An axiomatic approach to joint spectra I

by

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Abstract. We introduce and study axioms for joint spectra and related concepts in Banach algebras. In particular, we prove the existence of the largest spectrum and the minimal spectra and give some of their properties.

1. Introduction. Let $\mathcal{A}$ be a commutative complex unital Banach algebra with the unit $e$. For any subset $x_\lambda \in \mathcal{A}$, $x_\lambda = \{x_\lambda\}$ its joint spectrum (or shortly spectrum) in $\mathcal{A}$ is defined as

$$\sigma_{\mathcal{A}}(x_\lambda) = \{f(x_\lambda) \in C^* : f \in \mathbb{M}(\mathcal{A})\},$$

where $\mathbb{M}(\mathcal{A})$ is the set of all multiplicative-linear functionals of the algebra $\mathcal{A}$. The spectrum $\sigma_{\mathcal{A}}(x_\lambda)$ is always a non-void compact subset of $C^*$.

The joint spectrum is a basic concept for one of the most important chapters in the theory of commutative Banach algebras, namely for the functional calculus in these algebras. However, the concept of a joint spectrum is not so clear in the non-commutative case, even if we reduce our attention to commuting families of elements of the algebra in question. Thus different writers have adopted different concepts of a joint spectrum. Bonsall and Duncan propose in [2] to define a spectrum as the union of the left and the right spectrum. The same concept is adopted in the papers of Harte [6], [7]. Dash in [5] considers the bicommutant spectrum while Taylor in [11], before introducing his very interesting concept, starts with the commutant spectrum. All these concepts coincide in the case of a single element of the algebra in question. Some special subsets of joint spectra have also been studied, such as the left spectrum, the right spectrum, the defect spectrum, and, in particular, the approximate point spectrum (cf. [3], [4], [5], [6], [7], [10]). In Section 3 of this paper we propose axioms for the joint spectra and some of their subsets, extending our previous concept, called a spectral system ([10], Definition 2.1).

As the main axiom we assume here the spectral mapping property for polynomial maps. Since not all spectra considered before obey this axiom, we propose here the term "spectroid". So, in particular, the commutant and bicommutant spectra are not spectra but spectroids in the sense
of this paper. After the examples of Section 4 we give in Section 5 the 
functional representation of the sub spectra. We also obtain in this section 
the spectral mapping theorem for spectra under rational maps and give a 
description of some situations with a unique spectrum. In Section 6 we 
prove the existence of the largest spectrum and the minimal spectra and 
sub spectra. In Section 7 we consider the maps of spectra induced by homo-
morphisms; in particular, we show that the largest spectrum is invariant 
under automorphisms of the algebra in question.

In this paper we do not consider questions connected with the func-
tional calculi.

2. Notation. Let \( \mathcal{A} \) be a complex unital Banach algebra with the 
unit \( e \). The set of all invertible elements of \( \mathcal{A} \) will be denoted by \( G(\mathcal{A}) \).
The family of all non-void subsets of \( \mathcal{A} \) consisting of pairwise commuting 
elements will be designated by \( c(\mathcal{A}) \). The elements of \( c(\mathcal{A}) \) will be denoted by 
holomorphic characters \( \varphi \), where \( a \) is a (non-void) set of indices. So \( \varphi_a = \{ \varphi_{a,\alpha} \} \) is the set \( c(\mathcal{A}) \). \( \varphi \) is partially ordered by inclusion, its maximal 
elements coincide with the maximal commuting subalgebras of \( \mathcal{A} \). \( \varphi \) is the set of all maximal commutative subalgebras of \( \mathcal{A} \) will be denoted by \( \mathfrak{m}(\mathcal{A}) \) and \( \mathfrak{m}(\mathcal{A}) \subset c(\mathcal{A}) \). We shall write \( \varphi_q(\mathcal{A}) \) for the family of all 
finite elements of \( c(\mathcal{A}) \). As usual, \( \mathcal{C}^* = \bigcap \mathcal{C}_a \), where all \( \mathcal{C}_a \) are equal to 

3. Axioms and definitions. Suppose that to each family \( \varphi \in c(\mathcal{A}) \) 
there corresponds a non-void compact subset of \( \mathcal{C}^* \)
\( \varphi \rightarrow \tilde{\varphi}(\varphi) \subset \mathcal{C}^* \).

We shall formulate several conditions (axioms) for such a map
\( \tilde{\varphi}(\varphi) \subset \mathcal{C}^* \),

where \( \varphi = \{ \varphi_{a,\alpha} \} \subset c(\mathcal{A}) \), and \( \sigma(\varphi) \) is the usual spectrum of an element
\( \varphi \in \mathcal{A} \), defined as \( \sigma(\varphi) = \{ \varphi \in G(a) : \varphi \neq 0 \} \).

