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Smoothness properties in Banach spaces

by

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Abstract. In the present article we study the relationship between smoothness of a Banach space E and of its dual E^*, and also give the following criterion for the Banach space E to be Hilbert: if both E and E^* are H^2-smooth, then E is isomorphic to a Hilbert space.

Introduction. Through whole this paper we consider real Banach spaces, and understand differentiability in the sense of Fréchet.

DEFINITION 1. A Banach space E is called *C^r-smooth* if it admits a non-constant C^r function φ (that is, a function r times continuously differentiable) such that $\varphi(x) = 0$ when $\|x\| \geqslant 1$.

Let p be a real number greater than 1, and k the greatest integer strictly less than p.

DEFINITION 2. The function f on the Banach space E is called an *H^p-function* (*p-smooth in the sense of Hölder*) if f is k times continuously differentiable, and, for any $x \in E$, there is a neighbourhood U_x of x such that, for any $y, z \in U_x$,

$$\|D_y^k f - D_z^k f\| \leqslant C\|y-z\|^{p-k},$$

where the constant C depends only on the neighbourhood U_x. (Here $\|D_y^k f - D_z^k f\|$ denotes the norm of the k-linear map $D_y^k f - D_z^k f$.)

DEFINITION 3. The Banach space E is called *H^p-smooth* (*p-smooth in the sense of Hölder*) if it admits a non-constant H^p-function φ such that $\varphi(x) = 0$ when $\|x\| \geqslant 1$.

The following consequences of these definitions are evident:

(a) if E is H^p-smooth and $p' < p$, then E is also $H^{p'}$-smooth,

(b) if E is C^r-smooth, it is also H^r-smooth,

(c) if E is H^p-smooth, it is also C^r-smooth, for any natural number r strictly less than p,

(d) a subspace of an H^p-smooth Banach space is also H^p-smooth.

Touching the smoothness properties of the spaces l_p, $1 \leqslant p < \infty$, it is easy to see that l_p is H^p-smooth, and, furthermore, that it is C^∞-smooth if p is an even integer. For the proof, it is sufficient to consider the function φ, defined by

$$\varphi(x) = \psi(\|x\|^p),$$

where ψ is a C^∞ function of a real variable such that $\psi(0) = 1$ and $\psi(t) = 0$ when $|t| \geqslant 1$.

Let p not be an even integer, and r be the least integer greater than p. Kurzweil ([5], Theorem 4) showed that under these conditions the space l_p is not C^r-smooth. Bonic and Frampton sharpened Kurzweil's result and demonstrated the following theorem.

THEOREM A (R. Bonic and J. Frampton [2], Theorem 4). *If p is not an even integer, and $p' > p$, then l_p is not $H^{p'}$-smooth. If p is an odd integer, l_p is not C^p-smooth.*

When $p = 1$, we obtain, using property (d), the simple corollary.

COROLLARY. *If the Banach space B is H^p-smooth, where $p > 1$, then it does not contain a subspace isomorphic to l_1.*

From Theorem A it is clear that, if $E = l_p$ and $p \neq 2$, then E and E^* cannot be C^2-smooth simultaneously. It turns out that such a statement is valid in the general situation: we shall prove, in the second section of this work, the theorem below.

THEOREM 2.1. *If the Banach space E and its dual E^* are both C^2-smooth, then E is isomorphic to a Hilbert space.*

The following result in the same direction was previously known:

THEOREM B (R. Bonic and F. Reis (1); M. M. Rao, see [8], Theorem 1). *If the Banach space E has a C^2-smooth norm and the dual norm*

$$\|x^*\| = \sup_{\|x\|\leqslant 1} |x^*(x)|$$

in E^* is also C^2-smooth, then E is isomorphic to a Hilbert space.

(A norm is called *C^r-smooth* if it is C^r-smooth except at zero.)

We shall present here a proof of Theorem B (based on Lemma 1.2*, proved later), so as to illustrate by this simple example the methods used below to establish Theorem 2.1.

Set $\varphi(x) = \|x\|$, $\psi(x^*) = \|x^*\|$. Take $x_0 \in E$, $x_0^* \in E^*$ so that $\|x_0\| = 1$, $\|x_0^*\| = 1$, and $x_0^*(x_0) = 1$. Applying Taylor's theorem, we have

$$\varphi(x_0+h) = \varphi(x_0) + D^1_{x_0}\varphi(h) + \tfrac{1}{2}D^2_{x_0}\varphi(h, h) + o(\|h\|^2),$$
$$\psi(x_0^*+g) = \psi(x_0^*) + D^1_{x_0^*}\psi(g) + \tfrac{1}{2}D^2_{x_0^*}\psi(g, g) + o(\|g\|^2).$$

Let G be the subspace of E^* defined by the relation

$$G = \ker D^1_{x_0^*}\psi \cap \{x^*\colon x^*(x_0) = 0\}.$$

(1) The obtaining of this result is referred to in [2].

