Take now $f = \varphi_r$. Then (3) reduces to

$$\|\varphi_r * f\|_{L_p} \leqslant Cr^{1/p} r^{-1} \|\varphi\|_{L_\infty} = Cr^{1/p-1} \|\varphi\|_{L_\infty}$$

which precisely says that $f \in \mathscr{E}_p^{1/p-1,\infty;1}$. ■

Incidentally from Theorem III we also get a very simple proof of the multiplier theorem of Fefferman–Stein [4].

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Representation of random linear functionals on certain $S\{M_p\}$ spaces

by

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Abstract. We prove an integral representation theorem for random linear functionals on certain $S\{M_p\}$ spaces. With stronger conditions on $\{M_p\}$, the representation can be rewritten in terms of the Lebesgue integral over $\boldsymbol{R}^d$. In the process of proving the main theorems, we obtain a probabilistic Riesz–Radon representation theorem, which is of interest in it's own right.

I. Introduction. The purpose of this paper is to prove a representation theorem for generalized random processes on a rather large class of spaces, namely $S\{M_p\}$ spaces. Such spaces are introduced by Yamanaka in a note [6] in order to make possible a unified treatment of the $K\{M_p\}$ and S-type spaces of Gel'fand–Shilov [2]. For $K(a)$ (i.e. the space of $C^\infty(\boldsymbol{R})$-functions with supports contained in $[-a, a]$) such a representation theorem is obtained by Ullrich [5], and for certain $K\{M_p\}$ spaces, by Swartz and Myers [4]. Since the conditions we impose on $S\{M_p\}$ spaces are all satisfied by those $K\{M_p\}$ spaces considered by Swartz and Myers, our representation theorem is hence more general.

As far as method goes, Ullrich, Swartz and Myers all use a scheme of representing continuous linear functionals on σ-normed spaces parallel to the scheme employed by Gel'fand–Shilov [1], [2]. In this paper, we are working in the same spirit.

The organization of the paper is as follows: In II, we give the necessary preliminary definitions, then prove $S\{M_p\}$ is complete and σ-normed. In III, we lay down the basic assumptions on $\{M_p\}$, with the derivation of two straightforward consequences. And in IV, we prove a lemma which gives a "large" measurable set $B \in \mathscr{B}$, where $(\Omega, \mathscr{B}, \mu)$ is a fixed probability space, on which our random process is bounded. Then in V, we establish a probabilistic version of the classical Riesz–Radon theorem for continuous functions vanishing at ∞. This theorem, which is of independent interest, is an important ingredient in the proof of Theorem 6.1. An extension theorem, of a probabilistic Hahn–Banach type, is also stated in V. This theorem is proved in Hanš [3], and is used here on many occasions that follow. In VI, the main results of this paper are stated and proved. Under suitable decay conditions on $\{M_p(x,q)\}$ for large x, q, a random linear functional on the test function space $S\{M_p\}$ can be written as a sum $\sum_q$ of "random" integrals; such a representation is true for an arbitrary large measurable subset B of Ω. An example showing that $\mu(B)$ cannot be 1 in the case of $K(a)$ spaces is given by Ullrich [5]. With further assumptions on $\{M_p\}$, the original family of defining norms $\{\|\cdot\|_p\}$ can be replaced by a family $\{\|\cdot\|_{p'}\}$ which enables us to write the representation theorem in terms of Lebesgue measure with a random density rather than in terms of the "random" measure. This we do in VI. 3. The importance of this theorem lies in that by specializing the $\{M_p\}$, one obtains the known results of Ullrich [5], and of Swartz and Myers [4]. In VII, the Appendix, we supply a proof for the separability of the space $(\Gamma, \|\cdot\|)$ used in VI.1. Finally, we say a word on the use of μ in $(\Omega, \mathscr{B}, \mu)$. We use $(\Omega, \mathscr{B}, \mu)$ rather than $(\Omega, \mathscr{B})$ in Lemma 4.1 and hence Theorem 6.1 and Theorem 6.5. Elsewhere, however; the measurable space $(\Omega, \mathscr{B})$ is sufficient.

II. Preliminaries and basic theorems.

II. 1. A *probability space* is a triple $(\Omega, \mathscr{B}, \mu)$ consisting of a nonempty set Ω, a σ-algebra $\mathscr{B}$ of subsets of Ω, and a measure μ on $\mathscr{B}$ with $\mu(\Omega) = 1$. We fix $(\Omega, \mathscr{B}, \mu)$. We employ the following definition of generalized stochastic process, which we refer to as a *random linear functional*. Let X be a topological linear space and X' be its topological dual. A *random linear functional* (r.l.f.) is a map $\psi\colon \Omega\times X\to \boldsymbol{R}$ such that

(i) $\psi(\cdot, x)$ is Ω-measurable $\forall x\in X$, and

(ii) $\psi(\omega, \cdot)\in X'$.

Here $\boldsymbol{R}$ is the set of real numbers; it is equipped with the usual topology and the induced Borel structure.

II.2. Let d be a positive integer; let $\boldsymbol{R}^d$ be the d-dimensional Euclidean space with the usual inner product structure; and let Z^d_+ be the collection of d-tuples of nonnegative integers $q = (q_1, \ldots, q_d)$. D^q denotes the differentiation operator defined by the relation

$$D^q := \frac{\partial^{|q|}}{\partial x_1^{q_1}\ldots \partial x_d^{q_d}},$$

where $|q| := q_1+\ldots+q_d$.

II.3. Let $\overline{\boldsymbol{R}}$ denote the extended real numbers $\overline{\boldsymbol{R}} := \{-\infty\}\cup\boldsymbol{R}\cup\{\infty\}$. We shall use the following arithmetic in $\overline{\boldsymbol{R}}$: for $a\in\boldsymbol{R}$,

$$a+\infty = \infty+a = \infty,\quad a-\infty = -\infty+a = -\infty,$$

$$\infty+\infty = \infty,\quad -\infty-\infty = -\infty,\quad \infty-\infty \text{ is undefined},$$

$$b\cdot\infty = \infty\cdot b = \begin{cases}\infty & \text{if}\quad b\in\overline{\boldsymbol{R}},\ b>0,\\ -\infty & \text{if}\quad b\in\overline{\boldsymbol{R}},\ b<0,\end{cases}$$

$$\frac{a}{\infty} = \frac{a}{-\infty} = 0,\quad 0\cdot\infty = \infty\cdot 0 = 0,\quad \frac{\infty}{\infty} = 0.$$

Note that $\overline{\boldsymbol{R}}$ is not a field under those operations.

II.4. Let X be a topological space.

$C(X) :=$ the collection of continuous functions on X into $\boldsymbol{R}$,

$C_b(X) := \{f\in C(X)\colon f \text{ is bounded}\}$,

$C_0(X) := \{f\in C(X)\colon f \text{ has compact support}\}$.

