Outside B_{k+1},

$$|f(x)-0| < \varepsilon/4.$$

Therefore,

$$\sup_{x\in \boldsymbol{R}^d} |f(x)-\tilde{p}_k(x)| < \varepsilon,$$

i.e.,

$$\|f-\tilde{p}_k\|_\infty < \varepsilon.$$

Hence, $\{\tilde{p}_j: p \in \mathscr{P},\ j \in Z_+\}$ is a denumerable dense subset of $(C_\infty(\boldsymbol{R}^d),\ \|\cdot\|_\infty)$. Thus, $(C_\infty(\boldsymbol{R}^d),\ \|\cdot\|_\infty)$ is separable as claimed. ■

THEOREM 7.2. $(\Gamma, \|\cdot\|)$ *is separable.*

Proof. $\Gamma \subset \prod^\infty C_\infty(\boldsymbol{R}^d)$ and $C_\infty(\boldsymbol{R}^d)$ is separable by Lemma 7.1. A countable product of separable metric spaces is separable metric. A subspace of a separable metric space is separable. $(\Gamma, \|\cdot\|)$ is therefore separable. ■

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Generalized conjugate systems on local fields

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Abstract. The notion of a conjugate system of regular functions over $K^n \times \boldsymbol{Z}$, where K^n is the n-dimensional vector space over a local field and $\boldsymbol{Z}$ is a set of rational integers, is extended to that of a generalized conjugate system (GCS). Such systems are analogues of generalized Cauchy–Riemann systems of harmonic functions on Euclidean half-spaces. Examples of such GCS's are constructed by means of a system of operators, $\{R_l\}_{l=1}^n$, that are analogues of the Riesz transforms. An F. and M. Riesz theorem is proved. (If μ and $R_l\mu$, $l = 1, 2, \ldots, n$ are all finite Borel measures, then μ fs absolutely continuous.) A conjugate system definition of the Hardy space, $H^1(K^n)$, is proposed ($f \in H^1$ iff $f \in L^1$ and $R_l f \in L^1$ for all l) and it is shown that this definition is equivalent to other proposed definitions; namely, maximal function, Lusin area iunction, and atomic definitions.

§ 1. Introduction. Chao [1] and Chao and Taibleson [4] have given a definition of conjugate systems of functions on $K \times \boldsymbol{Z}$, K a local field and $\boldsymbol{Z}$ the rational integers, which gives rise to an F. and M. Riesz theorem: *Suppose the local class field of K is odd. Then there is a singular integral operator T on K with the property that if μ and $T\mu$ are both finite Borel measures then μ is absolutely continuous.* This operator is the local field version of the conjugate operator (Hilbert transform) on $\boldsymbol{R}$. In this paper we will extend the notion of *conjugate system* to *generalized conjugate system* (GCS) and we will construct examples which arise from systems of "Riesz" transforms, $\{R_l\}_{l=1}^n$ on K^n, the n-dimensional vector space over K.

For such a Riesz system we will establish an F. and M. Riesz theorem: *If μ and $R_l\mu$, $l = 1, 2, \ldots, n$ are all finite Borel measures then μ is absolutely continuous.* It will also be shown that a range of definitions for the Hardy space $H^1(K^n)$ are all equivalent. Thus, if H^1 is defined by the property: $f \in H^1$ iff f and $R_l f$, $l = 1, 2, \ldots, n$ are all integrable, then that

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conjugate system characterization is equivalent to maximal function characterizations, Lusin function characterizations, as well as atomic and molecular characterizations.

§ 2. Preliminaries and an example of a Riesz system. A convenient general reference for this paper is [7]. Let K be a local field, which is to say, a locally compact, non-discrete field that is not connected. Then K is totally disconnected. An example of such a field is a p-adic number field. The ring of integers, $\mathfrak{D}$, in K is the unique maximal compact subring of K. The prime ideal $\mathfrak{B}$, in $\mathfrak{D}$ is the unique maximal ideal in $\mathfrak{D}$. It is principal and is generated by an element $\mathfrak{p}$. $\mathfrak{D}/\mathfrak{B} = \mathrm{GF}(q)$, a finite field of order q. This field is called the *local class field* of K and we assume throughout that q is odd. For the p-adic field $\mathfrak{D}/\mathfrak{B} = \mathrm{GF}(p)$. There is a norm on K, $|\cdot|: K \to [0, \infty)$, such that $|x+y| \leqslant \max[|x|, |y|]$. An easy consequence of this "ultrametric" inequality is that if $|x| \neq |y|$ then $|x+y| = \max[|x|, |y|]$. In terms of this norm, $\mathfrak{D} = \{|x| \leqslant 1\}$ and $\mathfrak{B} = \{|x| < 1\} = \{|x| \leqslant q^{-1}\}$. $\mathfrak{D}^* = \{|x| = 1\}$ is the group of units in K^*, the multiplicative group of K. Note that $dx/|x|$ is a Haar measure on K^*, where dx is Haar measure on K^+, the additive group of K. We note also that if $x \in K$ and $x \neq 0$ then $|x| = q^k$ for some $k \in \boldsymbol{Z}$. The generator $\mathfrak{p}$ of the prime ideal $\mathfrak{B}$ can be any element $\mathfrak{p} \in \mathfrak{B}$ such that $|\mathfrak{p}| = q^{-1}$.

