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Two observations on a theorem by Coifman

by

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Abstract. We apply Coifman's characterization of H_p if $0 < p \leqslant 1$ (1) to prove anew the interpolation theorem of Fefferman–Rivière–Sagher and (2) to give a characterization of translation invariant operators in H_p if $0 < p \leqslant 1$.

Ronald Coifman recently established the following elegant characterization of the space $H_p = H_p(\boldsymbol{R})$ if $0 < p \leqslant 1$.

THEOREM I ([1]). *A distribution f is in H_p, where $0 < p \leqslant 1$ iff it admits the representation $f = \sum a_i b_i$ where b_i are measurable functions having their supports in intervals I_i and satisfying*

$$|b_i(x)| \leqslant |I_i|^{-1/p}, \qquad \int x^k b_i(x)\,dx = 0 \quad \text{for} \quad 0 \leqslant k < N$$

where N is an integer $\geqslant 1/p$ and a_i are real numbers subject to $\sum |a_i|^p < \infty$. Also we have the equivalence of norms

$$\|f\|_{H_p} \approx \inf\Big(\sum |a_i|^p\Big)^{1/p}.$$

On the other hand, Fefferman–Rivière–Sagher obtained the following interpolation.

THEOREM II ([3]). $(H_{p_0}, H_{p_1})_{\theta p}$ *where* $0 < p_0 < p_1 < \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $0 < \theta < 1$.

We now want to show how Theorem I can be used to give a simple proof of Theorem II. In a way, however, our argument is similar to the one in [1], where the main step in the proof of Theorem II is used to obtain Theorem I.

Before entering into the details we would like to point out the anafogy with the interpolation theory of L_p spaces. It has been well known lor a long time that

$$(1) \qquad (L_{p_0}, L_{p_1})_{\theta p} = L_p \quad \text{(in the same conditions as Theorem II).}$$

In Peetre–Sparr [6] a new approach to (1) was proposed. We introduced the space L_0 of measurable functions with support [1] of finite measure,

[1] "Support" here means the complement of the set on which the function vanishes. Since we identify functions differing on a nulset it is unique up to nulset.

equipped with norm $\|f\|_{L_0} = \operatorname{supp} f$ and we proved that (in the notation of [6])

$$(L_0, L_\infty)_{\alpha p;E} = L_p \quad \text{where } \alpha = 1/p. \tag{2}$$

Using the standard results of the abstract theory of interpolation spaces (as developed in [6]) we could obtain (1) from (2). Our present thesis is thus that Theorem I does about the same for Theorem II as (2) does it for (1).

Inspired by the above let us now for a distribution f with compact support and $\int x^k f(x)\,dx = 0$ if $0 \leqslant k < N$ put(2) $\|f\|_{H_0} = |I|$ where I is the smallest interval containing $\operatorname{supp} f$. If we write $f_i = a_i b_i$, Theorem I can be rephrased as follows: *$f \in H_p$, where $0 < p \leqslant 1$ iff $f = \sum f_i$ with $\sum \|f_i\|_{H_0} \|f_i\|^p_{L_\infty} < \infty$.* We can now supply as promised the

Proof of Theorem II. Since one inclusion $(H_{p_0}, H_{p_1})_{\theta p} \subset H_p$, as always in such situations, is trivial, we may concentrate on the opposite inclusion $H_p \subset (H_{p_0}, H_{p_1})_{\theta p}$. Also, in view of the reiteration theorem and the interpolation theorem of Rivière–Sagher [7], we may assume $p \leqslant 1$. We distinguish two cases.