II.5. A *Radon measure space* is a quadruple (X, τ, Σ, ν) where (X, Σ, ν) is a measure space and τ is a T_2 topology on X such that:

(i) $\tau\subseteq\Sigma$,

(ii) (X, Σ, ν) is complete,

(iii) ν is locally finite, i.e.,

$$\forall x\in X\,\exists G\in\tau\cdot\ni\cdot x\in G, \text{ and } \nu(G)<\infty,$$

(iv) ν is inner regular, i.e.,

$$\forall E\in\Sigma,\ \nu(E) = \sup_{\substack{K\subseteq E\\ K \text{ compact}}} \nu(K).$$

Such a ν is called a *Radon measure on* Σ. A *random Radon measure* is a map $\lambda\colon \Omega\times\Sigma\to\boldsymbol{R}$ such that $\lambda(\cdot, E)$ is Ω-measurable $\forall E\in\Sigma$, and $\lambda(\omega,\cdot)$ is a Radon measure on Σ $\forall\omega\in\Omega$. We shall denote such a measure by $\lambda(\omega, dx)$. $\lambda(\omega, dx)$ is *finite* if $\lambda(\omega, X)<\infty$, $\forall\omega\in\Omega$.

II.6. A topological linear space is *perfect* if bounded sets are precompact i.e., the closures are compact.

II.7. Let $\{M_p(x,q)\}_{p=1}^{\infty}$ $(q\in Z^d_+, x\in\boldsymbol{R}^d)$ be a sequence of $\overline{\boldsymbol{R}}$-valued functions on $\boldsymbol{R}^d\times Z^d_+$ such that

(i) $0\leqslant M_1(x,q)\leqslant M_2(x,q)\leqslant\ldots\leqslant M_p(x,q)\leqslant\ldots$, $\forall x\in\boldsymbol{R}^d$, $q\in Z^d_+$;

(ii) For each $(x,q)\in\boldsymbol{R}^d\times Z^d_+$, all $M_p(x,q)$ are either finite or infinite simultaneously $\forall_p$. $M_p(x,q)$ is continuous in x where it is finite;

(iii) For each p $\exists N_p\in\overline{\boldsymbol{R}}/\{0\}\cdot\ni\cdot N_p\to\infty$ as $p\to\infty$, $\inf_x M_p(x,q)>0$ whenever $|q|<N_p$, and $M_p(x,q)\equiv 0$ for $|q|\geqslant N_p$.

II.8.

Definition. $\varphi\in S\{M_p\}$ iff $\varphi\in C^\infty(\boldsymbol{R}^d)$ and $M_p(x,q)D^q\varphi(x)$ is everywhere defined, continuous in x and bounded in x and q, i.e.

$$\|\varphi\|_p := \sup_{x,q} M_p(x,q)\,|D^q(x)| < \infty, \qquad p = 1, 2, \ldots$$

Among examples of $S\{M_p\}$ spaces are the $K\{M_p\}$ spaces and S-type spaces of Gel'fand–Shilov ([2], Chapter 2 and Chapter 4).

We now study some basic topological properties of $S\{M_p\}$ spaces.

DEFINITION. Let p be fixed and denote by S_p the space $S\{M_p\}$ equipped with the norm $\|\cdot\|_p$.

Each S_p is a normed linear space. We will show it is indeed complete and therefore Banach. We need the following definition and lemmas.

DEFINITION. Let $\{\varphi_\nu\}$ be a sequence in $C^\infty(\boldsymbol{R}^d)$ and let $\varphi \in C^\infty(\boldsymbol{R}^d)$. $\{\varphi_\nu\}$ is said to *converge correctly to* φ iff for each q, $D^q\varphi_\nu(x)$ converges to $D^q\varphi(x)$ uniformly on compact subsets of $\boldsymbol{R}^d$.

LEMMA 2.1. *If $\{\varphi_\nu\}$ is a Cauchy sequence in S_p, then $\exists\,\varphi \in C^\infty(\boldsymbol{R}^d)$ such that $\{\varphi_\nu\}$ converges correctly to φ.*

Proof. It is obvious. ∎

LEMMA 2.2. *Let $\{\varphi_\nu\}$ be a sequence in S_p which converges correctly to $\varphi \in C^\infty(\boldsymbol{R}^d)$ with $\|\varphi_\nu\|_p \leqslant C$, $\forall \nu$ and for some positive constant C. Then $\|\varphi\|_p \leqslant C$.*

Proof. Let N_p be determined by p (see condition (iii) of II.7).

Case 1. $|q| \geqslant N_p$. Then $M_p(x, p) \equiv 0$ $\forall x$ and so

$$M_p(x,q)\,|D^q(x)| = 0 \qquad \forall x.$$

Case 2. $|q| < N_p$. Then $M_p(x,q) > 0$ $\forall x$ and so

$$M_p(x,q)\,|D^q\varphi_\nu(x)| \leqslant C \Rightarrow |D^q\varphi_\nu(x)| \leqslant \frac{C}{M_p(x,q)} \qquad \forall \nu,\ \forall x.$$

Therefore, by passing to limit, we obtain

$$|D^q\varphi(x)| \leqslant \frac{C}{M_p(x,q)}.$$

That is,

$$M_p(x,q)\,|D^q\varphi(x)| \leqslant C \qquad \forall x.$$

Thus $\|\varphi\|_p \leqslant C$ as claimed, and in particular, $\varphi \in S_p$. ∎

THEOREM 2.3. *S_p is a Banach space.*

Proof. Let $\{\varphi_\nu\}$ be a Cauchy sequence in S_p. Then $\{\varphi_\nu\}$ converges correctly to some φ. Thus $\{\varphi_\nu\}$ converges to φ in $\|\cdot\|_p$. Also, from the Cauchy condition, we have, $\|\varphi_\nu\|_p \leqslant C$ for some positive constant C. Thus, by the lemmas above, we get $\varphi \in S_p$ and $\|\varphi_\nu - \varphi\|_p \to 0$ as $\nu \to \infty$. Therefore S_p is complete. ∎

We now turn back to $(S\{M_p\}, \|\cdot\|_p;\ p = 1, 2, \ldots)$ and show the norms $\|\cdot\|_p$, $p = 1, 2, \ldots$ are pairwise consistent.

LEMMA 2.4. *Let $\{\varphi_\nu\}$ be a sequence which is Cauchy in $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$, and converges to zero in $\|\cdot\|_{p_1}$. Then $\{\varphi_\nu\}$ converges to zero in $\|\cdot\|_{p_2}$.*

Proof. Since $\{\varphi_\nu\}$ is Cauchy in $\|\cdot\|_{p_1}$ and converges to zero in $\|\cdot\|_{p_1}$, it converges to zero pointwise. But pointwise convergence to zero together with Cauchyness in $\|\cdot\|_{p_2}$ imply $\{\varphi_\nu\}$ converges to zero in $\|\cdot\|_{p_2}$. ∎

THEOREM 2.5. *$(S\{M_p\}, \|\cdot\|_p;\ p = 1, 2, \ldots)$ is a complete σ-normed space.*

Proof. Since $S\{M_p\} = \bigcap_{p=1}^{\infty} S_p$ and each S_p is complete (Theorem 2.3), $S\{M_p\}$ is therefore complete ([2], p. 17, § 3.2 Theorem); also the norms $\|\cdot\|_p$, $p = 1, 2, \ldots$ are pairwise consistent (Lemma 2.4). Hence $S\{M_p\}$ is a complete σ-normed space. ∎

III. Assumptions on $\{M_p(x,q)\}$.

III.1. The following conditions on $\{M_p\}$ (not always taken together) will be used in the sequel.