From this condition it follows, in particular, that
\( \tilde{\varphi}(\varphi) \subset \mathcal{C}^* \),

for every \( \varphi \in \mathcal{A} \). Here, for simplicity, we write \( \sigma(\varphi) \) instead of \( \tilde{\varphi}(\varphi) \).

Assume as the second axiom the following stronger version of formula
\( \mathcal{C}^* \),

\( \tilde{\varphi}(x) = \sigma(x) \),

for all \( x \in \mathcal{A} \).

The most essential will be the following axiom:

\( \tilde{\varphi}(\mathcal{C}^*) = \mathcal{C}^* \),

here \( \mathcal{A} \in c(\mathcal{A}) \) and \( \mathcal{P} \) is a system of complex polynomials in indeterminates
\( \mathcal{T} \), or a polynomial map. The property of the map (3) given by axiom
\( \mathcal{C}^* \),

will be called the spectral mapping property of \( \tilde{\varphi} \).

3.1. Definition. A map (2) is called a sub spectra on \( \mathcal{A} \) if axioms (I) and
\( \mathcal{C}^* \), and (III) are satisfied. It is called a spectrum, if, moreover, axiom
\( \mathcal{C}^* \), and (III) here since axioms (I) is their consequence. The set of all spectra on \( \mathcal{A} \) will be denoted by \( \text{Sp}(\mathcal{A}) \), and the set of all sub spectra by \( \text{Sp}_1(\mathcal{A}) \), so that \( \text{Sp}(\mathcal{A}) \subset \text{Sp}_1(\mathcal{A}) \).

Let us formulate some consequences of the spectral mapping property.

Suppose first that \( \mathcal{B} < \mathcal{A} \) and put \( \mathcal{P} = t_\beta \), for all \( \beta \in \mathcal{B} \). Axiom (III) then
implies

\( \tilde{\varphi}(\mathcal{P}) = \pi(\varphi) \),

where \( \mathcal{A} \in c(\mathcal{A}) \), \( \mathcal{B} \) is a non-void subset of \( \mathcal{A} \), and \( \pi \) is the projection of \( \mathcal{C}^* \),
onto \( \mathcal{C}^* \) given by \( \pi(\varphi) = \varphi \).

The property of \( \tilde{\varphi} \) given by formula (IV) will be called the projection
property of \( \tilde{\varphi} \).

Now put \( \beta = \alpha \) and \( \mathcal{P} = t_\beta = t_\alpha \), where \( \alpha \in G(a) \). Axiom (III) then
implies

\( \tilde{\varphi}(\alpha) = \tilde{\varphi}(t_\alpha) = \tilde{\varphi}(\varphi) \subset \mathcal{C}^* \),

where \( \mathcal{A} \in c(\mathcal{A}) \), \( \varphi = (\varphi_{a,\alpha} \in \mathcal{C}^* \), \( \alpha = \tilde{\varphi}(\varphi) \subset \mathcal{C}^* \).
3.3. Definition. A spectroid on $A$ is a map (2) satisfying axioms (I) and (V). If, moreover, axiom (IV) is also satisfied, a spectroid will be called a semi-spectroid. The set of all spectroids on $A$ will be denoted by $\text{Sp}(A)$, and the set of all semi-spectra by $\text{Sp}(A)$. Thus we have $\text{Sp}(A) \subset \text{Sp}(A) \subset \text{Sp}(A)$. Axiom (V) can be used for defining a spectroid by means of regularity. So we give the following

3.3. Definition. Let $\tilde{\sigma} \in \text{Sp}(A)$. An element $x_\sigma \in c(A)$ will be called $\tilde{\sigma}$-regular if $0 \neq \tilde{\sigma}(x_\sigma)$, where $0$ is the zero element of the linear space $C$. The set of all $\tilde{\sigma}$-regular elements will be denoted by $\text{Reg}(\tilde{\sigma})$. From axiom (V) it follows that each spectroid can be defined by means of $\tilde{\sigma}$-regular elements:

$$\tilde{\sigma}(x_\sigma) = \{a \in C^*: x_\sigma - a \neq 0 \} \subset \text{Reg}(\tilde{\sigma})$$

Let us remark that in order to define a semi-spectra on $c(A)$ it is sufficient to define it only on maximal subalgebras of $A$. In fact, for every $x_\sigma \in c(A)$ there is an element $a \in m(A)$, $x_\sigma \subset a$. If $a = x_\sigma$, then $a \subset b$ and axiom (IV) implies that $\tilde{\sigma}(x_\sigma) = \tilde{\sigma}(a)$ if $\tilde{\sigma} \in \text{Sp}(A)$. So all values of $\tilde{\sigma}$ are obtained by projecting the values $\tilde{\sigma}(a)$, $a \in m(A)$.