We now let K^n be the n-dimensional vector space over K. Then for $x \in K^n$, $x = (x_1, \ldots, x_n)$, $x_l \in K$, K^n is endowed with an ultrametric norm, $|x| = \max\limits_l |x_l|$. (The use of identical notation for different norms should cause no difficulty.) Haar measure on K^n, dx, is the product measure and for a measurable set E we write $|E| = \int_E 1\,dx$. Note that $dx/|x|^n$ is invariant with respect to scalar multiplication by elements of K. Let $P^k = \{|x| \leqslant q^{-k}\}$. The collection $\{P^k\}_{k\in Z}$ is a neighborhood system of 0 in K^n and each P^k is a subgroup of K^n. By convention $P \equiv P^1$, $R \equiv P^0$ and we let $R^* = R \sim {\sim} P = \{|x| = 1\}$. We normalize Haar measure on K^+ so $|\mathfrak{D}| = 1$. As a consequence $|P^k| = q^{-kn}$.

In [7], VII § 2 the examples of operators giving use to conjugate systems were singular integral operators on K of the Calderón–Zygmund type. They had kernels of the form $\pi(x)/|x|$, where π is an odd multiplicative character on K, ramified of degree 1 and homogeneous of degree zero. We will consider here, singular integral operators T on K^n where $Tf = (\mathrm{P.V.}\,\Omega(x)/|x|^n)*f$, Ω is homogeneous of degree zero $(\Omega(\mathfrak{p}^k x) = \Omega(x)$, $x \in K^n$, $k \in \boldsymbol{Z})$, is constant on cosets of P^{k+1} in $P^k \sim P^{k+1}$ $(\Omega(x+y) = \Omega(x)$, whenever $|y| < |x|)$ and Ω has mean value zero on R^* $\left(\int_{R^*} \Omega(x)\,dx = 0\right)$.

The main facts about these operators parallel those of the operators described in [7], VII § 2, and the proofs are essentially the same. We will outline those main facts, mostly without proof. The details can be filled in easily.

The distribution $\mathrm{P.V.}\,\Omega(x)/|x|^n$ has a Fourier transform $\hat{\Omega}$ that is homogeneous of degree zero, constant on cosets of P^{k+1} in $P^k \sim P^{k+1}$ and has mean value zero on R^*. Conversely, every such function arises as a Fourier transform of a distribution $\mathrm{P.V.}\,\Omega(x)/|x|^n$, as above. (For the most elementary facts about Fourier analysis on K^n see [7], III §§ 1–3.)

Recall the notion of a regular function on $K^n \times \boldsymbol{Z}$, (which is the local field analogue of a harmonic function on a Euclidean half-space). (See [7], IV for details.) $f(x, k)$ is regular on $K^n \times \boldsymbol{Z}$ if (i) $f(x, k)$ is constant on cosets of P^{-k} as a function of x and (ii)

$$\int_{y+P^{-l}} f(x, k)\,dx = \int_{y+P^{-l}} f(x, l)\,dx, \qquad y \in K^n,\ k \leqslant l.$$

If f satisfies (i) is real-valued and

$$\int_{y+P^{-l}} f(x, -k)\,dx \geqslant \int_{y+P^{-l}} f(x, l)\,dx, \qquad y \in K^n,\ k \leqslant l,$$

then f is said to be *subregular*.

Let

$$R(x, k) = \begin{cases} q^{-kn}, & x \in P^{-k},\\ 0, & x \notin P^{-k}.\end{cases}$$

The regularization of a distribution f is $f(x, k) = \big(f*R(\cdot, k)\big)(x)$ and is, indeed, a regular function. In fact, every regular function arises as the regularization of a distribution.

Consider the distribution $Q = \mathrm{P.V.}\,\Omega(x)/|x|^n$. Then

$$Q(x, k) = \begin{cases} \Omega(x)/|x|^n, & x \notin P^{-k},\\ 0, & x \in P^{-k}.\end{cases}$$

For "nice" functions on K^n we can define a singular integral operator T and a multiplier T' as follows:

$$Tf = \lim_{k\to-\infty} Q(\cdot, k)*f;\ (T'f)^{\hat{}} = \hat{\Omega}\hat{f}.$$

Since $\hat{Q} = \hat{\Omega}$ it follows that $T = T'$ in the sense that they agree on $\mathscr{S}$ (the class of test functions on K^n) and so have identical "extensions" on spaces such as L^p, $1 \leqslant p < \infty$ and the space of finite Borel measures.

Let us now suppose that a distribution is "nice enough" in that "$Tf(\cdot, k) = Tf*(\cdot, k) = f*Q(\cdot, k)$" makes sense.

Let $\{\varepsilon^j\}_{j=1}^{q^n-1}$ be representatives of the q^n-1 cosets of P in R^*, and set $\varepsilon^0 = 0$. $G = \{\varepsilon^j\}_{j=0}^{q^n-1}$ can be treated as an additive group that is the direct product of n groups of order q (each a copy of the additive group of $\mathrm{GF}(q)$). (It turns out that G can be given the field structure of $\mathrm{GF}(q^n)$, but we do not need that fact here. Details may be found in [8].)