(P) $\quad \forall p \exists p' > p \ni \forall \varepsilon \in]0,1], \exists N > 0 \ni M_p(x,q) \leqslant \varepsilon M_{p'}(x,q)$ whenever $|x| > N$ or $M_p(x,q) > N$.

(S) $\quad \forall p \exists p' > p \ni \forall \varepsilon \in]0,1], \exists q' \geqslant 0 \ni M_p(x,q) \leqslant \varepsilon M_{p'}(x,q)$ $\forall x, \forall q \geqslant q'$.

(N) $\quad \forall p \exists p' > p \ni m_{pp'}(x,q) := \dfrac{M_p(x,q)}{M_{p'}(x,q)} \in L^1(\boldsymbol{R}^d)$ and

$$\sup_q \int_{\boldsymbol{R}^d} m_{pp'}(x,q)\,dx < \infty.$$

(M) $\quad |x_j'| \geqslant |x_j''|$ and $x_j' x_j'' \geqslant 0$ imply $M_p(x_1, \ldots, x_j', \ldots, x_d, q) \geqslant M_p(x_1, \ldots, x_j'', \ldots, x_d, q)$.

Remark. (P), (S), (N), (M) stand for perfect, supplementary, nuclear, and monotonic, respectively.

II.2. We now derive some easy consequences of the above conditions concerning the behavior of $M_p(x,q)\,|D^q(x)|$ for large $|x|$.

LEMMA 3.1. *Condition* (P) *implies that*

$$M_p(x,q)\,|D^q\varphi(x)| \to 0 \quad \textit{as} \quad |x| \to \infty, \qquad \forall p, \forall q, \forall \varphi \in S\{M_p\}.$$

Proof. Suppose the contrary, i.e., $\exists p, q$ and $|x_\nu| \to \infty$ such that

$$M_p(x,q)\,|D^q\varphi(x_\nu)| \geqslant C > 0.$$

Corresponding to that p, $\exists p' > p$ by (P) such that $M_p(x_\nu, q) \leqslant \varepsilon_\nu M_{p'}(x,q)$ where $\varepsilon_\nu \to 0$. Thus,

$$M_{p'}(x,q)\,|D^q\varphi(x_\nu, q)| \geqslant \frac{C}{\varepsilon_\nu} \to \infty$$

which is a contradiction. Therefore,

$$M_p(x, q)\,|D^q\varphi(x)| \to 0 \quad \text{as} \quad |x| \to \infty. \blacksquare$$

Similarly, the behavior of $M_p(x, q)\,|D^q\varphi(x)|$ for large $|q|$, assuming condition (S) is given by

LEMMA 3.2. *Condition* (S) $\Rightarrow \forall p, \forall \varphi \in S\{M_p\}$

$$\limsup_{q\to\infty} \sup_x M_p(x, q)\,|D^q\varphi(x)| = 0.$$

Proof. Condition (iii) of II.7 trivially implies the lemma if $N_p < \infty$. Otherwise, fix p and $\varepsilon > 0$. We can find a $p' > p$ and a q' by condition (S) such that whenever $q \geq q'$ we have for every x,

$$M_p(x, q)\,|D^q\varphi(x)| \leq \varepsilon M_{p'}(x, q)\,|D^q\varphi(x)|.$$

Hence,

$$\sup_x M_p(x, q)\,|D^q\varphi(x)| \leq \varepsilon(\sup_x M_p(x, q)\,|D^q\varphi(x)|) \leq \varepsilon\,\|\varphi\|_p.$$

Thus, the lemma is proved. ■

IV. Properties of random linear functionals. We now prove an analogue of ([4], p. 663, Lemma 4).

LEMMA 4.1. *Let* $(X, \|\cdot\|_p;\ p = 1, 2, \ldots)$ *be a perfect, complete σ-normed linear space; and* Ψ *be a random linear functional on* X. *Let* $\varepsilon > 0$, *then*

(i) $\exists B \in \mathscr{B} \cdot \ni \cdot \mu(B) \geq 1-\varepsilon$,

(ii) $\exists$ *an integer* $r > 0$ *such that*

$$|\langle\varphi, \Psi(\omega)\rangle| \leq r\,\|\varphi\|_r, \qquad \forall \varphi \in X,\ \omega \in B.$$

Proof. Since $(X, \|\cdot\|_p;\ p = 1, 2, \ldots)$ is σ-normed, to each continuous linear functional Φ on X there exists an r such that

$$|\Phi(\varphi)| \leq \text{const}\,\|\varphi\|_r, \qquad \varphi \in X.$$

Since $\Psi(\omega) \in X'$, $\forall \omega \in \Omega$, $\exists\, \text{const}_\omega \geq 0$, $r_\omega > 0$ such that $|\langle\varphi, \Psi(\omega)\rangle| \leq \text{const}_\omega\,\|\varphi\|_{r_\omega}$, $\forall \varphi \in X$.

We may and will assume $r_\omega \geq \text{const}_\omega$, since $\{\|\cdot\|_p\}_{p=1}^{\infty}$ is increasingly directed. Therefore

$$|\langle\varphi, \Psi(\omega)\rangle| \leq r_\omega\,\|\varphi\|_{r_\omega}.$$

Define

$$A_N(\varphi) := \{\omega \in \Omega:\ |\langle\varphi, \Psi(\omega)\rangle| \leq N\,\|\varphi\|_N\}.$$

Obviously,

$$A_N(\varphi) \subseteq A_{N+1}(\varphi)$$

and

$$\Omega = \bigcup_{N=1}^{\infty} \bigcap_{\varphi\in X} A_N(\varphi).$$

Now, since X is perfect Fréchet, it is separable ([2], p. 58). There exists therefore a countable dense subset $D \subset X$ and thus

$$A_N := \bigcap_{\varphi\in X} A_N(\varphi) = \bigcap_{\varphi\in D} A_N(\varphi) \in \mathscr{B}.$$

Since $\Omega = \bigcup_{N=1}^{\infty} A_N$, for every $\varepsilon > 0$ $\exists$ integer $r > 0$ (independent of ω) such that

$$\mu(A_r) \geq 1-\varepsilon.$$

Now, take $B := A_r$. Since (ii) is satisfied, by construction, the lemma is thus proved. ■

COROLLARY. *Suppose* $\{M_p(x, q)\}$ *satisfies* (P) *and* Ψ *is a* r.l.f. *on* $(S\{M_p\}, \|\cdot\|_p;\ p = 1, 2, \ldots)$. *Let* $\varepsilon > 0$, *then*:

(1) $\exists B \in \mathscr{B} \cdot \ni \cdot \mu(B) \geq 1-\varepsilon$,

(2) $\exists$ *integer* $r > 0 \cdot \ni \cdot |\langle\varphi, \Psi(\omega)\rangle| \leq r\,\|\varphi\|_r$, $\forall \varphi \in S\{M_p\}$, $\omega \in B$.

Proof. Condition (P) on $(S\{M_p\}, \|\cdot\|_p;\ p = 1, 2, \ldots)$ implies perfectness ([6], p. 2; [2], p. 94), therefore the corollary follows. ■

Remark. This lemma shows that the size of $\langle\varphi, \Psi(\omega)\rangle$ is independent of ω if $\omega \in B$.