Case 1. $p_1 \leqslant 1$. Let thus $f \in H_p$ and let us (by Theorem I) write $f = \sum f_i$ with $\sum \|f_i\|_{H_0} \|f_i\|^p_{L_\infty} < \infty$. For every integer ν let us set

$$E_\nu = \{i \mid 2^\nu \leqslant \|f_i\|_{L_\infty} < 2^{\nu+1}\} \quad \text{and} \quad u_\nu = \sum_{i \in E_\nu} f_i.$$

Clearly, $f = \sum u_\nu$ and for every ν $u_\nu \in H_{p_0} \cap H_{p_1}$. More precisely, for $j = 0, 1$ we obtain (again by Theorem I)

$$\begin{aligned} \|u_\nu\|^{p_j}_{H_{p_j}} &\leqslant C \sum_{i \in E_\nu} \|f_i\|_{H_0} \|f_i\|^{p_j}_{L_\infty} \\ &\leqslant C 2^{\nu(p_j - p)} \sum_{i \in E_\nu} \|f_i\|_{H_0} \|f_i\|^p_{L_\infty} \\ &\leqslant C 2^{\nu(p_j - p)} a_\nu \end{aligned}$$

with $a_\nu = \sum_{i \in E_\nu} \|f_i\|_{H_0} \|f_i\|^p_{L_\infty}$, or (in the notation of [6])

$$2^{\nu(p - p_0)} J(2^{\nu(p_1 - p_0)}, u_\nu; H^{[p_0]}_{p_0}, H^{[p_1]}_{p_1}) \leqslant C a_\nu.$$

Since $\sum a_\nu < \infty$, it follows that $f \in (H^{[p_0]}_{p_0}, H^{[p_1]}_{p_1})_{\eta 1;J}$ with $p = (1-\eta) p_0 + \eta p_1$, $0 < \eta < 1$. But (see [6])

$$(H^{[p_0]}_{p_0}, H^{[p_1]}_{p_1})_{\eta 1;J} = (H_{p_0}, H_{p_1})^{[p]}_{\theta p;K}$$

and the proof is complete.

(2) This is not a "norm", not even in the sense of [6].

Case 2. $p_1 > 1$. We proceed as in case 1 but make first the additional remark that from the proof of Theorem I (in [1]) follows that actually $\|u_\nu\|_{L_\infty} \leqslant C 2^\nu$. From the Hölder type inequality

$$\|u_\nu\|_{H_{p_1}} \leqslant C \|u_\nu\|^{1-\lambda}_{H_p} \|u_\nu\|^\lambda_{L_\infty} \quad \text{with} \quad 1/p_1 = (1-\lambda)/p + \lambda/\infty$$

follows now

$$\|u_\nu\|_{H_{p_1}} \leqslant C 2^{\nu(1 - p/p_1)} a_\nu^{1/p_1}.$$

The rest of the proof is the same. ∎

So much for our first "observation"(3). The second one is concerned with multipliers. In May 1968 I wrote a survey article [5] concerned with the Stampacchia spaces $\mathscr{L}_{p\lambda}$. The importance of this family stems from the fact that it encompasses a number of classical spaces, the most famous being B.M.O. $= \mathscr{E} = \mathscr{L}_{p,-n}$ $(1 \leqslant p < \infty)$(4). I also proposed an alternative notation $\mathscr{E}^{\alpha;p} = \mathscr{L}_{p\lambda}$ (with $\lambda = n + \alpha p$) and suggested the introduction of more general spaces $\mathscr{E}^{\alpha q;p}_P$ (by the analogy with Besov spaces $B^{\alpha q}_P$): $f \in \mathscr{E}^{\alpha q;p}_P$ iff

$$\Big(\int_0^\infty (\|\varphi_r * f\|_{L_p}/r^\alpha)^q\, dr/r\Big)^{1/q} < \infty$$

for all $\varphi \in L_p$ with the support contained in the unit interval $[-1, 1]$ and $\int x^k \varphi(x)\,dx = 0$ for $0 \leqslant k < N$, $\varphi_r = (1/r)\varphi(x/r)$. I further gave a characterization of translation invariant operators $T\colon L_\infty \to$ B.M.O. (analogous to a classical result of Hardy–Littlewood for Besov spaces): *If $Tf = k * f$ then $T\colon L_\infty \to$ B.M.O. iff $k \in \mathscr{E}^{0\infty;p}_1$* $(1 \leqslant p < \infty)$. In view of the duality theorem of Fefferman–Stein [4], to the effect that $H_1 =$ B.M.O. (which helas was not availble in '68) this can also be viewed upon as a characterization of translation invariant operators $T\colon H_1 \to H_1$. We now extend the latter result to H_p, $0 < p < 1$. This is implicit in [1], [2].