Let us also remark that in many concrete situations we have a semi-spectra defined only on elements of $c(A)$. In this case we can extend it to $c(A)$, using the following result of [10], which in our terminology reads as follows:

3.4. Proposition. Let $\tilde{\sigma}$ be a semi-spectra defined on elements of $c(A)$. There exists a unique semi-spectra on $c(A)$, which, restricted to $c(A)$, equals $\tilde{\sigma}$. If $\tilde{\sigma}$ is a subspectra on the semi-spectra, then its extension is also a subspectra, or a semi-spectra, respectively.

We have assumed here a Banach algebra convention, i.e. all concepts are related to a Banach algebra $A$. However, many important spectroids are defined in the so called spatial convention, i.e. they are defined on the algebra $L(X)$ of all continuous endomorphisms of a complex Banach space $X$, and in the definition elements of $X$ are involved. If such a definition does not depend upon a particular choice of the space $X$, we can have a spectroid also in the Banach algebra convention. For a given Banach algebra $A$ we simply put $X = A$ and interpret elements of $A$ as endomorphisms of $X$. We can do it in two ways; interpreting elements of $A$ as operators either of left multiplications or of right multiplication. Thus we embed $A$ in $L(X)$ in two ways, and so a spectroid on $L(X)$ gives rise to two spectroids defined on $A$, namely the left one and the right one (we are not interested in the iteration of this procedure, since we can embed $L(X)$ in $L(L(X))$ again in two ways, etc.)

In the next section, recalling the known and important examples of spectroids, we shall exploit both conventions, though the general convention accepted in this paper is the Banach algebra one.

4. Examples of spectroids.

4.0. The usual joint spectrum (1) in a commutative Banach algebra $A$. It is an element of $\text{Sp}(A)$.

4.1. The left spectrum $\sigma_l$ and the right spectrum $\sigma_r$. An element $x_\sigma \in c(A)$ is $\sigma_l$-regular ($\sigma_r$-regular) if the left (right) ideal in $A$ generated by the set $x_\sigma$ is improper and coincides with the whole of $A$. Both the left and the right spectra are in $\text{Sp}(A)$, but not, in general, in $\text{Sp}(A)$ (cf. [6], [10]).

4.2. The spectrum $\sigma$. It is defined by $\sigma(x_\sigma) = \sigma_l(x_\sigma) \cup \sigma_r(x_\sigma), x_\sigma \in c(A)$. It belongs to $\text{Sp}(A)$ (cf. [3], [7]).

4.3. The product $x_\sigma \rightarrow \sigma(x_\sigma)$ is a semi-spectra but not a spectrum (cf. [10]).

4.4. The biocommutant spectrum $\sigma'$. For any $x_\sigma \in c(A)$ its biocommutant $(x_\sigma)' = \sigma(x_\sigma)$ is the intersection $\bigcap \{a \in m(A): x_\sigma \subset a\}$. It is a commutative subalgebra of $A$ and $\sigma'(x_\sigma)$ is defined as the usual joint spectrum 4.0 in this subalgebra. We have $o' \in \text{Sp}(A)$, but, in general, it is not a semi-spectra (cf. [10]).

4.5. The commutant spectrum $\sigma'$. For any $x_\sigma \in c(A)$ its commutant $(x_\sigma)'$ is the union $\bigcup \{a \in m(A): x_\sigma \subset a\}$. An element $x_\sigma$ is $\sigma'$-regular if there are elements $a_1, a_2, \ldots, a_n \in c(A)$ and $y_1, y_2, \ldots, y_n \in c(A)'$ such that $\sum a_i y_i = x_\sigma$. The commutant spectrum is a spectroid but not, in general, a semi-spectra (cf. [10]).

4.6. The defect spectrum $\sigma_d$. We exploit here the spatial convention. An element $x_\sigma \in c(L(X))$ is $\sigma_d$-regular if there are elements $a_1, a_2, \ldots, a_n \in c(A)$ such that $\sum a_i \xi = x_\sigma$. In the Banach algebra convention we then have two spectra: the left defect spectrum $\sigma_d$ and the right defect spectrum $\sigma_d^\prime$. The defect spectrum is a subspectra (cf. [10], where it was denoted by $\sigma_r$ (Definition 3.9)).

4.7. The approximate point spectrum $\sigma_p$. Here we again apply the spatial convention. An element $x_\sigma \in c(L(X))$ is $\sigma_p$-regular if it is not $\sigma_d$ singular, and it is $\sigma_r$-singular if there is a net $(\xi_i)$ of elements of $X$, with $\|\xi_i\| = 1$, such that $\lim_{i \to \infty} \xi_i = 0$ for all $a \in A$. Passing to the Banach algebra convention, we obtain two concepts of a left and right approximate point spectrum, both being subspectra (cf. [9], [10]).