V. A probabilistic Riesz–Radon representation theorem. We prove a probabilistic Riesz-Radon representation theorem for continuous functions vanishing at ∞ (see definition below) on a locally compact, σ-compact metric space X (e.g. $\boldsymbol{R}^d$). We begin by quoting the well-known Riesz–Radon representation theorem for compact metric spaces.

LEMMA 5.1. *Let* X *be a compact metric space and* $(C(X), \|\cdot\|_\infty)$ *be the space of real continuous functions on* X, *equipped with the uniform norm* $\|\cdot\|_\infty := \sup_{x\in X} |\cdot(x)|$. *Then, for every positive linear functional* Φ *on* $C(X)$ *there exists a unique finite positive Radon measure* ν *on* $\mathscr{B}(X)$ *such that* Φ *is the integral with respect to* ν, *i.e.*,

$$\Phi f = \int_X f\,d\nu, \qquad f \in C(X).$$

LEMMA 5.2. *Let* Ψ *be a random linear functional on* $(C(X), \|\cdot\|_\infty)$ *where* X *is compact metrizable. Then* $\exists!$ *random finite Radon measure* $\nu(\omega, dx)$ *such that*

$$\Psi f = \int_X f(x)\,\nu(\omega, dx), \qquad \forall \omega \in \Omega,\ f \in C(X).$$

Proof. (A) *Reduction step.* We show that it is sufficient to prove the lemma for the case of positive random linear functionals on $C(X)$. For, assume the lemma is true for such functionals Ψ. Then, for each

$\omega \in \Omega$, $\Psi(\omega, \cdot)$ has a canonical decomposition:

$$\Psi(\omega, \cdot) = \Psi^+(\omega, \cdot) - \Psi^-(\omega, \cdot)$$

where $\Psi^{\pm}(\omega, \cdot)$ are positive linear functionals on $C(X)$ for each $\omega \in \Omega$. Recall that

$$\Psi^+(\omega, f) := \sup_{0 \leqslant g \leqslant f} \Psi(\omega, g); \quad f, g \in C(X).$$

Using separability of $C(X)$, we conclude that $\Psi^+(\cdot, f)$ is Ω-measurable, and hence $\Psi^-(\cdot, f)$ is Ω-measurable also. They also have the smoothness property, viz., $f_n \downarrow 0 \Rightarrow \Psi^{\pm}(\omega, f_n) \to 0$. $\forall \omega \in \Omega$ (Dini's lemma).

Now, by our assumption and Lemma 5.1, we obtain positive finite random Radon measure

$$\nu^{\pm}(\omega, E) := \Psi^{\pm}(\omega, 1_E), \quad E \in \mathscr{B}(X),$$

such that $\Psi^{\pm}$ is the integral of $\nu^{\pm}$.

We define the signed random Radon measure $\nu(\omega, \cdot)$:

$$\nu(\omega, \cdot) = \nu^+(\omega, \cdot) - \nu^-(\omega, \cdot).$$

Then the representation continues to hold for this ν.

(B) So assume $\forall \omega \in \Omega$, $\Psi(\omega, \cdot)$ is a positive linear functional on $C(X)$. Therefore by Lemma 5.1, $\exists!$ finite positive Radon measure $\nu(\omega, \cdot)$ on $\mathscr{B}(X)$ such that, for that particular $\omega \in \Omega$,

$$\Psi(\omega, f) = \int_X f(x)\nu(\omega, dx), \quad \forall f \in C(X).$$

We must show $\nu(\cdot, dx)$ is Ω-measurable. Now, $1_X \in C(X)$,

$$\nu(\omega, X) = \int_X 1_X \nu(\omega, dx) = \Psi(\omega, 1_X)$$

and hence $\nu(\cdot, X)$ is Ω-measurable. Let $G \subset X$ be open. We claim that

$$\nu(\omega, G) = \sup_{\substack{f \in C_0(X) \\ f \leqslant 1_G}} \{\Psi(\omega, f)\}.$$

Proof of claim.

$$\nu(\omega, G) = \sup_{\substack{F \subseteq G \\ F \text{ compact}}} \{\nu(\omega, F)\}, \quad \forall \omega \in \Omega$$

by inner regularity of the Radon measure ν. If $F \subset G$ is compact, $\exists f \in C_0(X)$ such that $1_F \leqslant f \leqslant 1_G$. Therefore,

$$\nu(\omega, F) \leqslant \int f \nu(\omega, dx) \leqslant \nu(\omega, G)$$

by positivity of ν. Hence

$$\nu(\omega, G) \leqslant \sup_{\substack{F \subset G \\ F \text{ compact}}} \{\nu(\omega, F)\} \leqslant \sup_{\substack{f \in C_0(X) \\ f \leqslant 1_G}} \int f \nu(\omega, dx) \leqslant \nu(\omega, G).$$

Therefore,

$$\nu(\omega, G) = \sup_{\substack{f \in C_0(X) \\ f \leqslant 1_G}} \int f \nu(\omega, dx)$$

as claimed. But X is compact, hence $(C_0(X), \|\cdot\|_\infty)$ is a Banach space. Since X is metrizable, $C_0(X)$ is separable. Let $D \subset C_0(X)$ be countable and dense (in $\|\cdot\|_\infty$) in $C_0(X)$. We have

$$\nu(\omega, G) = \sup_{\{f \in C_0(X), f \leqslant 1_G\} \cap D} \{\Psi(\omega, f)\}.$$

Therefore, $\nu(\cdot, G)$ is Ω-measurable, $\forall G$ open in X, as it is a countable supremum of Ω-measurable functions $\Psi(\cdot, f)$. Hence $\nu(\cdot, E)$ is Ω-measurable for every Borel set E.

Remark. Lemma 5.1 is proved by Ullrich ([5], p. 662, Lemma 3), for the case $X := [0, 1]$, $\mu(\cdot, E) := \int_{[0,1]} 1_E \, dg(\cdot, x)$, where $g \in BV[0, 1]$.

Definition. Let $f \in C(X)$, where X is locally compact. f is said to *vanish at* ∞ if $\{x: |f(x)| \geqslant \varepsilon\}$ is compact, $\forall \varepsilon > 0$.

We now state a lemma which will be used very frequently in the sequel. It says that for separable Banach spaces, a probabilistic Hahn–Banach Theorem is valid.