Let $\varepsilon_m^j = \mathfrak{p}^{-(m+1)}\varepsilon^j$. Then $\{\varepsilon_m^j\}_{j=0}^{q^n-1}$ are coset representatives of P^{-m} in $P^{-(m+1)}$. Ω is completely defined by the q^n-1 values $\{\Omega(\varepsilon^j)\}_{j=1}^{q^n-1}$

and we set $\Omega(0) \equiv \Omega(\varepsilon^0) = 0$. Similarly, $\hat{\Omega}$ is defined by the values $\{\hat{\Omega}(\varepsilon^j)\}_{j=1}^{q^n-1}$ and we set $\hat{\Omega}(0) \equiv \hat{\Omega}(\varepsilon^0) = 0$. Note that $\Omega(\varepsilon_m^j) = \Omega(\varepsilon^j)$, $\hat{\Omega}(\varepsilon_m^j) = \hat{\Omega}(\varepsilon^j)$ for $j = 0, 1, \ldots, q^{n-1}$ and $m \in \mathbf{Z}$.

The trick is to observe that if we normalize Haar measure on G so that the mass of each point is q^{-n}, then the dual group of G with Plancherel measure is G again but with each point having mass 1. It is easy to check that the mapping $\Omega \to \hat{\Omega}$ is given precisely by the Fourier transform on G.

We continue as in [7], p. 245. We have

$$Tf(x, k) = \sum_{m=k}^{\infty} q^{-n} \sum_{j=0}^{q^n-1} \Omega(\varepsilon^j) f(x - \varepsilon_m^j, m).$$

If we let $d_k f = f(\cdot, k) - f(\cdot, k+1)$, then $T(d_k f) = d_k Tf$, which is to say,

$$T(d_k f)(x) = q^{-k} \sum_{j=0}^{q^n-1} \Omega(\varepsilon^j) d_k f(x - \varepsilon_k^j).$$

Note that regularity and subregularity can be defined in terms of the behaviour of $d_k f$. Specifically, if $f(x, k)$ is constant on cosets of P^{-k} then f is regular iff $\sum_{j=0}^{q^n-1} d_k f(x - \varepsilon_k^j) = 0$ for all x and k and is subregular ff f is real-valued and $\sum_j d_k f(x - \varepsilon_k^j) \geqslant 0$ for all x and k.

Consider operators T_l, associated functions Ω_l, and their transforms $\hat{\Omega}_l$, $l = 1, 2, \ldots, m$. Fix a coset $y + P^{-(k+1)}$. Let $a_0 = f(y, k+1)$, $a_0^j = d_k f(y + \varepsilon_k^j)$, $a_l = T_l f(y, k+1)$, $a_l^j = T_l d_k f(y + \varepsilon_k^j)$, $l = 1, \ldots, m$, $j = 0, 1, \ldots, q^n - 1$. Then for $x \in y + \varepsilon_k^j + P^{-k}$ we have

$$a_k^j = T(d_k f)(x) = q^{-n} \sum_{i=0}^{q^n-1} \Omega_l(\varepsilon^j - \varepsilon^i) d_k f(y + \varepsilon_k^i) = q^{-n} \sum_{i=0}^{q^n-1} \Omega_l(\varepsilon^j - \varepsilon^i) a_0^i.$$

Thus, the map from $d_k f$ to $d_k T_l f$ is realized on G as a convolution operator. Specifically, if $a_0 = \{a_0^j\}$, $a_l = \{a_l^j\}$, and $\Omega_l = \{\Omega_l(\varepsilon^j)\}$ we have $a_l = a_0 * \Omega_l$. Using the fact that the Fourier transform of Ω_l is $\hat{\Omega}$ we see that $\hat{a}_l = \{\hat{a}_0^j \hat{\Omega}_l(\varepsilon^j)\}$. The following results are then simple consequences of Fourier analysis on G.

$$0 = q^{-n} \sum_j d_k(y + \varepsilon_k^j) = q^{-n} \sum a_0^j = \hat{a}_0^0, \tag{2.1}$$

$$q^{-n} \sum_j a_l^j = \hat{a}_l^0 = \hat{a}_0^0 \hat{\Omega}_l(\varepsilon^0) = 0, \tag{2.2}$$

$$\|a_0\|^2 = q^{-n} \sum_j |a_0^j|^2 = \sum_j |\hat{a}_0^j|^2, \tag{2.3}$$

$$\|a_l\|^2 = q^{-n} \sum_j |a_l^j|^2 = \sum_j |\hat{a}_0^j|^2 |\hat{\Omega}_l(\varepsilon^j)|^2, \tag{2.4}$$

$$q^{-n} \sum_j a_0^j a_l^j = \sum_j \hat{a}_0^j \hat{a}_l^{-j} = \sum \hat{a}_0^j \hat{a}_0^{-j} \hat{\Omega}_l(\varepsilon^{-j}), \tag{2.5}$$

where, by convention, we arrange the $\{\varepsilon^j\}$ in such a way that $\varepsilon^{-j} = -\varepsilon^j$.