4.8. The Taylor spectrum $\sigma_t$. Let $X$ be a complex Banach space and consider the linear spaces $X_0, X_1, \ldots, X_n$ of homogeneous exterior forms of degree, respectively, 0, 1, ..., $n$, in indeterminates $e_1, e_2, \ldots, e_n$.
with coefficients from \( X \). Thus
\[
X_k = \left\{ \sum_{i_1 < i_2 < \ldots < i_k} \xi_{i_1} \lambda_{i_2} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} : \xi_{i_1} \in X \right\}
\]
and \( X_k \) may be identified with \( X \). Let \( \langle x_1, x_2, \ldots, x_n \rangle \) be a fixed \( n \)-tuple of pairwise commuting operators in \( L(X) \). This \( n \)-tuple gives rise to linear maps
\[
\partial_k: X_k \to X_{k+1} \quad (k = 0, 1, \ldots, n, \text{ with } X_{n+1} = 0)
\]
defined by
\[
\partial_k(\xi e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}) = \sum_{k+1}^n \xi_j (\xi \xi_j e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k})
\]
exploiting the usual convention \( e_i \wedge e_j = -e_j \wedge e_i \). The \( n \)-tuple \( \langle x_1, \ldots, x_n \rangle \) is \( \sigma \)-regular if the sequence
\[
0 \to X_0 \to X_1 \to \cdots \to X_k \to X_{k+1} \to \cdots \to X_n \to 0
\]
is exact. This defines a spectrum on \( \sigma(L(X)) \), which, by Proposition 3.4 can be extended to a spectrum on \( \lambda(L(X)) \). If we want to have \( \sigma \)-defined on \( \lambda(A) \), we proceed as at the end of Section 3, obtaining the left and right Taylor spectra (cf. [11], [12]).

For more examples cf. [9], where some families of subspectra contained in the Taylor spectrum are considered.

5. The functional representation of a subspectrum. If we have a semi-spectrum on \( A \), then from the projection property (axiom (IV)) it follows that we know it on the whole of \( \lambda(A) \) if we know it on \( m(A) \). For this reason we want to know how subspectra behave on the maximal commutative subalgebras of \( A \). In this section we study the subspectra on \( A \) restricted to the elements of \( m(A) \).

5.1. Theorem. Let \( \hat{\sigma} \) be a subspectrum on \( A \). Then
\[
\hat{\sigma}(x) = \sigma_{\hat{\sigma}}(x)
\]
for every \( x \in \lambda(A) \) and every \( \hat{\sigma} \in m(A) \) with \( x \in \hat{\sigma} \). (Here \( \sigma_{\hat{\sigma}}(x) \) is the usual joint spectrum in the commutative Banach algebra \( \hat{\sigma} \).)

Proof. We first show that for any \( \hat{\sigma} \in m(A) \) we have
\[
\hat{\sigma}(a) \subseteq \sigma_{\hat{\sigma}}(a).
\]
In fact, let \( \hat{\sigma} = \sigma_{\hat{\sigma}} \) and suppose that there is in \( C^0 \) an element \( \lambda \) such that \( \lambda \in \hat{\sigma}(\hat{\sigma}) \), \( \lambda \notin \sigma_{\hat{\sigma}}(a) \). Since \( \sigma_{\hat{\sigma}}(a) \) is a polynomially convex subset of \( C^0 \), it follows that there exists a polynomial \( p(\lambda) \), such that
\[
\|p(\lambda)\| = \sup \{|p(x)| : x \in \sigma_{\hat{\sigma}}(a)\}.
\]
Applying this polynomial to \( \hat{\sigma} \), we obtain an element \( a = p(\hat{\sigma}) \in \hat{\sigma} \), and, by axiom (III), we have
\[
\hat{\sigma}(a) = \sigma_{\hat{\sigma}}(p(\hat{\sigma})) = \sigma_{\hat{\sigma}}(a).
\]
On the other hand, we also have
\[
\sigma_{\hat{\sigma}}(a) = \sigma_{\hat{\sigma}}(p(\hat{\sigma})) = p(\sigma_{\hat{\sigma}}(a)),
\]
which in view of (6), shows that \( \hat{p}(a) \notin \sigma_{\hat{\sigma}}(a) \). Together with relation (7) this shows that \( \hat{\sigma}(a) \subseteq \sigma_{\hat{\sigma}}(a) \), which contradicts formula (5). Thus we have established formula (5). Formula (4) now follows by projecting \( C^0 \) onto \( C^0 \) and using axiom (IV).

5.2. Remark. From Theorem 3 of [10] it immediately follows, that if \( \sigma \) is a semispectrum on \( A \) and the relation (4) holds true for each \( x_\lambda \in \lambda(A) \) and each \( \hat{\sigma} \in m(A) \) satisfying \( x_\lambda \in \hat{\sigma} \), then \( \sigma \) is a subspectrum.