Lemma 5.3 ([3], p. 1154, Theorem 2): *Let $(\Omega, \mathscr{B})$ be a measurable space, X a real separable normed linear space and $M \subset X$ a linear subspace. Let F be a random linear functional on M. Then $\exists$ a random linear functional $\tilde{F}$ on X agreeing with F on M with preservation of bounds.*

Theorem 5.4. *Let (X, τ) be a locally compact, σ-compact, metrizable space. Let $C_\infty(X)$ be the Banach space of continuous functions on X vanishing at ∞. Let F: $\Omega \times C_\infty(X) \to \mathbf{R}$ be a random linear functional. Then $\exists!$ random finite Radon measure $\nu(\omega, \cdot)$ on $\mathscr{B}(X)$, $\forall \omega \in \Omega$ such that F is the integral with respect to $\nu(\omega, dx)$.*

Proof. (A) *Existence.* Let $\tilde{X} := X \cup \{\infty\}$ be the one-point compactification of X. Then $\tilde{X}$ is a compact metric space. Since every $f \in C_\infty(X)$ has a unique continuous extension to $\tilde{X}$ with $f(\infty) = 0$, $C_\infty(X)$ is isometrically (with respect to $\|\cdot\|_\infty$) isomorphic to the (closed) linear subspace Γ of $C(\tilde{X})$ defined by $\Gamma := \{f \in C(\tilde{X}): f(\infty) = 0\}$. Hence $C_\infty(X)' \approx \Gamma'$ and every random linear functional F on $C_\infty(X)$ corresponds uniquely to a random linear functional on Γ', also denoted by F (by abuse of language). Now, by Lemma 5.3, $\exists$ a random linear functional $\tilde{F}$ on $C(\tilde{X})$ agreeing with F on Γ with preservation of bounds. Hence by Lemma 5.2, $\exists!$ random finite Radon measure $\nu(\omega, dx)$ such that

$$\tilde{F}(\omega, f) = \int_{\tilde{X}} f(x)\nu(\omega, dx), \quad \forall \omega \in \Omega, \ f \in C(\tilde{X}).$$

In particular,

$$F(\omega, f) = \int_X f(x)\nu(\omega, dx), \qquad \forall f \in C_\infty(X).$$

Since the Borel structure $\mathscr{B}(X)$ if X is a σ-additive class in $\mathscr{B}(\tilde{X})$, the measure $\nu(\omega, dx)$ is σ-additive on $\mathscr{B}(X)$, $\forall \omega \in \Omega$. Ω-measurability of ν is guaranteed by Lemma 5.2.

(B) *Uniqueness of $\nu(\omega, dx)$ on $\mathscr{B}(X)$.* Let $\lambda(\omega, dx)$ be determined by another extension of F to $C(\tilde{X})$, then

$$F(\omega, f) = \int_X f(x)\lambda(\omega, dx), \qquad \forall f \in C_\infty(X).$$

We claim $\lambda = \nu$ on $\mathscr{B}(X)$.

Proof of the claim. Since $C_0(X)$ is dense (in $\|\cdot\|_\infty$) in $C_\infty(X)$, it is sufficient to work in $C_0(X)$ (e.g., by monotone convergence, one can pass to elements of $C_\infty(X)$ by elements of $C_0(X)$). So, letting $f \in C_0(X)$, we have

$$\begin{aligned} F(\omega, f) &= \int f(x)\nu(\omega, dx) \\ &= \int f(x)\lambda(\omega, dx). \end{aligned}$$

But $\nu(\omega, G) = \sup_{\substack{f \in C_0(X) \\ f \leqslant 1_G}} F(\omega, f) = \lambda(\omega, G)$ for every $G \in \tau$ (see the proof of Lemma 5.2). Therefore $\lambda = \nu$ on $\mathscr{B}(X)$. ■

VI. Representation theorems.

VI.1. We are now in a position to prove a main result of this paper.

THEOREM 6.1. *Let Ψ be a random linear functional on $S\{M_p\}$ with conditions* (P) *and* (S) *assumed on $\{M_p\}$. Let $\varepsilon > 0$, then there exist $B \in \mathscr{B}$ and an integer $r > 0$ such that*

(i) $\mu(B) \geqslant 1 - \varepsilon$,

(ii) $\forall \omega \in B$, $\varphi \in S\{M_p\}$

$$\langle \varphi, \Psi(\omega)\rangle = \sum_q \int_{\mathbf{R}^d} M_r(x, q) D^q \varphi(x) \nu_q(\omega, dx)$$

where $\nu_q(\omega, dx)$ is a random finite Radon measure for each q.

Proof. (A) Fix $\varepsilon > 0$. By the corollary of Lemma 4.1, $\exists B \in \mathscr{B}$, integer $r > 0$ such that (i) is satisfied and

$$|\langle \varphi, \Psi(\omega)\rangle| \leqslant r\|\varphi\|_r, \qquad \forall \varphi \in S\{M_p\},\ \omega \in B.$$

Define

$$\langle \varphi, \tilde{\Psi}(\omega)\rangle := \begin{cases} \langle \varphi, \Psi(\omega)\rangle, & \omega \in B, \\ 0, & \omega \notin B \end{cases}$$

and

$$S(\omega) := \sup_{\substack{\varphi \in S\{M_p\} \\ \|\varphi\|_r \leqslant 1}} |\langle \varphi, \tilde{\Psi}(\omega)\rangle|.$$

But $S\{M_p\}$ is separable, so let D be a countable set dense in $S\{M_p\}$. We have

$$S(\omega) = \sup_{\varphi \in D, \|\varphi\|_r \leqslant 1} |\langle \varphi, \tilde{\Psi}(\omega)\rangle|.$$

Hence $S(\omega)$ is Ω-measurable and $|\langle \varphi, \tilde{\Psi}(\omega)\rangle| \leqslant S(\omega)\|\varphi\|_r$.

(B) Let $C_\infty(\mathbf{R}^d)$ be the bounded continuous functions $C_b(\mathbf{R}^d)$ on $\mathbf{R}^d$ vanishing at ∞, provided with the sup norm:

$$\|f\|_\infty := \sup_x |f(x)|.$$

Form the countable product

$$\prod^\infty C_\infty(\mathbf{R}^d)$$

and consider the following subspace Γ:

$$\{f_j\} \in \Gamma \Leftrightarrow \{f_j\} \in \prod^\infty C_\infty(\mathbf{R}^d) \quad \text{and} \quad \lim_{j\to\infty} \|f_j\|_\infty = 0.$$

Equip Γ with norm:

$$\|\{f_j\}\| := \sup_j \|f_j\|_\infty.$$

Then $(\Gamma, \|\cdot\|)$ is a separable Banach space, (a proof is provided in VII, the Appendix).

(C) We next construct a linear map

$$\Theta\colon (S\{M_p\}, \|\cdot\|_r) \to (\Gamma, \|\cdot\|)$$

where r is the integer determined in Lemma 4.1. Let

$$\Theta(\varphi) := \{M_r(x, q) D^q \varphi(x)\}, \qquad \varphi \in (S\{M_p\}, \|\cdot\|_r).$$

This map is obviously injective and isometric (recall that $M_r(x, q) D^q \varphi(x) \to 0$ as $|x| \to \infty$, and $M_r(x, q) D^q \varphi(x) \to 0$ as $q \to \infty$). Define $F(\omega, \cdot)$ on $\Theta(S\{M_p\})$ by

$$F(\omega, \Theta(\varphi)) := \langle \varphi, \tilde{\Psi}(\omega)\rangle.$$

Note $F(\omega, \cdot) \in [\Theta(S\{M_p\})]'$ with

$$|F(\omega, \Theta(\varphi))| \leqslant S(\omega)\|\Theta(\varphi)\|.$$

Since $(\Gamma, \|\cdot\|)$ is separable and $F(\omega, \cdot)$ is a random linear functional on $\Theta(S\{M_p\})$, hence by the probabilistic Hahn–Banach extension theorem (Lemma 5.3), there is a random linear functional $\tilde{F}(\omega, \cdot)$ on $(\Gamma, \|\cdot\|)$ with

$$|\tilde{F}(\omega, \cdot)| \leqslant S(\omega)\|\cdot\|, \qquad \forall (\cdot) \in \Gamma.$$