According to Lemma 1.2* below, one may find in E a subspace X of finite codimension such that

(*) for any $x \in X$ there exists $y \in G$, with $y \neq 0$, such that

$$y(x) \geqslant \tfrac{1}{3}\|y\|\,\|x\|.$$

Define $H = X \cap \ker D^1_{x_0}\varphi \cap \ker x_0^*$. It is clear that

$$\operatorname{codim} H \leqslant \operatorname{codim} X + 2 < \infty.$$

Suppose now that E is not isomorphic to a Hilbert space; then we claim that

$$\inf\{|D^2_{x_0}\varphi(h, h)|\colon h \in H \text{ and } \|h\| = 1\} = 0.$$

For consider the bounded quadratic form $D^2_{x_0}\varphi(h, h)$ on the subspace H of E. If it assumes on the unit sphere of H both positive and negative values, then, in view of the connectedness of this sphere, it must also assume the value 0. If on the other hand its values on the sphere are all positive, but $D^2_{x_0}\varphi(h, h) > a > 0$ for all $h \in H$ of unit norm, this would signify that the Hilbertian norm

$$|||h||| = [D^2_{x_0}\varphi(h, h)]^{1/2}$$

is equivalent to the original norm $\|\ \|$, and H would be isomorphic to a Hilbert space. As $\operatorname{codim} H < \infty$, E would also be isomorphic to a Hilbert space, which contradicts our hypothesis. The case where $D^2_{x_0}\varphi(h, h) < 0$ at all points on the unit sphere reduces to the preceding one on changing the form $D^2_{x_0}\varphi(h, h)$ to $-D^2_{x_0}\varphi(h, h)$.

Take the number M so that $|D^2_{x_0^*}\psi(g, g)| \leqslant M\|g\|^2$ for all $g \in E^*$.

Choose $h \in H$ such that $h \neq 0$ and $|D^2_{x_0}\varphi(h, h)| \leqslant \dfrac{1}{100M}\|h\|^2$. Using (*), choose $g \in G$ such that $\|g\| = \dfrac{1}{10M}\|h\|$ and

$$g(h) \geqslant \tfrac{1}{3}\|g\|\,\|h\| = \frac{1}{30M}\|h\|^2.$$

Then, for sufficiently small τ, we have estimate for the numbers $\|x_0 + \tau h\|$ and $\|x_0^* + \tau g\|$ as follows:

$$\|x_0+\tau h\| = \varphi(x_0+\tau h) \leqslant 1 + \frac{1}{200M}\|\tau h\|^2 + o(\tau^2),$$
$$\|x_0^*+\tau g\| = \psi(x_0^*+\tau g) \leqslant 1 + \tfrac{1}{2}M\|\tau g\|^2 + o(\tau^2).$$

Furthermore, the definition of the norm of $\|x_0^*+\tau g\|$ in E^* entails

$$(x_0^*+\tau g)(x_0+\tau h) \leqslant \|x_0^*+\tau g\|\,\|x_0+\tau h\|.$$

Therefore

$$x_0^*(x_0)+\tau^2 g(h) \leqslant \left(1+\frac{\tau^2\|h\|^2}{200M}+o(\tau^2)\right)\left(1+\frac{\tau^2\|h\|^2}{200M}+o(\tau^2)\right),$$

and

$$1+\frac{\tau^2\|h\|^2}{30M} \leqslant 1+\frac{\tau^2\|h\|^2}{100M}+o(\tau^2).$$

As $\|h\| \neq 0$, this last inequality cannot hold when τ is sufficiently small. The contradiction demonstrates Theorem B.

From Theorem A it is also clear that, if $E = l_p$, where $p \neq 2n/(2n-1)$ for any positive integer n, then E^* cannot be H^q-smooth when $q > p/(p-1)$. Again, a similar result holds in the general case: in §3 we shall prove the following theorem:

THEOREM 3.1. *If the Banach space E is H^p-smooth, and its dual E^* is H^q-smooth, and neither of these spaces is C^∞-smooth, then $1/p+1/q \geqslant 1$.*

T. Figiel read in detail a preliminary version of this paper, and contributed a series of profound observations, which made possible a fundamental simplification of the arguments; in particular, Lemma 1.4 is his. The author is deeply indebted to him, and also wishes to thank B. S. Mitjagin for much helpful advice and for his unfailing interest in the work.