We can now give the functional representation of a subspectrum.

5.3. Theorem. Let \( \hat{\sigma} \) be a subspectrum on \( A \). For each \( x \in \lambda(A) \), there exists a compact subset \( \hat{\sigma}(x) \subseteq \sigma_{\hat{\sigma}}(x) \) such that
\[
\hat{\sigma}(x) = \{f(x) : f \in \hat{\sigma}(\hat{\sigma}) \}
\]
for all \( x \in \lambda(A) \) with \( x \in \hat{\sigma} \).

Proof. Fix an element \( y \in \lambda(A) \). By formula (5) for each \( \lambda \in \hat{\sigma}(\hat{\sigma}) \), \( \lambda = (\lambda \in \hat{\sigma}(\hat{\sigma})) \), there exists a unique function \( f \in \sigma_{\hat{\sigma}}(x) \) such that \( f(x) = \lambda \), for all \( x \in \hat{\sigma} \). The correspondence \( \lambda \mapsto f \) is a homeomorphism between \( \hat{\sigma}(\hat{\sigma}) \), treated, by (5), as a subset of \( \sigma_{\hat{\sigma}}(x) \) and a subset of \( \sigma_{\hat{\sigma}}(\hat{\sigma}) \) (which, in turn, is homeomorphic to \( \sigma_{\hat{\sigma}}(\hat{\sigma}) \)). Denote this subset by \( \hat{\sigma}(x) \). Since \( \hat{\sigma}(x) \) is compact \( \hat{\sigma}(x) \) is also compact. We have
\[
\hat{\sigma}(x) = \{f(x) : f \in \hat{\sigma}(\hat{\sigma}) \},
\]
where \( f \) is a suitable set of indexes. Formula (8) now follows from formula (9) and the projection property of \( \hat{\sigma} \).

From this result we obtain the following spectral mapping theorem for elements of \( \lambda(A) \), which also gives a characterization of spectra among the subspectra:

5.4. Theorem. Let \( \hat{\sigma} \) be a subspectrum on \( A \). Then \( \hat{\sigma} \) is a subspectrum on \( A \) if and only for each \( x_\lambda \in \lambda(A) \) and each system of rational functions \( r(\lambda) = p_\beta(\lambda) \), \( \beta \in \beta \), such that
\[
\hat{\sigma}(x_\lambda) \subseteq \{x_\beta \in C^0 : g_\beta(x_\beta) \neq 0 \text{ for all } \beta \in \beta \}
\]
we have
\[
\hat{\sigma} = \{x_\beta \in C^0 : g_\beta(x_\beta) \neq 0 \text{ for all } \beta \in \beta \}
\]
5.7. Theorem. Let $H$ be a complex Hilbert space and let $A = L(H)$. Then all spectra on $A$ coincide on elements of $c(A)$ consisting of normal operators.

Proof. Take any element $a_0$ in $c(A)$ consisting of normal operators.

Let $\mathcal{A}$ be a maximal subset of $A$ consisting of pairwise commuting normal operators and containing the set $a_0$. By the Fuglede–Putnam theorem we have $\mathcal{A}^* = \mathcal{A}$, and by the maximality of $\mathcal{A}^*$ is a commutative $C^*$-algebra. We claim that $\mathcal{A} \subset m(A)$. In fact, if $a$ commutes with all elements of $\mathcal{A}$, then, by the Fuglede–Putnam theorem $a^* a$ also commutes and so do the elements $(a + a^*)/2$ and $(a - a^*)/2$. The latter element, being hermitian, belongs to $\mathcal{A}^*$, and so $a \in \mathcal{A}^*$. Hence $\mathcal{A} \subset m(A)$, and $M(\mathcal{A}) = C(\mathcal{A})$, since $\mathcal{A}^*$ is the unique spectral set for $\mathcal{A}$, and so $d(\mathcal{A}, \mathcal{A}) = 0$ for all $\delta \in Sp(A)$. This means that $\tilde{\delta}(\mathcal{A}) = s_{\mathcal{A}}(\mathcal{A}^*)$ and so, by the projection property, $\tilde{\delta}(\mathcal{A}) = s_{\mathcal{A}}(\mathcal{A})$ for each $\delta \in Sp(A)$, which gives the desired result.

6. The largest spectrum and the minimal spectra and subspectra.

6.1. Definition. For two subspaces $\tilde{a}, \tilde{a}_0$ defined on a Banach algebra $A$ we shall write $\tilde{a} \leq \tilde{a}_0$ if $s_{\mathcal{A}}(\mathcal{A}) = s_{\mathcal{A}}(\mathcal{A})$ for all $a \in c(A)$.