(D) Now, each continuous linear functional ξ on Γ is of the form

$$\langle\{f_j\}, \xi\rangle = \sum_{j=1}^{\infty} \langle f_j, \xi_j\rangle.$$

where ξ_j $(j = 1, 2, \ldots)$ is a continuous linear functional on

$$\Gamma_j := \{(0, \ldots, f_j, 0, \ldots): f_j \in C_\infty(\boldsymbol{R}^d)\}.$$

We have

$$\|\xi\| = \sum_{j=1}^{\infty} \|\xi_j\|.$$

(E) Each $\tilde{F}_j$ $(j = 1, 2, \ldots)$ is therefore a random linear functional on Γ_j. Hence by the probabilistic Riesz–Radon theorem (Theorem 5.4), there exist random finite Radon measures $\nu_j(\omega, dx)$ on $\boldsymbol{R}^d$ such that

$$\tilde{F}_j(\omega, f_j) = \int_{\boldsymbol{R}^d} f_j(x)\nu_j(\omega, dx), \quad j = 1, 2, \ldots$$

Hence, $\forall \omega \in B$

$$F(\omega, \Theta(\varphi)) = \langle\varphi, \Psi(\omega)\rangle = \sum_q \int_{\boldsymbol{R}^d} M_r(x, q) D^q\varphi(x)\nu_q(\omega, dx).$$

The theorem is thus proved. ■

VI.2. We will derive an equivalent set of norms on $S\{M_p\}$ under additional conditions; namely, (M) and (N), together with (B) and (S). Under these conditions on $\{M_p\}$, we can rewrite Theorem 6.1. Instead of random finite Radon measures $\nu_q(\omega, dx)$, we can use the Lebesgue measure dx. This is worked out in VI.3.

THEOREM 6.2. *Suppose* $\{M_p(x, q)\}$ *satisfies assumptions* (P), (S), (N) *and* (M). *Let*

$$\|\varphi\|_p' := \sup_q \int_{\boldsymbol{R}^d} M_p(x, q)|D^q\varphi(x)|\,dx, \quad \forall \varphi \in S\{M_p\}.$$

Then $\{\|\cdot\|_p'\}$ *and* $\{\|\cdot\|_p\}$ *are equivalent.*

Proof. (i) Let $\|\varphi\|_p = \sup_{x,q} M_p(x, q)|D^q\varphi(x)| < \infty$, $p = 1, 2, \ldots$ Condition (N) gives a $p' > p$ such that

$$\begin{aligned} M_p(x, q)|D^q\varphi(x)| &= m_{pp'}(x, q)M_{p'}(x, q)|D^q\varphi(x)| \\ &\leqslant m_{pp'}(x, q)\sup_{x,q} M_{p'}(x, q)|D^q\varphi(x)| \\ &= m_{pp'}(x, q)\|\varphi\|_{p'}. \end{aligned}$$

Thus

$$\|\varphi\|_p' = \sup_q \int M_p(x, q)|D^q\varphi(x)|\,dx \leqslant \sup_q B_p(q)\|\varphi\|_{p'} < \infty,$$

since $\sup_q B_p(q) := \sup_q \int m_{pp'}(x, q)\,dx < \infty$ by (N). Hence

$$\|\varphi\|_p' \leqslant \mathrm{const}_p \|\varphi\|_{p'}.$$

(ii) Consider

$$\|\varphi\|_p = \sup_{x,q} M_p(x, q)|D^q\varphi(x)| = \sup_q \sup_x M_p(x, q)|D^q\varphi(x)|.$$

Since $M_p(x, q)|D^q\varphi(x)|$ is continuous in x and goes to 0 for every q as $|x| \to \infty$ (Lemma 3.1), $\exists x_0^q$ such that

$$M_p(x_0^q, q)|D^q(x_0^q)| = \sup_x M_p(x, q)|D^q\varphi(x)|.$$

Therefore,

$$\|\varphi\|_p = \sup_q M_p(x_0^q, q)|D^q\varphi(x_0^q)|.$$

Now

$$\lim_{q\to\infty}\sup_x M_p(x, q)|D^q\varphi(x)| = \lim_{q\to\infty} M_p(x_0^q, q)|D^q\varphi(x_0^q)| = 0$$

by Lemma 3.2. Hence $\exists q_0$ such that

$$\|\varphi\|_p = M_p(x_0^{q_0}; q_0)|D^{q_0}\varphi(x_0^{q_0})| = \sup_x M_p(x, q_0)|D^{q_0}\varphi(x)|.$$

We claim

$$\sup_x M_p(x, q_0)|D^{q_0}\varphi(x)| \leqslant \sup_x M_p(x, q_0)\Big|\int_x^\infty D^{q_0+1}\varphi(\xi)\,d\xi\Big|.$$

Proof of claim. First observe that the improper integral

$$\int_x^\infty D^{q_0+1}\varphi(\xi)\,d\xi$$

exists. For,

$$\lim_{|y|\to\infty} D^{q_0}\varphi(y) = 0,$$

and thus

$$\int_x^\infty D^{q_0+1}\varphi(\xi)\,d\xi = \lim_{|y|\to\infty}\big(D^{q_0}\varphi(y) - D^{q_0}\varphi(x)\big) = -D^{q_0}\varphi(x).$$

Now,

$$M_p(x, q_0)|D^{q_0}\varphi(\infty) - D^{q_0}\varphi(x)| \geqslant -M_p(x, q_0)|D^{q_0}\varphi(\infty)| + M_p(x, q_0)|D^{q_0}\varphi(x)|$$

and

$$M_p(x, q_0)|D^{q_0}\varphi(\infty)| = 0,$$

because by using (M), we get

$$M_p(x, q_0) \leqslant M_p(t, q_0), \quad |t| \geqslant |x|$$

and

$$0 \leqslant \lim_{|t|\to\infty} M_p(x, q_0)\, |D^{q_0}\varphi(t)|\, \frac{M_p(t, q_0)}{M_p(t, q_0)} \leqslant \lim_{|t|\to\infty} M_p(t, q_0)\, |D^{q_0}\varphi(t)| = 0.$$

Therefore

$$M_p(x, q_0)\, |D^{q_0}\varphi(\infty) - D^{q_0}\varphi(x)| \geqslant M_p(x, q_0)\, |D^{q_0}\varphi(x)|$$

and our claim is established.

Now,

$$\|\varphi\|_p \leqslant \sup_x \int_x^\infty M_p(x, q_0)\, |D^{q_0+1}\varphi(\xi)|\, d\xi = \mathrm{const}'_p \sup_x \int_x^\infty M_p(x, q_0+1)\, |D^{q_0+1}\varphi(\xi)|\, d\xi$$

where $\mathrm{const}'_p \in \boldsymbol{R}$ is such that

$$M_p(x, q_0) = \mathrm{const}'_p M_p(x, q_0+1).$$

Thus, by (M),

$$\|\varphi\|_p \leqslant \mathrm{const}'_p \sup_x \int_x^\infty M_p(\xi, q_0+1)\, |D^{q_0+1}\varphi(\xi)|\, d\xi = \mathrm{const}''_p \sup_q \int M_p(\xi, q)\, |D^q\varphi(\xi)|\, d\xi = \mathrm{const}''_p \|\varphi\|'_p.$$

Hence $\{\|\cdot\|_p\}$ and $\{\|\cdot\|'_p\}$ are equivalent. ■

VI.3. We need the following lemmas:

LEMMA 6.3 ([4], p. 236, Proposition 1). *Let ξ be a random linear functional on $L^1(\boldsymbol{R}^d, dx)$. Then there exists $f\colon \Omega\times\boldsymbol{R}^d \to \boldsymbol{R}$ such that*

(i) $f(\cdot, x) \in L^0(\Omega)$, $\forall x \in \boldsymbol{R}^d$;

(ii) $f(\omega, \cdot)$ *is essentially bounded with respect to dx*, $\forall \omega \in \Omega$;

(iii) $\forall \omega \in \Omega$, $\varphi \in L^1(\boldsymbol{R}^d, dx)$

$$\langle\varphi, \xi(\omega)\rangle = \int_{\boldsymbol{R}^d} \varphi(x) f(\omega, x)\, dx.$$

($L^0(\Omega)$ is the set of measurable functions on Ω.)