1. Basic lemmas. In this section we discuss a few lemmas essential in the proofs of Theorems 2.1 and 3.1.

LEMMA 1.1 (Davis, Dean, and Singer [3], Remark 1 to Theorem 1). *If B_1 is a finite-dimensional subspace of the Banach space B, then, for any $\varepsilon > 0$, one may find a finite-codimensional subspace B_2 of B such that*

(1) $B_1 \cap B_2 = 0$,

(2) *the projection P_{B_1} in the subspace B_1+B_2 of B, whose image is B_1 and kernel B_2, has norm less than $1+\varepsilon$.*

(Hence it follows that the complementary projection in B_1+B_2, with image B_2 and kernel B_1, $P_{B_2} = 1_{B_1+B_2}-P_{B_1}$, has norm less than $2+\varepsilon$.)

A proof of this lemma may also be found in [4], Lemma 2.3.

LEMMA 1.2. *Let X be a finite-codimensional subspace in the Banach space B. Then there is a finite-codimensional subspace Y of B^* such that, for each $y \in Y$, there exists a non-zero $x \in X$ with $y(x) \geqslant \frac{1}{3}\|y\|\,\|x\|$.*

Proof. Let $X^\perp = Z$ in B^*. Applying Lemma 1.1 with $\varepsilon = \frac{1}{2}$, we may choose a finite-codimensional subspace Y of B^* such that $Z \cap Y = 0$ and the projection P_Y in $Y+Z$ with kernel Z and image Y has norm less than $\frac{5}{2}$.

Let $y \in Y$, and choose a functional $\tilde{x}$ on Y such that $\|\tilde{x}\| = 1$ and $\tilde{x}(y) = \|y\|$. Then the functional $\tilde{x}\cdot P_Y$ is defined on $Y+Z$, is zero on Z, and has norm less than $\frac{5}{2}$. Extend it by the Hahn–Banach theorem to B^* without increasing its norm; we obtain a functional in B^{**}. By Helly's theorem ([9], Chapter 4, §6, Theorem 5), there exists $x \in B$ such that $z(x) = 0$ whenever $z \in Z$, $y(x) = \|y\|$ and $\|x\| \leqslant 3$. Evidently, therefore, $x \in X$ and $y(x) \geqslant \frac{1}{3}\|y\|\,\|x\|$.

LEMMA 1.2*. *Let Y be a finite-codimensional subspace of B^*. Then there is a finite-codimensional subspace X of B such that, for any $x \in X$, there exists a non-zero $y \in Y$ with $y(x) \geqslant \frac{1}{3}\|y\|\,\|x\|$.*

Proof. According to Lemma 1.2, one may find in B^{**} a subspace $\tilde{X}$, $\operatorname{codim}\tilde{X} < \infty$, such that for each $x^{**} \in \tilde{X}$ there is a non-zero $y \in Y$ with $x^{**}(y) \geqslant \frac{1}{3}\|x^{**}\|\,\|y\|$. Let $i\colon B \to B^{**}$ be the canonical embedding; then the conclusion of the lemma is valid for $X = i^{-1}(\tilde{X} \cap i(B))$.

LEMMA 1.3. *Let the Banach space E be H^p-smooth and let E^* be H^q-smooth, where $p > 1$ and $q > 1$. Suppose that (x_n) is a sequence of points all lying either in E or in E^*, such that the set of partial sums $\sum_{n=1}^{N} \varepsilon_n x_n$ is bounded for all positive integers N and sequences (ε_n) taking values $\varepsilon_n = \pm 1$. Then, for any such sequence (ε_n), the infinite series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges, and the set of all possible partial sums $\sum_{n=1}^{N} \varepsilon_n x_n$, for $N = 1, 2, \ldots$ and all possible (ε_n), is compact.*

The following well-known theorems will be of use:

THEOREM C (Bessaga and Pełczyński [1], Theorem 5). *If the Banach space B does not contain a subspace isomorphic to c_0, then every weakly absolutely convergent series in B converges unconditionally.*

(Recall that a series $\sum_{n=1}^{\infty} x_n$ is called *weakly absolutely convergent* if, for any $f \in B^*$, the numerical series $\sum_{n=1}^{\infty} |f(x_n)|$ converges; and is called *unconditionally convergent* if, for any sequence (ε_n) taking values $\varepsilon_n = \pm 1$, the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges.)