If $T_l f(x, k)$ is defined it follows from (2.2) that it is regular. If Ω_l is odd, then so is $\hat{\Omega}_l$ and it follows from (2.5) that $q^{-n} \sum_j a_0^j a_l^j = 0$. Suppose there are constants, $0 < M_1 \leqslant 1 \leqslant M_2$ such that

$$M_1 \leqslant \sum_l |\hat{\Omega}_l(\varepsilon^j)|^2 \leqslant M_2, \quad j = 1, \ldots, q^n - 1.$$

Then if we set $B = \min[1/(1+M_1), M_2/(1+M_2)]$ we see that $1/2 \leqslant B < 1$, and

$$\max\left[\|a_0\|^2, \sum_{l=1}^{m} \|a_l\|^2\right] \leqslant B \sum_{l=0}^{m} \|a_l\|^2.$$

Furthermore, $(\|a_0\|^2 = 0)$ iff $\left(\sum_{l=1}^{m} \|a_l\|^2 = 0\right)$. These last observations follow from (2.3) and (2.4).

If, for instance, $|\hat{\Omega}_l(\varepsilon^j)| \leqslant 1$ and for each $j \neq 0$ there is an l so that $\hat{\Omega}_l(\varepsilon^j)$ is not zero we have the conditions above where

$$M_1 = \min\{|\hat{\Omega}_l(\varepsilon^j)|^2 \colon \hat{\Omega}_l(\varepsilon^j) \neq 0\} \quad \text{and} \quad M_2 = m.$$

Examples of such systems abound. We will describe one such system which is an analogue of the Riesz system on $\mathbf{R}^n$. For $l = 1, 2, \ldots, n$ write $x \in K^n$ as $x = (x_l, x')$, $x_l \in K$, $x' \in K^{n-1}$. Let $\Omega_l(x) = (q^{n-1}/\Gamma(\pi))\pi(x_l)$ when $|x_l| > |x'|$ and is zero otherwise; where π is a unitary multiplicative character on K that is odd, homogeneous of degree zero, and ramified of degree 1; $\Gamma(\pi)$ is the gamma function ([7], II § 5). With some calculation we find that $\hat{\Omega}_l(x) = \pi^{-1}(x_l)$ if $|x_l| \geqslant |x'|$ and is zero otherwise. While there are many such possible analogues of the Riesz system, in the sequel we refer to this system as the *Riesz system* and the operators $R_l f = (\mathrm{P.V.}\, \Omega_l/|x|^n) * f$, $l = 1, 2, \ldots, n$ will be called the *Riesz operators*.

§ 3. Generalized conjugate systems, F. and M. Riesz theorems. Let $\{a_l^j\}$, $l = 0, 1, \ldots, m$; $j = 1, \ldots, k$. Then set

$$a_l = (a_l^1, \ldots, a_l^k) \in \mathbf{C}^k; \quad \|a_l\| = \left[\sum_{j=1}^{k} |a_l^j|^2\right]^{1/2}, \quad l = 0, \ldots, m.$$

$$a^j = (a_0^j, \ldots, a_m^j) \in \mathbf{C}^{m+1}; \quad |||a^j||| = \left[\sum_{l=0}^{m} |a_l^j|^2\right]^{1/2}; \quad j = 1, \ldots, k.$$

For any $a = (a_0, \ldots, a_m) \in C^{m+1}$ we set $|||a||| = \left[\sum_{l=0}^{m} |a_l|^2\right]^{1/2}$. Note that the definition of $\|a_l\|$ differs from the definition given in § 2 by a constant.

We saw above how such systems arise. Somewhat more generally we consider a vector valued function $F(x, k) = (f_0(x, k), \ldots, f_m(x, k))$ on $K^n \times Z$, which is constant on cosets of P^{-k}, as a function of x, for each $k \in Z$. Then, as above, for each coset $y + P^{-(k+1)}$ let $a_l = f_l(y, k+1)$, $a_l^j = d_k f_l(y + \varepsilon_k^j)$.

DEFINITION. Let F be as above with each f_l regular. F is a *generalized conjugate system* if there is a non-trivial partition of $\{0, 1, \ldots, m\} = D \cup E$, and constants A and B, $1/2 \leqslant B < 1$, $0 \leqslant A < 1$, $B(1+A) < 1$ such that for every coset $y + P^{-(k+1)}$,

$$\max\left[\sum_{l \in D} \|a_l\|^2, \sum_{s \in E} \|a_s\|^2\right] \leqslant B \sum_{l=0}^{m} \|a_l\|^2, \tag{3.1}$$

$$\sum_{l \in D} \|a_l\|^2 = 0 \quad \text{iff} \quad \sum_{s \in E} \|a_s\|^2 = 0, \tag{3.2}$$

$$\sum_{j=0}^{q^n-1} a_l^j a_s^j \leqslant A \|a_l\| \|a_s\|, \quad l \in D,\ s \in E. \tag{3.3}$$

We emphasize that if f is a regular function and $\{R_l\}$ is the Riesz system and $\{R_l f(x, k)\}$ is defined (if f is a finite Borel measure, it is), then $F = (f, R_1 f, \ldots, R_n f)$ is a generalized conjugate system with $D = \{0\}$, $E = \{1, \ldots, n\}$, $m = n$, $A = 0$, and $B = n/(n+1)$.