LEMMA 6.4. *Let ξ be a random linear functional on $L^1(\boldsymbol{R}^d, W(x)\,dx)$ where $W\colon \boldsymbol{R}^d \to [1, \infty]$ is a weight function such that W is bounded continuous on $\{x \in \boldsymbol{R}^d\colon W(x)$ is finite$\} \subset \boldsymbol{R}^d$, i.e. $W \in C_b$. Then $\exists f\colon \Omega\times\boldsymbol{R}^d\to\boldsymbol{R}$ such that*

(i) $f(\cdot, x) \in L^0(\Omega)$, $\forall x \in \boldsymbol{R}^d$,

(ii) $f(\omega, \cdot)$ *is essentially bounded w.r.t. dx*, $\forall \omega \in \Omega$;

(iii) $\forall \omega \in \Omega$ *and* $\varphi \in L^1(\boldsymbol{R}^d, W(x)\,dx)$

$$\langle\varphi, \xi(\omega)\rangle = \int_{\boldsymbol{R}^d} \varphi(x) f(\omega, x) W(x)\, dx.$$

Proof. $\varphi \in L^1(\boldsymbol{R}^d, W(x)\,dx)$ means $\varphi(x)W(x) \in L^1(\boldsymbol{R}^d, dx)$. Denote the norm in $L^1(\boldsymbol{R}^d, dx)$ by $\|\cdot\|_1$ and in $L^1(\boldsymbol{R}^d, W(x)\,dx)$ by $|||\cdot|||_1$.

Clearly, the map $v\colon \varphi\to\varphi W$ is injective and linear. Also,

$$|||\varphi|||_1 \leqslant K \int_S |\varphi|\, dx \leqslant K\|\varphi\|_1$$

where $S := \{x \in \boldsymbol{R}^d\colon W(x)$ is finite$\} \subset \boldsymbol{R}^d$ and K is a positive constant $\geqslant W(x)$, $\forall x \in S$. On the other hand, if $\psi \in L^1(\boldsymbol{R}^d, dx)$

$$\|\psi\|_1 := \int_{\boldsymbol{R}^d} |\psi(x)|\, dx = \int |\psi(x)|\, \frac{W(x)}{W(x)}\, dx \leqslant \int |\psi(x)| W(x)\, dx := |||\psi|||_1.$$

Hence $\|\cdot\|_1$ and $|||\cdot|||_1$ are equivalent on $v[L^1(\boldsymbol{R}^d, W(x)\,dx)] \subset L^1(\boldsymbol{R}^d, dx)$. Now let ξ be a random linear functional on $L^1(\boldsymbol{R}^d, W(x)\,dx)$. First we identify $L^1(\boldsymbol{R}^d, W(x)\,dx)$ with $v[L^1(\boldsymbol{R}^d, W(x)\,dx)]$, then we extend ξ to $L^1(\boldsymbol{R}^d, dx)$ by the standard argument. Thus, for $\varphi \in L^1(\boldsymbol{R}^d, W(x)\,dx)$, $\exists f\colon \Omega\times\boldsymbol{R}^d\to\boldsymbol{R}$ satisfying (i) and (ii). Furthermore, by Lemma 6.3

$$\langle\varphi, \xi(\omega)\rangle = \langle v(\varphi), \xi(\omega)\rangle = \int \varphi(x) W(x) f(\omega, x)\, dx = \int \varphi(x) f(\omega, x) W(x)\, dx. \;■$$

THEOREM 6.5. *Let Ψ be a random linear functional on $S\{M_p\}$ with assumptions* (P), (S), (M), (N) *and* $1 \leqslant M_p(x, q)$, $\forall x, p, q$. *Let $\varepsilon > 0$. Then there exist $B \in \mathscr{B}$, an integer $r > 0$ and $\boldsymbol{R}$-valued functions $\{f_q\}$ on $\Omega\times\boldsymbol{R}^d$, such that*

(i) $\mu(B) \geqslant 1-\varepsilon$,

(ii) (a) $f_q(\omega, \cdot)$ *is essentially bounded with respect to dx for each* $\omega \in \Omega$,
(b) $f_q(\cdot, x) \in L^0(\Omega)$, $\forall x \in \boldsymbol{R}^d$,

(iii) $\forall \omega \in B$, $\varphi \in S\{M_p\}$

$$\langle\varphi, \Psi(\omega)\rangle = \sum_q \int_{\boldsymbol{R}^d} M_r(x, q) D^q\varphi(x) f_q(\omega, x)\, dx.$$

Proof. (A) Let r be the integer determined in Lemma 4.1 and let $L^1[M_r(x, q)\,dx]$ be the space of functions on $\boldsymbol{R}^d$ integrable w.r.t. the weight $M_r(x, q)$, provided with the usual norm

$$\|\cdot\|_q := \int_{\boldsymbol{R}^d} |(\cdot)| M_r(x, q)\, dx, \qquad \forall q.$$

Form the countable product

$$\prod_{q}^{\infty} L^1[M_r(x, q)\,dx]$$

and define a subspace

$$\Delta \subset \prod_{q}^{'\infty} L^1[M_r(x, q)\,dx]$$

as follows:

$$\{g_q\} \in \Delta \Leftrightarrow \sup_q \int |g_q| M_r(x, q)\,dx < \infty.$$

Now equip Δ with the uniform norm

$$\|\{g_q\}\| := \sup_q \int |g_q| M_r(x, q)\,dx.$$

$(\Delta, \|\cdot\|)$ is a Banach space. Recall the Banach space $(S\{M_p\}, \|\cdot\|_r')$ where

$$\|\varphi\|_r' := \sup_q \int M_r(x, q)|D^q\varphi(x)|\,dx < \infty.$$

The map

$$(S\{M_p\}, \|\cdot\|_r') \ni \varphi \to \{D^q\varphi(x)\} \in \Delta$$

is clearly injective, linear and isometric

$$(D^q\varphi(x) \in L^1[M_r dx] \quad \text{because} \quad \int_{\mathbf{R}^d} |D^q\varphi(x)| M_r(x, q)\,dx < \infty, \forall q).$$

Therefore, the image of that map is a closed linear subspace of $(\Delta, \|\cdot\|)$. Hence every continuous linear functional on $(S\{M_p\}, \|\cdot\|_r')$ can be extended to a continuous linear functional on $(\Delta, \|\cdot\|)$ with norm conserved, by the Hahn–Banach theorem.