THEOREM D (Bessaga and Pełczyński [1], Theorem 4). *Let B^* be the dual of the Banach space B. If B^* contains a subspace isomorphic to c_0, then B contains a complemented subspace isomorphic to l_1.*

Proof of Lemma 1.3. If E contained a subspace isomorphic to c_0, it would follow easily that E^* has a subspace isomorphic to l_1; whilst, if we assumed E^* to contain a subspace isomorphic to c_0, we should have from Theorem D that E has a subspace isomorphic to l_1. But from the corollary to Theorem A, neither E nor E^* contains a subspace isomorphic to l_1.

Now, if the partial sums of the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ are uniformly bounded for all sequences (ε_n), it is obvious that the series is weakly absolutely convergent. Thus, according to Theorem C, it converges for all sequences (ε_n), $\varepsilon_n = \pm 1$. From this it follows easily that, for given $\varepsilon > 0$, there exists N such that, for all sequences (ε_n) with values ± 1, $\|\sum_{n=N}^{\infty} \varepsilon_n x_n\| < \varepsilon$. Hence the set of all partial sums $\sum_{n=1}^{N} \varepsilon_n x_n$, $N = 1, 2, 3, \ldots$, $\varepsilon_n = \pm 1$, is totally bounded, and its closure is compact.

LEMMA 1.4 (Figiel). *Let Q_j, $j = 1, 2, \ldots, m$, be a finite class of bounded symmetric bilinear forms on the Banach space E, such that for all $x \in E$ the following inequality holds:*

$$\|x\|^2 \leqslant \sup\{|Q_j(x, x)|: 1 \leqslant j \leqslant m\}. \tag{1}$$

Then E is isomorphic to a Hilbert space.

We shall exploit the theorem below:

THEOREM E (Kwapień [6], Proposition 3.1). *A Banach space E is isomorphic to a Hilbert space if and only if there exist positive constants A and B such that for arbitrary $x_1, x_2, \ldots, x_n \in E$ the inequality*

$$A\Big(\sum_{i=1}^{n} \|x_i\|^2\Big) \leqslant \int_0^1 \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\|^2 dt \leqslant B\Big(\sum_{i=1}^{n} \|x_i\|^2\Big)$$

is satisfied, where r_i denotes the classical i-th Rademacher function $r_i(t) = \operatorname{sign}(\sin(2^i \pi t))$.

Now, using Hölder's inequality and condition (1), we obtain

$$\Big(\int_0^1 \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\|^2 dt\Big)^2 \leqslant \int_0^1 \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\|^4 dt \tag{2}$$
$$\leqslant \int_0^1 \Big(\sum_{j=1}^{m} Q_j^2\Big(\sum_{i=1}^{n} x_i r_i(t), \sum_{i=1}^{n} x_i r_i(t)\Big)\Big) dt.$$

From the properties of Rademacher functions we have the equalities

$$\int_0^1 Q_j^2\Big(\sum_{i=1}^{n} x_i r_i(t), \sum_{i=1}^{n} x_i r_i(t)\Big) dt$$
$$= \sum_{i,k,l,m=1}^{n} Q_j(x_i, x_k) Q_j(x_l, x_m) \int_0^1 r_i(t) r_k(t) r_l(t) r_m(t)\, dt$$
$$= \Big(\sum_{i=1}^{n} Q_j(x_i, x_i)\Big)^2 + 2 \sum_{\substack{k,l=1\\ k\neq l}}^{n} Q_j^2(x_k, x_l).$$

Hence we get

$$\int_0^1 Q_j^2\Big(\sum_{i=1}^{n} x_i r_i(t), \sum_{i=1}^{n} x_i r_i(t)\Big) dt \leqslant 3\|Q_j\|^2 \Big(\sum_{i=1}^{n} \|x_i\|^2\Big)^2. \tag{3}$$

Let $M = \max_{j=1,2,\ldots,m} \|Q_j\|$. From inequalities (2) and (3) results the estimate from above

$$\int_0^1 \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\|^2 dt \leqslant (3m)^{1/2} M \Big(\sum_{i=1}^{n} \|x_i\|^2\Big)^2.$$

Now we obtain an estimate from below for the integral. We have

$$\sum_{i=1}^{n} \|x_i\|^2 \leqslant \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} |Q_j(x_i, x_i)|\Big)$$
$$\leqslant m \max_j \sum_{i=1}^{n} |Q_j(x_i, x_i)| = m \sum_{i=1}^{n} |Q_{j_0}(x_i, x_i)|$$

for some j_0 between 1 and m inclusive.