THEOREM 1. *If F is a generalized conjugate system with A and B as above, then there is a P_0, independent of F, $2 - (1/B(1+A)) < p_0 < 1$ such that whenever $p \geqslant p_0$, $|F(x, k)|^p = \left[\sum_{l=0}^{m} |f_l(x, k)|^2\right]^{p/2}$ is subregular.*

The theorem is a direct corollary of the following elementary arithmetic lemma.

LEMMA 2. *Let $\{a_l^j\}$, $l = 0, \ldots, m$; $j = 1, \ldots, k$; and $a \in C^{m+1}$ be given. Suppose we are given a non-trivial partition of $\{0, 1, \ldots, m\} = D \cup E$ and constants A and B, as in the definition of generalized conjugate systems and that (3.1), (3.2), and (3.3) are satisfied. Suppose further that*

$$\sum_{j=0}^{k} a_l^j = 0, \quad l = 0, 1, \ldots, m. \tag{3.4}$$

Then there is a p_0, $2 - (1/B(1+A)) < p_0 < 1$ such that

$$|||a|||^p \leqslant (1/k) \sum_{j=1}^{k} |||a + a^j|||^p, \quad p \geqslant p_0. \tag{3.5}$$

Proof. We borrow from [1], 4.6 which is also [7], VII (2.5). We assume $|||a||| \neq 0$, $0 < p < 1$. Note that

$$\sum_j \Big|\sum_l \bar{a}_l a_l^j\Big|^2 \leqslant \sum_j |||a|||^2 |||a^j|||^2 = |||a||| \sum_j |||a^j|||^2 = |||a|||^2 \sum_l \|a_l\|^2.$$

Write

$$\sum_j |||a + a^j|||^p$$

$$= |||a|||^p \sum_j \left\{1 + \frac{2 \operatorname{Re} \sum_l \bar{a}_l a_l^j}{|||a|||^2} + \frac{|||a^j|||^2}{|||a|||^2}\right\}^{p/2} \cdot \left|2 \operatorname{Re} \sum_l \bar{a}_l a_l^j + |||a^j|||^2\right|$$

$$\leqslant 2\,|||a|||\left[\sum_l \|a_l\|^2\right]^{1/2} + \sum_l \|a_l\|^2.$$

Hence, if $\left[\sum_l \|a_l\|^2\right]^{1/2} \leqslant (1/3)|||a|||$, we may apply the binomial theorem and obtain:

$$\sum_j |||a + a^j|||^p = |||a|||^p \sum_j \left\{1 + p \frac{\operatorname{Re} \sum_l \bar{a}_l a_l^j}{|||a|||^2} + \frac{p}{2} \frac{|||a^j|||^2}{|||a|||^2} - \right.$$

$$\left. - \frac{p(2-p)}{8\,|||a|||^4}\left[4\left(\operatorname{Re} \sum_l \bar{a}_l a_l^j\right)^2 + 4\,|||a^j|||^2 \left(\operatorname{Re} \sum_l \bar{a}_l a_l^j\right) + |||a^j|||^4\right] + \mathcal{R}_{3j}\right\}.$$

Using (3.4) we see that

$$\sum_j p \frac{\sum_l \bar{a}_l a_l^j}{|||a|||^2} = 0.$$

From the definition of $\|\cdot\|$ and $|||\cdot|||$ we see that

$$\sum_j \frac{p}{2} \frac{|||a^j|||^2}{|||a|||^2} = \frac{p}{2} \frac{\sum_l \|a_l\|^2}{|||a|||^2}.$$

If we observe that

$$\sum_j \left(\operatorname{Re} \sum_l \bar{a}_l a_l^j\right)^2 \leqslant \sum_j \Big|\sum_{l \in D} \bar{a}_l a_l^j + \sum_{s \in E} a_s \bar{a}_s^j\Big|^2$$

$$= \sum_j \Big|\sum_{l \in D} \bar{a}_l a_l^j\Big|^2 + \sum_j \Big|\sum_{s \in E} a_s a_s^j\Big|^2 + 2 \operatorname{Re} \sum_{\substack{l \in D \\ s \in E}} \bar{a}_l \bar{a}_s \left[\sum_j a_l^j a_s^j\right]$$

(which, after some calculation, and a judicious use of (3.1) and (3.3), leads to)

$$\leqslant B(1+A)\,|||a|||^2 \sum_l \|a_l\|^2.$$

Consequently,

$$\frac{p(2-p)}{8|||a|||^2}\sum_j 4\Big(\operatorname{Re}\sum_l \bar{a}_l a_l^j\Big)^2 \leqslant (p/2)(2-p)B(1+A)\Big(\sum_l \|a_l\|^2\Big)/|||a|||^2.$$

An easy calculation shows that the remaining terms in the curly brackets are bounded by $C[(\sum\|a_l\|^2)^{1/2}/|||a|||]^3$ with $C>0$, independent of p. Thus,

$$(1/k)\sum_j |||a+a^j|||^p$$
$$\geqslant |||a|||^p\Big\{1+\frac{\sum_l\|a_l\|^2}{|||a|||^2}\Big(\frac{p}{2}(2-p)B(1+A)+\frac{2C}{p}\frac{(\sum\|a_l\|^2)^{1/2}}{|||a|||}\Big)\Big\}.$$

We may then obtain the following intermediate result: Fix p_1, $2-1/B(1+A)<p_1<1$, and require that

$$\Big(\sum_l\|a_l\|^2\Big)^{1/2}/|||a|||\leqslant\min\big[1/3,\ p_1(1-(2-p_1)B(1+A))/2C\big].$$

In this case we have that $|||a|||^p\leqslant(1/k)\sum_j|||a+a^j|||^p$, whenever $p\geqslant p_1$.

