are analytic in \(K\) and coincide with \(w_j\) on \(G_j\) \((j = 1, \ldots, N)\) (cf. [7], Theorem L.0.5). By (9) we obtain
\[
\sum_{j=1}^{N} (w_j - \pi_j)\eta_j(x) = 1 \quad \text{for all } x \in K.
\]

This is only possible if \(w \not\in K\). Thus \(\sigma_{\mathcal{A}}(K) = K\).

(b) Let \(A(\mathcal{A})\) be the space of all non trivial multiplicative linear functionals on \(\mathcal{A} = (S)'\). Then
\[
\sigma_{\mathcal{A}}(S) = \left\{ (\varphi(S_1), \ldots, \varphi(S_N)) : \varphi \in A(\mathcal{A}) \right\}
\]
\[
= \left\{ (\varphi(T_1), \ldots, \varphi(T_N)) : \varphi \in A(\mathcal{A}) \right\}
\]
\[
= \sigma_{\mathcal{A}}(T) = K
\]
by (a) and because of \(\varphi(D_j) = 0\) for all \(\varphi \in A(\mathcal{A})\) (as \(D_j = 0\) for \(j = 1, \ldots, N\). Together with Lemma 3 (b) this proves (b).

7. Remarks. (a) Part (b) in the preceding theorem shows that for \(N \geq 2\) the Taylor spectrum may be strictly smaller than the commutant spectrum (cf. Theorem 4.1 in [8] for \(N \geq 5\)). Moreover, in our example \(H(\sigma_{\mathcal{A}}(S)) = H(K) \subseteq H(G) = H(\sigma(S, X))\). For example, the germ of the function \(h\), which vanishes in a neighbourhood of \(\partial G\), and is identical to 1 in a neighbourhood of \(\partial G\), is in \(H(G)\) but not in \(H(K)\). The operator \(\Phi(h)\) is in the algebra generated by Taylor's analytic functional calculus ([9]) but not in the algebra generated by the analytic functional calculus (in the sense of [4]) in any closed commutative subalgebra of \(L(X)\) containing \(I, S_1, \ldots, S_N\). This shows that, in general, Taylor's analytic functional calculus is richer than the analytic functional calculus in closed commutative subalgebras.

(b) There is no admissible algebra \(\mathcal{A}\) of functions (in the sense of [3]) such that there exists a homomorphism \(\mathcal{V} : \mathcal{A} \to L(X)\) with \(\mathcal{V}(1) = I\) and \(\mathcal{V}(w_j) = \delta_j\) for \(j = 1, \ldots, N\). Otherwise, by Theorem 6 in [3] we would have \(\sigma(S, X) = \sigma_{\mathcal{A}}(S)\) in contradiction to part (b) in the preceding theorem. In the case \(N = 1\), a corresponding example has been given in [1].