Let $I^+ = \{i: Q_{j_0}(x_i, x_i) > 0\}$, $I^- = \{i: Q_{j_0}(x_i, x_i) \leqslant 0\}$. Suppose in the first place that

$$\sum_{i\in I^+} Q_{j_0}(x_i, x_i) > -\sum_{i\in I^-} Q_{j_0}(x_i, x_i).$$

Then we obtain $\sum_{i=1}^{n} \|x_i\|^2 \leqslant 2m \sum_{i\in I^+} Q_{j_0}(x_i, x_i)$.

Relabel the x_i, for $i \in I^+$, as $x_1, x_2, \ldots, x_k$, so that

$$\sum_{i=1}^{n} \|x_i\|^2 \leqslant 2m\Big(\sum_{i=1}^{k} Q_{j_0}(x_i, x_i)\Big) = 2m \int_0^1 Q_{j_0}\Big(\sum_{i=1}^{k} x_i r_i(t), \sum_{i=1}^{k} x_i r_i(t)\Big) dt$$
$$\leqslant 2mM \int_0^1 \Big\|\sum_{i=1}^{k} x_i r_i(t)\Big\|^2 dt = 2mM \sum_{\varepsilon_i=\pm 1} \frac{1}{2^k} \Big\|\sum_{i=1}^{k} \varepsilon_i x_i\Big\|^2$$
$$\leqslant 2mM \sum_{\varepsilon_i=\pm 1} \frac{1}{2^n} \Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\|^2 = 2mM \int_0^1 \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\|^2 dt.$$

The last inequality in this chain results from the observation that, for any x and y in E, $\|x+y\|^2 + \|x-y\|^2 \geqslant 2\|x\|^2$.

If, on the other hand,

$$\sum_{i\in I^+} Q_{j_0}(x_i, x_i) \leqslant -\sum_{i\in I^-} Q_{j_0}(x_i, x_i),$$

then, writing the form $-Q_{j_0}$ as R_{j_0}, we shall have the inequality

$$\sum_{i=1}^{n} \|x_i\|^2 \leqslant 2m \sum_{i\in I^-} R_{j_0}(x_i, x_i).$$

Transforming this inequality as above, we obtain in this case also

$$\sum_{i=1}^{n} \|x_i\|^2 \leqslant 2mM \int_0^1 \Big\| \sum_{i=1}^{n} x_i r_i(t) \Big\|^2 dt. \ \blacksquare$$

COROLLARY. *Let the Banach space E not be isomorphic to a Hilbert space, and let H be a finite-codimensional subspace in E; let $Q_1, Q_2, \ldots, Q_m$ be bounded symmetric bilinear forms on E. Then for any $\varepsilon > 0$ there is an $h \in H$, with $\|h\| = 1$, such that, for all $j = 1, 2, \ldots, m$, $|Q_j(h, h)| < \varepsilon$.*

2. Proof of Theorem 2.1. As E is C^2-smooth, there exists on E, by definition, a non-constant C^2-smooth function g, such that $g(x) = 0$ when $\|x\| \geqslant 1$. Take $z \in E$ such that $g(z) = a \neq 0$. Let f be a nonnegative C^∞ function of one real variable, such that $f(a) = 0$ and $f(0) = 2$. Set $\varphi(x) = f\big(g(2x+z)\big)$. It is easily verified that $\varphi(x) \geqslant 0$, $\varphi(0) = 0$, and $\varphi(x) = 2$ when $\|x\| \geqslant 1$. Analogously, on E^* as well exists a C^2 function ψ such that $\psi(0) = 0$ and $\psi(x^*) = 2$ when $\|x^*\| \geqslant 1$.

Consider $E \times E^*$ with the norm $\|(x, x^*)\| = \|x\| + \|x^*\|$ and the C^2 function $\Phi\colon E\times E^* \to R$, defined by $\Phi(x, x^*) = \varphi(x) + \psi(x^*)$. From the definition of Φ it follows that $\Phi(0, 0) = 0$ and $\Phi(x, x^*) \geqslant 2$ when $\max(\|x\|, \|x^*\|) \geqslant 1$.

LEMMA 2.1. *Let K be a compact set in the Banach space B and f a C^2 function on B. Then*

$$f(x+h) = f(x) + D_x^1 f(h) + \tfrac{1}{2} D_x^2 f(h, h) + r(f)(x, h),$$

where $\dfrac{r(f)(x, h)}{\|h\|^2} \to 0$ *as $h \to 0$, uniformly for $x \in K$.*

This lemma is an immediate corollary of Taylor's formula in the form given in [7], Chapter I, 4,

$$f(x+h) = f(x) + D_x^1 f(h) + \int_0^1 (1-t) D_{x+th}^2 f(h, h)\, dt.$$

Let us assume that E is not isomorphic to a Hilbert space. Then the following lemma holds.