(B) Now, let Ψ be a random linear functional on $S\{M_p\}$ with assumptions (P), (S), (M), (N) and $1 \leqslant M_p(x, q)$, $\forall x, p, q$, on $\{M_p\}$. Ψ can be extended to a random linear functional on Δ by the probabilistic Hahn–Banach theorem. By Lemma 6.4, and the corollary of Lemma 4.1, we have the required representation

$$\langle \varphi, \Psi(\omega)\rangle = \sum_q \int_{\mathbf{R}^d} D^q\varphi(x) f_q(\omega, x) M_r(x, q)\,dx$$

$\forall \omega \in B, \varphi \in S\{M_p\}$. ■

Remarks. 1. Theorem 6.5 says essentially that with additional assumptions on $\{M_p\}$ (compare Theorem 6.5 with Theorem 6.1), one can write $\nu_q(\omega, dx)$ as $f_q(\omega, x)\,dx$, the Lebesgue measure on $\mathbf{R}^d$ with density $f_q(\omega, x)$ where the "randomness" enters only into the density.

2. By making suitable choices for the $\{M_p\}$, one obtains the results of Ullrich [5] and of Swartz and Myers [4].

VII. Appendix. In this appendix, we prove that $(\Gamma, \|\cdot\|)$ is separable. We need the following lemma which is the main ingredient in the proof. For definitions of terms, see VI.

LEMMA 7.1. *$(C_\infty(\mathbf{R}^d), \|\cdot\|_\infty)$ is separable.*

Proof. Write

$$\mathbf{R}^d := \bigcup_{j=1}^{\infty} B_j$$

where $B_j := \{x \in \mathbf{R}^d\colon |x| \leqslant j\}$.

Now, for each $j \in Z_+ \setminus \{0\}$ (positive integers), construct, by Urysohn's lemma, a continuous function g_j which is 1 on B_j, 0 outside B_{j+1} and is between 0 and 1 on $B_{j+1}\setminus B_j$. Fix one such g_j for each j.

Let $\mathscr{P}$ denote the set of all polynomials on $\mathbf{R}^d$ with rational coefficients. For each $p \in \mathscr{P}$ and $j \in Z_+$, we define

$$p_j := \begin{cases} p & \text{on } B_{j+1}, \\ 0 & \text{elsewhere}, \end{cases}$$

and

$$\tilde{p}_j = p_j g_j.$$

Clearly, $\tilde{p}_j \in C_0(\mathbf{R}^d)$ and in fact, $\tilde{p}_j \in C_0(\mathbf{R}^d, B_{j+1})$ i.e., it is supported on B_{+1}. Furthermore,

$$\tilde{p}_j = \begin{cases} p & \text{on } B_j, \\ 0 & \text{outside } B_{j+1}, \end{cases}$$

and

$$|\tilde{p}_j(x)| \leqslant |p(x)|.$$

Fix $\varepsilon > 0$. Let $f \in C_\infty(\mathbf{R}^d)$, then $K := \{x \in \mathbf{R}^d\colon |f(x)| \geqslant \varepsilon/4\}$ is compact. Let B_k be the smallest B_j containing K. Then, by the classical Weierstrass's theorem, there exists a $p_k \in B_{k+1}$ which uniformly approximates $f\restriction B_{k+1}$ ("$\restriction$" means "restricted to") unto $\varepsilon/4$.

On B_k,

$$|f(x) - \tilde{p}_k(x)| = |f(x) - p(x)| < \varepsilon/4.$$

On $B_{k+1}\setminus B_k$, we have

$$|p(x)| \leqslant |f(x)| + \varepsilon/4 \leqslant \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

and so,

$$|f(x) - \tilde{p}_k(x)| \leqslant |f(x)| + |p_k(x)|\,|g_k(x)| \leqslant \frac{\varepsilon}{4} + \left(\frac{\varepsilon}{2}\right)\cdot 1 = \frac{3\varepsilon}{4}.$$

Outside B_{k+1},

$$|f(x)-0| < \varepsilon/4.$$

Therefore,

$$\sup_{x\in \mathbf{R}^d} |f(x)-\tilde{p}_k(x)| < \varepsilon,$$

i.e.,

$$\|f-\tilde{p}_k\|_\infty < \varepsilon.$$

Hence, $\{\tilde{p}_j\colon p\in\mathscr{P},\ j\in Z_+\}$ is a denumerable dense subset of $(C_\infty(\mathbf{R}^d),\ \|\cdot\|_\infty)$. Thus, $(C_\infty(\mathbf{R}^d),\ \|\cdot\|_\infty)$ is separable as claimed. ∎

THEOREM 7.2. *$(\Gamma, \|\cdot\|)$ is separable.*

Proof. $\Gamma\subset\prod^\infty C_\infty(\mathbf{R}^d)$ and $C_\infty(\mathbf{R}^d)$ is separable by Lemma 7.1. A countable product of separable metric spaces is separable metric. A subspace of a separable metric space is separable. $(\Gamma, \|\cdot\|)$ is therefore separable. ∎

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Generalized conjugate systems on local fields

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Abstract. The notion of a conjugate system of regular functions over $K^n\times \mathbf{Z}$, where K^n is the n-dimensional vector space over a local field and $\mathbf{Z}$ is a set of rational integers, is extended to that of a generalized conjugate system (GCS). Such systems are analogues of generalized Cauchy–Riemann systems of harmonic functions on Euclidean half-spaces. Examples of such GCS's are constructed by means of a system of operators, $\{R_l\}_{l=1}^n$, that are analogues of the Riesz transforms. An F. and M. Riesz theorem is proved. (If μ and $R_l\mu$, $l=1,2,\ldots,n$ are all finite Borel measures, then μ fs absolutely continuous.) A conjugate system definition of the Hardy space, $H^1(K^n)$, is proposed ($f\in H^1$ iff $f\in L^1$ and $R_lf\in L^1$ for all l) and it is shown that this definition is equivalent to other proposed definitions; namely, maximal function, Lusin area iunction, and atomic definitions.

§ 1. Introduction. Chao [1] and Chao and Taibleson [4] have given a definition of conjugate systems of functions on $K\times\mathbf{Z}$, K a local field and $\mathbf{Z}$ the rational integers, which gives rise to an F. and M. Riesz theorem: *Suppose the local class field of K is odd. Then there is a singular integral operator T on K with the property that if μ and $T\mu$ are both finite Borel measures then μ is absolutely continuous.* This operator is the local field version of the conjugate operator (Hilbert transform) on $\mathbf{R}$. In this paper we will extend the notion of *conjugate system* to *generalized conjugate system* (GCS) and we will construct examples which arise from systems of "Riesz" transforms, $\{R_l\}_{l=1}^n$ on K^n, the n-dimensional vector space over K.

For such a Riesz system we will establish an F. and M. Riesz theorem: *If μ and $R_l\mu$, $l=1,2,\ldots,n$ are all finite Borel measures then μ is absolutely continuous.* It will also be shown that a range of definitions for the Hardy space $H^1(K^n)$ are all equivalent. Thus, if H^1 is defined by the property: $f\in H^1$ iff f and R_lf, $l=1,2,\ldots,n$ are all integrable, then that

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