Since we assume that $|||a|||>0$ and it is trivial that $(1/k)\sum|||a+a^j|||\geqslant|||a|||$, we assume

(3.6) $\quad (1/k)\sum|||a+a^j|||=1;\qquad \Big(\sum_l\|a_l\|^2\Big)^{1/2}\geqslant D|||a|||>0,\qquad D>0$

where D depends on p_1, but is otherwise independent of p.

Note that if $\sum_l\|a_l\|^2=0$, then $a^j\equiv 0$ so that $|||a|||=(1/k)\sum|||a+a^j|||=1$. But then, $0=\sum\|a_l\|^2\geqslant D|||a|||>0$, a contradiction. Note also that if either $\sum_{l\in D}\|a_l\|^2=0$ or $\sum_{s\in E}\|a_s\|^2=0$ then (by (3.2)) both are equal to zero so we get the same contradiction.

Consider the set $B=\{b=\{a+a^j\}\}\subset C^{(m+1)k}$, consisting of vectors satisfying (3.1), (3.2), (3.3), (3.4) and (3.6). B is compact.

It will suffice to show

(3.7) $\quad |||a|||\leqslant\delta,\quad$ some δ, $0<\delta<1$ and all $\{a+a^j\}\in B$.

An easy argument shows that (3.7) implies that (3.5) holds for $\{a+a^j\}\in B$ with $p\geqslant p_2=(1-\log_k\delta)^{-1}$. Then (3.5) holds in general with $p\geqslant p_0=\max[p_1,p_2]$.

Since $|||a|||\leqslant 1$, for all $\{a+a^j\}\in B$ we see that if (3.7) does fail then $|||a|||=1=(1/k)\sum|||a+a^j|||$, and so there exists $\{\lambda^j\}$, real, such that $a+a^j=\lambda^j a$, and so $(1-\lambda^j)a=a^j$, $a_l^j=(1-\lambda^j)a_l$. From (3.3) we get

$$|a_l|\,|a_s|\sum(\lambda^j-1)^2=\Big|\sum a_l^j a_s^j\Big|\leqslant A\|a_l\|\|a_s\|=A|a_l|\,|a_s|\sum(\lambda^j-1)^2$$

for an A, $0\leqslant A<1$, whenever $l\in D$, $s\in E$. Thus, $a_l a_s\sum(\lambda^j-1)^2=0$ whenever $l\in D$, $s\in E$. There are three possibilities:

I. $\sum(\lambda^j-1)^2=0$. Then $\lambda^j\equiv 1$ and so $a_l^j\equiv 0$ which implies that $\sum\|a_l\|^2=0$, a contradiction.

II. $a_l\equiv 0$, $l\in D$. Then $a_l^j\equiv 0$, $l\in D$ which implies that $\sum_{l\in D}\|a_l\|^2=0$, a contradiction.

III. $a_s\equiv 0$, $s\in E$, and argue as in II arriving at a contradiction.

Thus, (3.7) does hold. This proves Lemma 2, and Theorem 1 is an immediate consequence.

The following results are obtained as before ([7], VII § 3).

THEOREM 3. *Suppose $F(x,k)=(f_0(x,k),\dots,f_m(x,k))$ is a generalized conjugate system on $K^n\times Z$ and $\int_{K^n}|F(x,k)|\,dx\leqslant A$, for all $k\in Z$, $A>0$ independent of k. Then $\lim_{k\to-\infty}F(x,k)=F(x)$ exists in L^1 and* a.e. $f_j^*(x)=\sup_{k\in Z}|f_j(x,k)|\in L^1$, $j=0,1,\dots,m$ *and so then does $f_j(x)=\lim_{k\to-\infty}f_j(x,k)$.*

COROLLARY 4. *Suppose $\mu_0,\mu_1,\dots,\mu_m$ are finite Borel measures on K^n. If $F(x,k)=(\mu_0(x,k),\dots,\mu_m(x,k))$ is a generalized conjugate system, then μ_j is absolutely continuous, $j=0,1,\dots,m$.*

COROLLARY 5 (F. and M. Riesz theorem). *Suppose μ is a finite Borel measure and $R_l\mu$ are also finite Borel measures, $l=1,2,\dots,n$. Then μ is absolutely continuous.*

§ 4. A generalized conjugate system induced by a field structure on K^n.

In [8] it is shown that there is a model K' for K^n that is, itself a local field and that harmonic analysis on K' is identical to harmonic analysis on K^n if the norms are adjusted for homogeneity ($|\cdot|_{K^n}^n=|\cdot|_{K'}$), in the same way that the complex numbers is a model for the Euclidean plane.