In this section we shall show that every Banach algebra possesses a largest spectrum $c_m$, i.e. a spectrum such that $\tilde{a} \leq c_m$ for all $\tilde{a} \in Sp(A)$.

We shall show also that there exist minimal spectra and subspectra.

The proof of the following lemma is an easy exercise on compact sets and continuous maps.

6.2. Lemma. Let $p$ be a continuous map from $C^*$ into $C^*$, and let $\Omega$ be a subset of $C^*$ with compact closure $\Omega$. Then $p(\Omega) = p(\Omega)$.

This lemma will be used in the proof of the following

6.3. Theorem. There exists a largest spectrum $c_m$ on $A$.

Proof. For a fixed $a_0 \in c(A)$ define

\[ s_{c_m}(a_0) = \bigcup \{ \tilde{\delta}(a_0) : \tilde{a} \in Sp(A) \} \]

it is a compact subset of $C^*$ satisfying axiom (I).

Take any family of complex polynomials $p_j(a_0)$. Since it defines a continuous map from $C^*$ into $C^*$, we can apply Lemma 6.2, taking as $\Omega$ the union $\bigcup \{ \tilde{\delta}(a_0) : \tilde{a} \in Sp(A) \}$ and as $p$ the map defined by the family $p_j(a_0)$. Thus

\[ p_j(s_{c_m}(a_0)) = p_j(\bigcup \{ \tilde{\delta}(a_0) : \tilde{a} \in Sp(A) \}) \]

\[ = p_j(\bigcup \{ \tilde{\delta}(a_0) : \tilde{a} \in Sp(A) \}) = \bigcup \{ p_j(\tilde{\delta}(a_0)) : \tilde{a} \in Sp(A) \} \]

\[ = \bigcup \{ p_j(\tilde{\delta}(a_0)) : \tilde{a} \in Sp(A) \} \]

and so $c_m$ satisfies axiom (III). Axiom (II) is satisfied as well, since $c_m$ is larger than some spectrum, e.g. the spectrum of Example 4.2.
By Theorem 3.1 we obtain the following corollary:

6.4. Corollary. For all \( x_\bullet \in C_{\mathcal{A}} \) we have

\[
c_{\mathcal{A}}(x_\bullet) \subseteq \bigcap \{ \mathbf{c}_\mathcal{A}(x): \mathcal{A} \in \mathbf{m}(A), x_\bullet \in \mathbf{c}(x) \} = \mathbf{c}''(x_\bullet).
\]

Remark. For the algebra \( C_{\mathcal{A}} \) in which \( \phi_{\mathcal{A}}(x) \) does not depend upon \( \mathcal{A} \), we have \( \mathbf{c}_\mathcal{A}(x) = \mathbf{c}(x) \). Such a situation holds e.g. for the situation described in Corollary 3.6. In general, however, the value of \( \phi_{\mathcal{A}}(x) \) depends upon \( \mathcal{A} \), as shown in paper [1]. Thus the problem arises of a more effective description of the largest spectrum. In the commutative case the largest spectrum equals \( \mathbf{c}(x) \), but in the non-commutative we know it only in situations described by Corollary 5.6.

We now prove a theorem concerning the existence of minimal spectra and sub-semi-spectra.

6.5. Theorem. \( \mathbf{Sp}(A) \) and \( \mathbf{Sp}_{1}(A) \) contain the minimal elements.

Proof. Suppose that we have a linearly ordered subset \( \{x_\bullet\} \subseteq \mathbf{Sp}(A) \). We shall show that there exists an element \( x_\bullet \in \mathbf{Sp}_{1}(A) \) such that \( x_\bullet \leq x \) for all \( x \). To this end put \( \phi_{\mathcal{A}}(x_\bullet) = \bigcap \{ \phi_{\mathcal{A}}(x): \mathcal{A} \in \mathbf{m}(A) \} \), then \( \phi_{\mathcal{A}}(x_\bullet) \) satisfies the axioms for \( \mathcal{A} \). By the finite intersection property \( \phi_{\mathcal{A}}(x_\bullet) \) is a non-empty compact subset of \( C_{\mathcal{A}} \), which clearly satisfies axiom (I). If \( (\mathcal{A}) \subseteq \mathbf{Sp}(A) \), then \( \mathbf{c}(x_\bullet) \) satisfies also axiom (II). In fact, let us fix an index \( a \in \mathcal{A} \). For each \( x \in \mathbf{c}(x_\bullet) \) and each \( k \) there is a point \( \lambda \in \phi_{\mathcal{A}}(x_\bullet) \) such that \( \pi(\lambda) = l_\lambda \), where \( \pi \) is the projection of \( C_{\mathcal{A}} \) onto the \( \mathfrak{a} \)th coordinate plane. The set \( \{ \lambda \} \) contains a subset convergent to a point \( \lambda \in C_{\mathcal{A}} \). Since \( \lambda \in \phi_{\mathcal{A}}(x_\bullet) \) for all \( k \in \mathcal{A} \), the ordering of indexes \( \lambda \) is the same as the ordering of the corresponding spectra and we write \( \kappa \prec \lambda \) if \( \kappa \leq \lambda \), it follows that \( \lambda \) is in the intersection \( \phi_{\mathcal{A}}(x_\bullet) \). We have \( \pi(\lambda) = \pi(\mathbf{c}(x_\bullet)) = \pi(\mathbf{c}(x_\bullet)), \) and \( \mathbf{c}(x_\bullet) \) contains the spectrum \( \phi_{\mathcal{A}}(x_\bullet) \). Together with the relation given by (I), this implies that \( \phi_{\mathcal{A}}(x_\bullet) \) satisfies axiom (II).