LEMMA 2.2. *Let K and K^* be compact sets in E and E^*, respectively. Suppose that $F \subset K$ and $F^* \subset K^*$ are finite sets, and H, G are finite-codimensional subspaces in E and E^* respectively. Denote by $Q(F, F^*, H, G)$ the set of pairs (h, g), where $h \in H$, $g \in G$, $\|h\| \leqslant \frac{1}{2}$, $\|g\| \leqslant \frac{1}{2}$, which satisfy the inequality*

$$|\Phi(x \pm h, x^* \pm g) - \Phi(x, x^*)| \leqslant g(h)$$

for all $x \in F$ and $x^* \in F^*$. Define

$$a(F, F^*, H, G) = \sup\{g(h)\colon (h, g) \in Q(F, F^*, H, G)\}.$$

Then there is a constant $a_0(K, K^*)$, depending only on K and K^*, such that $a(F, F^*, H, G) \geqslant a_0(K, K^*) > 0$.

Proof. Let $\tilde{G} = G \cap \bigcap_{x^* \in F^*} \ker D_{x^*}^1 \psi$. By Lemma 1.2*, there is a finite-codimentional subspace X of E, such that

(*) for every $x \in X$, there exists a nonzero $g \in \tilde{G}$ such that

$$g(x) \geqslant \tfrac{1}{3}\|g\|\,\|x\|.$$

Let $\tilde{H} = H \cap X \cap \bigcap_{x \in F} \ker D_x^1 \varphi$, $M = \sup\{\|D_{x^*}^2 \psi\|\colon x^* \in K^*\}$. We may without loss of generality assume that $M \geqslant 1$. By Lemma 2.1 there exists δ, $0 < \delta \leqslant \frac{1}{2}$, such that if $\|h\| \leqslant \delta$ and $\|g\| \leqslant \delta$, then

$$r(\varphi)(x, h) \leqslant \frac{1}{100M}\|h\|^2 \quad \text{and} \quad r(\psi)(x^*, g) \leqslant \frac{1}{100M}\|g\|^2$$

for all $x \in K$ and $x^* \in K^*$.

By virtue of the Corollary to Lemma 1.4, there is an $h \in \tilde{H}$, with $\|h\| = \delta$, such that $|D_x^2 \varphi(h, h)| \leqslant \dfrac{1}{100M}\delta^2$ for all $x \in F$. For this vector h, choose $g \in \tilde{G}$ so that

$$\|g\| = \frac{1}{10M}\delta \quad \text{and} \quad g(h) \geqslant \tfrac{1}{3}\|g\|\,\|h\| > \frac{1}{30M}\delta^2.$$

Then, for $x \in F$ and $x^* \in F^*$ we have the inequality

$$\begin{aligned} &|\Phi(x \pm h, x^* \pm g) - \Phi(x, x^*)| \\ &\quad = |\tfrac{1}{2} D_x^2 \varphi(h, h) + \tfrac{1}{2} D_{x^*}^2 \psi(g, g) + r(\varphi)(x, \pm h) + r(\psi)(x^*, \pm g)| \\ &\quad \leqslant \frac{1}{200M}\delta^2 + \frac{1}{200M}\delta^2 + \frac{1}{100M}\delta^2 + \frac{1}{100M}\,\frac{1}{100M^2}\delta^2 \\ &\quad \leqslant \frac{3}{100M}\delta^2 < \frac{1}{30M}\delta^2 < g(h). \end{aligned}$$

Thus we might take the number $\dfrac{1}{30M}\delta^2$ for $a_0(K, K^*)$. This demonstrates Lemma 2.2.

Let us proceed directly to the proof of Theorem 2.1. On the basis of Lemmas 1.3 and 2.2, we shall construct a contradiction. We define inductively a sequence of vectors $\{h_n\}_{n=0}^{\infty}$ in E and a sequence of functionals $\{g_n\}_{n=0}^{\infty}$ in E^*.