THEOREM III. *If $Tf = k * f$, then $T\colon H_p \to H_p$, where $0 < p < 1$ iff $k \in \mathscr{E}^{1/p-1,\infty;1}_p$. (In particular, we must thus have $k \in B^{1/p-1,\infty}_p$.)*

Proof. As remarked in [1] it suffices to establish $T\colon H_p \to L_p$. Again, in view of Theorem I, this is equivalent to the inequality

$$\|k * f\|_{L_p} \leqslant C \|f\|^{1/p}_{H_0} \|f\|_{L_\infty}. \tag{3}$$

(3) Actually in [3] is proved a little bit more, namely the in itself interesting formula:

$$K(t, f; H_{p_0}, H_{p_1}) \approx K(t, f^\#; L_{p_0}, L_{p_1}) \tag{$*$}$$

where $f^\#$ denotes the "grand maximal function" of [4]. It would be interesting to know if $(*)$ could be derived using the present approach, too.

(4) It is perhaps not so well known, but B.M.O. actually stands for (the initial of) my children Benjamin, Mikaela, Opi (Jakob).

Take now $f = \varphi_r$. Then (3) reduces to

$$\|\varphi_r * f\|_{L_p} \leqslant Cr^{1/p} r^{-1} \|\varphi\|_{L_\infty} = Cr^{1/p-1} \|\varphi\|_{L_\infty}$$

which precisely says that $f \in \mathscr{E}_p^{1/p-1,\infty;1}$. ■

Incidentally from Theorem III we also get a very simple proof of the multiplier theorem of Fefferman–Stein [4].

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Representation of random linear functionals on certain $S\{M_p\}$ spaces

by

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Abstract. We prove an integral representation theorem for random linear functionals on certain $S\{M_p\}$ spaces. With stronger conditions on $\{M_p\}$, the representation can be rewritten in terms of the Lebesgue integral over $\boldsymbol{R}^d$. In the process of proving the main theorems, we obtain a probabilistic Riesz–Radon representation theorem, which is of interest in it's own right.

I. Introduction. The purpose of this paper is to prove a representation theorem for generalized random processes on a rather large class of spaces, namely $S\{M_p\}$ spaces. Such spaces are introduced by Yamanaka in a note [6] in order to make possible a unified treatment of the $K\{M_p\}$ and S-type spaces of Gel'fand–Shilov [2]. For $K(a)$ (i.e. the space of $C^\infty(\boldsymbol{R})$-functions with supports contained in $[-a, a]$) such a representation theorem is obtained by Ullrich [5], and for certain $K\{M_p\}$ spaces, by Swartz and Myers [4]. Since the conditions we impose on $S\{M_p\}$ spaces are all satisfied by those $K\{M_p\}$ spaces considered by Swartz and Myers, our representation theorem is hence more general.

As far as method goes, Ullrich, Swartz and Myers all use a scheme of representing continuous linear functionals on σ-normed spaces parallel to the scheme employed by Gel'fand–Shilov [1], [2]. In this paper, we are working in the same spirit.

The organization of the paper is as follows: In II, we give the necessary preliminary definitions, then prove $S\{M_p\}$ is complete and σ-normed. In III, we lay down the basic assumptions on $\{M_p\}$, with the derivation of two straightforward consequences. And in IV, we prove a lemma which gives a "large" measurable set $B \in \mathscr{B}$, where $(\Omega, \mathscr{B}, \mu)$ is a fixed probability space, on which our random process is bounded. Then in V, we establish a probabilistic version of the classical Riesz–Radon theorem for continuous functions vanishing at ∞. This theorem, which is of independent interest, is an important ingredient in the proof of Theorem 6.1. An extension theorem, of a probabilistic Hahn–Banach type, is also stated in V. This theorem is proved in Hanš [3], and is used here on many occasions that follow. In VI, the main results of this paper are stated and