Axiom (III) is also satisfied since for any element \( x_\bullet \in \mathbf{Sp}(A) \) and any system of complex polynomials \( p_{\mathcal{A}}(x_\bullet) \) we have

\[
\phi_{\mathcal{A}}(x_\bullet) = \bigcap \{ p_{\mathcal{A}}(x_\bullet): \mathcal{A} \in \mathbf{m}(A) \} = \bigcap \{ p_{\mathcal{A}}(x_\bullet): \mathcal{A} \in \mathbf{m}(A) \} = \bigcap \{ p_{\mathcal{A}}(x_\bullet): \mathcal{A} \in \mathbf{m}(A) \} = \bigcap \{ p_{\mathcal{A}}(x_\bullet): \mathcal{A} \in \mathbf{m}(A) \}.
\]

Applying the Kuratowski–Zorn lemma, we see that there are minimal elements both in \( \mathbf{Sp}(A) \) and \( \mathbf{Sp}_{1}(A) \).

If \( \mathcal{A} \) is a commutative Banach algebra, then the largest spectrum coincides with the usual joint spectrum, while the minimal spectra are of the form \( \mathbf{c}(x_\bullet) = \{ (x_\bullet)(x): \mathfrak{a} \in \mathfrak{a} \} \), where \( \mathfrak{a} \) is a minimal spectral set. Of course, each minimal sub-spectrum of a commutative algebra consists of a single point for each \( x_\bullet \in \mathbf{c}(\mathcal{A}) \). (1)

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(1) In paper [16] there is described the situation when there exist single point sub-semi-spectra on a Banach algebra \( \mathcal{A} \).

We now give two examples of minimal spectra for a commutative algebra.

6.6. Example. Let \( \mathcal{A} = \{ x_\bullet \in C_{\mathcal{A}}: |x_\bullet| < 1, |x_\bullet| < 1 \} \), and let \( \mathcal{A} \) be the algebra of all functions holomorphic on \( \mathcal{A} \) and continuous on \( \mathcal{A} \) provided with the supremum norm. The set \( \mathcal{B} = \{ (x_\bullet, x_\bullet) \in C_{\mathcal{A}}: |x_\bullet|^2 + |x_\bullet|^2 = 1 \} \) is a spectral set for \( \mathcal{A} \), which coincides with its Shilov boundary. Thus \( \mathcal{A} \) possesses a unique minimal spectral set, and so a unique minimal spectrum.

6.7. Example. Let \( \mathcal{A} = \{ x_\bullet, y_\bullet \in C_{\mathcal{A}}: |x_\bullet| < 1, |y_\bullet| < 1 \} \), and let \( \mathcal{A} \) be the algebra of all functions holomorphic on \( \mathcal{A} \) and continuous on \( \mathcal{A} \) provided with the supremum norm. The algebra \( \mathcal{A} \) possesses two minimal spectral sets,

\[
\mathcal{A}_1 = \{ (x_\bullet, x_\bullet) \in C_{\mathcal{A}}: x_\bullet = 0, |x_\bullet| < 1 \}
\]

and

\[
\mathcal{A}_2 = \{ (x_\bullet, y_\bullet) \in C_{\mathcal{A}}: |x_\bullet| = 1, |y_\bullet| = 1 \}
\]

and

\[
\mathcal{A}_3 = \{ (x_\bullet, y_\bullet) \in C_{\mathcal{A}}: |x_\bullet| = 1, |y_\bullet| = 1 \}
\]

consequently there are two different minimal spectra.