Set $h_0 = 0$, $g_0 = 0$. Now, suppose that h_k, g_k for $0 \leqslant k \leqslant n$ have already been constructed, and denote by F_n and F_n^* the sets, in E and E^*, respectively,

$$F_n = \Big\{\sum_{k=0}^{n} \varepsilon_k h_k \colon \varepsilon_k = \pm 1,\ k = 0, 1, 2, \ldots, n\Big\},$$

$$F_n^* = \Big\{\sum_{k=0}^{n} \varepsilon_k g_k \colon \varepsilon_k = \pm 1,\ k = 0, 1, 2, \ldots, n\Big\}.$$

Let $H_n = \bigcap_{i=0}^{n} \ker g_i$, $G_n = \bigcap_{i=0}^{n} \{x^* \colon x^*(h_i) = 0\}$, $Q_{n+1} = Q(F_n, F_n^*, H_n, G_n)$, $a_{n+1} = a(F_n, F_n^*, H_n, G_n)$, and choose $(h_{n+1}, g_{n+1}) \in Q_{n+1}$ so that $g_{n+1}(h_{n+1}) \geqslant a_{n+1}/2$.

Notice that these constructions ensure fulfilment of the conditions

(1) $g_k(h_n) = 0$, when $k \neq n$;

(2) $g_k(h_k) \leqslant \frac{1}{4}$.

Let us show first that $\sum_{k=0}^{\infty} g_k(h_k) \geqslant 2$. If $\sum_{k=0}^{\infty} g_k(h_k) < 2$, then

$$\Phi\Big(\sum_{k=0}^{N} \varepsilon_k h_k, \sum_{k=0}^{N} \varepsilon_k g_k\Big) \leqslant \sum_{k=0}^{N-1} \Big|\Phi\Big(\sum_{n=0}^{k+1} \varepsilon_n h_n, \sum_{n=0}^{k+1} \varepsilon_n g_n\Big) - \Phi\Big(\sum_{n=0}^{k} \varepsilon_n h_n, \sum_{n=0}^{k} \varepsilon_n g_n\Big)\Big| \leqslant \sum_{k=1}^{N} g_k(h_k) < 2.$$

Consequently, $\|\sum_{k=0}^{N} \varepsilon_k h_k\| < 1$, $\|\sum_{k=0}^{N} \varepsilon_k g_k\| < 1$ for all N and $\{\varepsilon_k\}$, where $\varepsilon_k = \pm 1$. By Lemma 1.3, there are compact sets K in E and K^* in E^* such that, for all n, $F_n \subset K$ and $F_n^* \subset K^*$. Thus, by Lemma 2.2, $a_n \geqslant a_0(K, K^*)$. As $g_n(h_n) \geqslant a_n/2 \geqslant a_0(K, K^*)/2 > 0$ for all n, this contradicts the convergence of the series $\sum_{k=0}^{\infty} g_k(h_k)$.

Now take N so that $1 < \sum_{m=0}^{N} g_m(h_m) < 2$. Then, from the definition of the sequences $\{h_m\}$ and $\{g_m\}$, we have

$$\Phi\Big(\sum_{m=0}^{N} h_m, \sum_{m=0}^{N} g_m\Big) \leqslant \sum_{m=0}^{N} g_m(h_m) < 2,$$

so that $\|\sum_{m=0}^{N} h_m\| < 1$ and $\|\sum_{m=0}^{N} g_m\| < 1$, whilst $\big(\sum_{m=0}^{N} g_m\big)\big(\sum_{m=0}^{N} h_m\big) = \sum_{m=0}^{N} g_m(h_m) > 1$. This contradiction establishes Theorem 2.1.

3. Proof of Theorem 3.1. In the same fashion as in §2 we may construct a function $\Phi' \colon E \times E^* \to \mathbf{R}$, enjoying the following properties:

(1) $\Phi'(x, x^*) = \varphi(x) + \psi(x^*)$, where φ is H^p-smooth on E and ψ H^q-smooth on E^*,

(2) $\Phi'(0, 0) = 0$, and $\Phi'(x, x^*) \geqslant 2$ when $\max(\|x\|, \|x^*\|) \geqslant 1$.

LEMMA 3.1. *Let K be a compact set in the Banach space E and f an H^p-function on a neighbourhood of K. Then there exist δ, $M > 0$ such that, if $\|h\| < \delta$ and $x \in K$,*

$$f(x+h) = f(x) + D_x^1 f(h) + \frac{1}{2!} D_x^2 f(h, h) + \ldots + \frac{1}{k!} D_x^k f(h, h, \ldots, h) + r(f)(x, h),$$

where k is the greatest natural number strictly less than p, and $|r(f)(x, h)| \leqslant M\|h\|^p$.