The theorem of Chao, referred to in the introduction, says that on any local field (with local class field of odd order) there is at least one singular integral operator of the Calderón–Zygmund type (denote it $f\to\tilde f$, and name it the *conjugate operator* and $\tilde f$ the *conjugate* of f) such that if $\tilde f$ is defined then $(f(x,k),\tilde f(x,k))$ is a conjugate system (which is a particular case of a generalized conjugate system). Furthermore, we may choose this operator so $(\tilde f)^\sim=f$. Consequently,

THEOREM 5. *If μ and $\tilde\mu$ are both finite Borel measures then μ and $\tilde\mu$ are absolutely continuous.*

Thus, we may construct generalized conjugate system with n operators $\{R_l\}$ that behave like the Riesz operators, or may use the one conjgate operator. There is a simple extension of these ideas that shows that we never need more than q^n-2 independent operators.

All the operators we are considering are determined by kernels $\Omega_l(x)/|x|^n$ where Ω_l is homogeneous of degree zero so it is determined by its values on R^*. On R^* it is constant on cosets of P so it is determined by the q^n-1 numbers $\{\Omega_l(\varepsilon^j)\}$. Finally,

$$\int_{R^*} \Omega_l = \sum_{j=1}^{q^n-1} \Omega_l(\varepsilon^j) = 0$$

so the collection of such functions (and, so also, the collection of such operators) is a vector space over C of dimension (q^n-2).

COROLLARY 6. *Let $\{T_l\}$ be a system of q^n-2 independent Calderón–Zygmund operators on K^n of the sort described in the paragraph above. If μ and $T_l\mu$ are finite Borel measures then μ is absolutely continuous.*

Using a similar argument shows that we restrict ourselves to operators with odd kernels we only need $(q^n-1)/2$ such odd independent operators.

§ 5. Equivalence of characterizations of the Hardy space, H^1, for K^n. We gather, in this section, a collection of various definitions and norms that have been suggested for the Hardy space $H^1(K^n)$ and show that they all give the same space with equivalent norms.

In the following definitions, where we assume that $f\in L^1$, we could just as well assume that f was a finite Borel measure and obtain the integrability of f as a conclusion.

A. *Maximal function.* Let $f\in L^1$ and $f(x,k)$ be the regularization of f. Let $f^*(x) = \sup_{k\in Z} |f(x,k)|$. If $f^*\in L^1$ we say that $f\in H^1_{\mathrm{A}}$ and set $\|f\|_{\mathrm{A}} = \|f^*\|_1$.

B. *Lusin-function.* Let $f\in L^1$ and $f(x,k)$ be the regularization of f. Let $Sf(x) = \big[\sum_{k\in Z} |f(x,k)-f(x,k+1)|^2\big]^{1/2}$. (On a local field this is the natural analogue of the Lusin Area Integral as well as the Littlewood–Paley Operator.) If $Sf\in L^1$ we say that $f\in H^1_{\mathrm{B}}$ and set $\|f\|_{\mathrm{B}} = \|f\|_1+\|Sf\|_1$.

C. *Conjugate system.* We say that $\{T_l\}_{l=0}^m$ is a *generalized conjugate system* of operators if $T_0 = \mathrm{Id}$ and if, whenever T_lf is defined for all l, $F(x,k) = (f(x,k), T_1f(x,k), \ldots, T_mf(x,k))$ is a generalized conjugate system, where $T_lf(x,k)$ is the regularization of T_lf (the T_l as in the paragraph preceding Corollary 6).

If for some generalized conjugate system of operators $\{T_l\}$, $T_lf\in L^1$ for all l we say that $f\in H^1_{\mathrm{C}}$ and set $\|f\|_{\mathrm{C}_1} = \sum_{l=0}^m \|T_lf\|_1$. Alternately we may require only that $\sup_k \int_{K^n} |F(x,k)|\,dx = \|f\|_{\mathrm{C}_2} < \infty$. $\|\cdot\|_{\mathrm{C}_1}$ and $\|\cdot\|_{\mathrm{C}_2}$ are equivalent norms. Furthermore, the norms derived from any pair of generalized conjugate systems are equivalent.

D. *Atoms.* A coset of some P^k in K^n is called a *sphere*. An *atom*, a, is a function that is supported on a sphere I, $|a|\leqslant |I|^{-1}$, and $\int_I a = 0$. If $f(x) = \sum c_ia_i(x)$, a_i an atom for all i and $\sum|c_i| < \infty$, then we say that $f\in H^1_{\mathrm{D}}$ and set

$$\|f\|_{\mathrm{D}} = \inf\Big\{\sum|c_i|:\ f = \sum c_ia_i,\ a_i \text{ an atom for all } i\Big\}.$$

E. *Molecules.* A function M is a *molecule* about a point x_0 if

$$\int_{K^n} M = 0 \quad\text{and}\quad \int_{K^n} |M(x)|^2dx \int_{K^n} |M(x)|^2|x-x_0|^{2n}dx \leqslant 1.$$

If $f(x) = \sum d_iM_i(x)$, M_i a molecule for all i and $\sum|d_i| < \infty$, then we say that $f\in H^1_{\mathrm{E}}$ and a set

$$\|f\|_{\mathrm{E}} = \inf\Big\{\sum|d_i|:\ f = \sum d_iM_i,\ M_i \text{ a molecule for all } i\Big\}.$$

THEOREM 7. *The space H^1_{A} to H^1_{E} are all the same space and the norms $\|\cdot\|_{\mathrm{A}}$ to $\|\cdot\|_{\mathrm{E}}$ are equivalent.*

Before proceeding we should note that, contrary to appearances, the list above is really quite narrow. For example, there are "non-tangential" versions of the maximal function and Lusin function which give equivalent norms. (See [2], [3].) For another, the atoms described above are $(1,\infty)$ atoms. There are $(1,q)$, $1<q\leqslant\infty$, atoms that give the same space. The molecules described above are 1-molecules. There are ε-molecules, $\varepsilon>0$ that give the same space. (See [5] for details.) There are endless variants to consider including Lipschitz space characterizations of the Fourier transforms of molecules.