7. Maps of sub-semi-spectra defined by homomorphisms.

7.1. Proposition. Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be unit Banach algebras. Let \( \mathcal{A}_2 \) be a spectral (seminispectrum, subspectrum) on \( \mathcal{A}_2 \). Let \( \mathcal{A}_2 \) be a unit Banach algebra. Let \( \mathcal{A}_2 \) be a spectral (seminispectrum, subspectrum) on \( \mathcal{A}_2 \). Define \( \mathcal{A}_2 \) by setting

\[
\mathcal{A}_2 = \mathcal{A}_1 \times \mathcal{A}_2,
\]

where \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are the algebras of all functions holomorphic on \( \mathcal{A}_2 \) and continuous on \( \mathcal{A}_2 \) provided with the supremum norm. The algebra \( \mathcal{A}_2 \) possesses two minimal spectral sets,

\[
\mathcal{A}_4 = \{ (x_\bullet, x_\bullet) \in C_{\mathcal{A}_2}: |x_\bullet| < 1, |x_\bullet| < 1 \}
\]

and

\[
\mathcal{A}_5 = \{ (x_\bullet, y_\bullet) \in C_{\mathcal{A}_2}: |x_\bullet| < 1, |y_\bullet| < 1 \}
\]

and

\[
\mathcal{A}_6 = \{ (x_\bullet, y_\bullet) \in C_{\mathcal{A}_2}: |x_\bullet| = 1, |y_\bullet| = 1 \}
\]

consequently there are two different minimal spectra.
Remarks. It can occur that $\sigma$ fails to be a spectrum on $A$, while $\tilde{\sigma}$ is a spectrum on $A$ and one can easily create a situation in which $\sigma$ is the largest spectrum and $\tilde{\sigma}$ is a minimal subspectrum. Even if $h$ is a homomorphism onto, or an isomorphism into, $h^*e$ may fail to be a spectrum for a spectrum $e$. Only when $h$ is an isomorphism onto and $e$ is a spectrum, $h^*e$ is also a spectrum. This is because $\sigma(e) = \sigma(ha)$ for any $a \in A$ and so axiom (12) is satisfied for $h^*e$.

The next remark we formulate as

7.2. COROLLARY. Let $h$ be an automorphism of a Banach algebra $A$ and let $\tilde{\sigma}$ be a spectrum on $A$. Then $h^*\sigma$ is also a spectrum on $A$.

From Proposition 7.1 we shall obtain the following.

7.3. THEOREM. The maximal spectrum $\sigma_m$ on a Banach algebra $A$ is invariant under automorphisms of $A$, i.e.

\[ \sigma_m = h^*\sigma_m \]

for each automorphism $h$ of $A$.

Proof. By Proposition 7.1, $h^*\sigma_m$ is a subspectrum (a spectrum by Corollary 7.2), and so from the definition of $\sigma_m$ we see that

\[ (h^*\sigma_m)(x) = \sigma_m(x) \]

for all $x \in c(A)$. Taking $h^{-1}$ instead of $h$, we have, by (18),

\[ (h^{-1})^*\sigma_m(x) = \sigma_m(x) \]

for all $x \in c(A)$. Relation (19) implies

\[ \sigma_m(x) = (h^*\sigma_m)(x) = (h^*\sigma_m)(x) \]

Relations (18) and (20) imply (17).

7.4. DEFINITION. A spectroid $\tilde{\sigma}$ on the Banach algebra $A$ is called regular if it is invariant under automorphisms of $A$, i.e. if $h^*\tilde{\sigma} = \tilde{\sigma}$ for each automorphism $h$ of $A$.

The spectroids $\sigma$, $\sigma'$, $\sigma_1$, $\sigma_2$ and $\sigma_m$ are regular (we do not know whether spatially defined spectroids $\sigma_1$, $\sigma_2$, $\sigma_m$ are regular; we only know that they are invariant under inner automorphism, or inner regular). However, we have not added the requirement of regularity, or inner regularity, because we do not see any consequences of this property. Moreover, the minimal spectrums, in general, fail to be regular.

7.5. EXAMPLE. Let $A$ be the bicylinder algebra of Example 6.7 and let $\sigma_i$ be the minimal spectrum defined by the spectral set $\Delta_i$, $i = 1, 2$. Let $h$ be the automorphism of $A$ given by $\langle h \rangle$ $\langle \sigma_1, \sigma_2 \rangle = \sigma(\sigma_1, \sigma_2)$. Then $h^*\sigma_i = \sigma_2 \neq \sigma_1$.

7.6. PROPOSITION. For each $\tilde{\sigma} \in \text{Sp}_i(A)$ there exists a smallest regular subspectrum $\tilde{\sigma}$, larger than $\tilde{\sigma}$.

Proof. We define

\[ \tilde{\sigma}(x) = \bigcup \{ h^*\sigma(x) : h \in \text{Aut}(A) \} \]

for all $x \in c(A)$, where $\text{Aut}(A)$ is the set of all automorphisms of $A$.

Then we proceed as in proofs of Theorem 6.3 and Proposition 7.1.

References