We also pause here to show the power of these results by showing how the conjugate characterization (H^1_C) gives a result which superficially has nothing to do with conjugate functions.

COROLLARY 8. *Let J be a multiplier transform on K^n; i.e., $(Jf)\hat{} = m\hat{f}$, m a bounded function. Then J maps H^1 boundedly into H^1 iff J maps H^1 into L^1 and the two operator norms are equivalent.*

Proof. We use the conjugate operator described in § 4, $f\to\tilde{f}$ and consider the H^1 norm $\|f\|_{H_1} = \|f\|_1+\|\tilde{f}\|_1$. If $\|Jf\|_{H^1}\leqslant M_1\|f\|_{H_1}$ we see that $\|Jf\|_1\leqslant \|Jf\|_{H_1}\leqslant M_1\|f\|_{H^1}$, so the result in one direction is trivial. It is easy to see that J commutes with the conjugate operator when operating on distributions with Fourier transforms that are functions so if $\|Jf\|_1 \leqslant M_2\|f\|_{H^1}$ for all $f \in H^1$ we see that

$$\|Jf\|_{H^1} = \|Jf\|_1 + \|(Jf)^{\sim}\|_1 = \|Jf\|_1 + \|J\tilde{f}\|_1 \leqslant M_2\|f\|_{H^1} + M_2\|\tilde{f}\|_{H^1}.$$

But $(\tilde{f})^{\sim} = f$, so $\|\tilde{f}\|_{H^1} = \|f\|_{H^1}$ and hence, $\|Jf\|_{H^1} \leqslant 2M_2\|f\|_{H^1}$. This completes the proof of the corollary.

Now to sketch a proof of Theorem 7.

The equivalence of D and E is found in Coifman and Weiss [5] since K^n is a space of homogeneous type.

The equivalence of D and A is due to Herz [6] since it is easy to see that $\{f(x, k)\}$, the regularization of f, is a regular (and so regulated) martingale. (Definitions are found in Herz's paper.)

The equivalence of A and B is found in Chao [3].

Next we show that for a given generalized conjugate system, the two norms described in C give the same space and are equivalent. We note from [7], IV (1.8)–(1.9) that $g \in L^1$ then $\sup_k \|g(\cdot, k)\|_1 = \|g\|_1$. Let $F(x) = (f(x), T_1 f(x), \ldots, T_m f(x))$. If $T_l f \in L^1$, $l = 0, 1, \ldots, m$, we see that $\|F\|_1 = \|f\|_{C_2}$ and $(1/(m+1))\|f\|_{C_1} \leqslant \|F\|_1 \leqslant \|f\|_{C_1}$. In the one case we are given that the $T_l f$ are in L^1. In the other case it follows from Theorem 3.

We now finish by showing that A $\Rightarrow$ C for each generalized conjugate system and if C holds for any particular generalized conjugate system then that implies A.

A $\Rightarrow$ C. Chao [2] has shown that for operators like the T_l, we have that $\|(T_l f)^*\|_1 \leqslant B_l\|f^*\|_1$. (Actually, the proof is for a subclass of operators, but it extends trivially, to the class we are considering here.) Thus, if $f^* \in L^1$ each $T_l f \in L^1$ and we see that

$$\|f\|_{C_1} = \sum_{l=0}^{m} \|T_l f\|_1 \leqslant \|f^*\|_1 + \Big(\sum_{l=1}^{m} B_l\Big)\|f^*\|_1 = \Big(1 + \sum_{l=1}^{m} B_l\Big)\|f\|_{A}.$$

C $\Rightarrow$ A. From Theorem 3 we see that $f^* = (T_0 f)^* \in L^1$, so we are done if we show that there is a constant $M > 0$ such that $\|f^*\|_1 \leqslant M\|f\|_{C_1} = M\sup_k \int |F(x, k)|\,dx$.

An examination of the proof of Theorem 3 shows that then additional information is available. In the proof of Theorem 3, a least regular majorant, m is constructed. This function is the least regular majorant of $|F(x, k)|^p$, for a p, $0 < p < 1$, such that $|F(x, k)|^p$ is subregular. $m \in L^{1/p}$ and $\|m\|_{1/p} = \|f\|_{C_1}^p$ by [7], IV (3.7). By [7], IV (1.7) it follows that $m^* \in L^{1/p}$ and $\|m^*\|_{1/p} \leqslant M_p\|f\|_B^p$. But an easy calculation shows that

$$\|f^*\|_1^p \leqslant \|F^*\|_1^p = \||F^*|^p\|_{1/p} \leqslant \|m^*\|_{1/p} \leqslant M_p\|f\|_{C_1}^p,$$

so $\|f^*\|_1 \leqslant (M_p)^{1/p}\|f\|_{C_1}$ and the proof is complete.

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