Lemma 3.1 results straightforwardly from Taylor's formula, as given in [7], Chapter I, 4,

$$f(x+h) = f(x) + D_x^1 f(h) + \frac{1}{2!} D_x^2 f(h, h) + \ldots$$
$$\ldots + \frac{1}{(k-1)!} D_x^{k-1}(h, h, \ldots, h) + \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} D_{x+th}^k f(h, \ldots, h)\, dt.$$

In §2 we proved Lemma 2.2 for the function Φ, on the hypothesis that E and E^* are C^2-smooth, and, furthermore, that E is not isomorphic to a Hilbert space; now, however, we shall demonstrate the conclusion of Lemma 2.2 for the function Φ', on the hypothesis that $1/p + 1/q < 1$ and that neither of the spaces E, E^* is C^∞-smooth. In the proof we shall employ the notations of Lemma 2.2.

If both p and q exceed 2, then by Theorem 2.1 we know E is isomorphic to a Hilbert space, and consequently C^∞-smooth, which contradicts the hypothesis of Theorem 3.1. Accordingly one of the numbers p, q does not exceed 2. Let us suppose first that $p \leqslant 2$.

Let k be the greatest integer strictly less than q. Then

$$|\Phi'(x \pm h, x^* \pm g) - \Phi'(x, x^*)| = |D_x^1 \varphi(\pm h) + r(\varphi)(x, \pm h) + P_{x^*}(\pm g) + r(\psi)(x^*, \pm g)|,$$

where

$$P_{x^*}(g) = D_{x^*}^1 \psi(g) + \frac{1}{2!} D_{x^*}^2 \psi(g, g) + \ldots + \frac{1}{k!} D_{x^*}^k \psi(g, g, \ldots, g).$$

Set $\tilde{H} = H \cap \bigcap_{x \in F} \ker D_x^1 \varphi$. By Lemma 1.2 we may choose in E^* a finite-codimensional subspace Y such that

$(*)$ for any $g \in Y$ there exists a nonzero $h \in \tilde{H}$, such that $g(h) \geqslant \frac{1}{3}\|g\|\,\|h\|$.

Take $\tilde{G} = G \cap Y$.

Let us now show that for arbitrary positive β and ε there exists $g_0 \in \tilde{G}$ such that $\|g_0\| = \beta$ and, for any $x^* \in F^*$, $|P_{x^*}(\pm g_0)| < \varepsilon$. For this, consider the C^∞ function f on $\tilde{G}$

$$f(g) = \sum_{x^* \in F^*} \big(P^2_{x^*}(g) + P^2_{x^*}(-g)\big).$$

It will suffice to show that $\inf\{f(g)\colon g \in \tilde{G},\ \|g\| = \beta\} = 0$.

Certainly $f(0) = 0$. Therefore, if $\inf\{f(g)\colon g \in \tilde{G},\ \|g\| = \beta\}$ were positive, $\tilde{G}$ would be C^∞-smooth, and as $\tilde{G}$ has finite codimension in E^*, E^* also would be C^∞-smooth, which contradicts the hypothesis of Theorem 3.1.

From Lemma 3.1, there exist positive M and δ (with $\delta \leqslant \frac{1}{2}$) such that $r(\varphi)(x, h) \leqslant M\|h\|^p$ and $|r(\psi)(x^*, g)| \leqslant M\|g\|^q$ for all $x \in K$ and $x^* \in K^*$, provided $\|h\| \leqslant \delta$ and $\|g\| \leqslant \delta$. Without loss of generality we may suppose that $M \geqslant 1$. If $1/p + 1/q < 1$, then $p(q-1) = q + \gamma$ where $\gamma > 0$. Take $\beta > 0$ so that the following conditions are satisfied:

(1) $\beta \leqslant \delta$,
(2) $10M\beta^{q-1} \leqslant \delta$,
(3) $M(10M)^p \cdot \beta^\gamma \leqslant 1$.

Take $g \in \tilde{G}$ such that $\|g\| = \beta$ and, for any $x^* \in F^*$, $|P_{x^*}(\pm g)| \leqslant \beta^q$. By virtue of $(*)$, let us choose $h \in \tilde{H}$ such that $\|h\| = 10M\beta^{q-1}$ and $g(h) \geqslant \frac{1}{3}\|g\|\,\|h\| = \frac{10}{3}M\beta^q$. Then, for any $x \in F$ and $x^* \in F^*$,

$$\begin{aligned}|\Phi'(x \pm h, x^* \pm g) - \Phi'(x, x^*)| &\leqslant |r(\varphi)(x, \pm h) + P_{x^*}(\pm g) + r(\psi)(x^*, \pm g)| \\ &\leqslant M\|h\|^p + \beta^q + M\|g\|^q \\ &\leqslant M(10M)^p \beta^q \beta^\gamma + \beta^q + M\beta^q \\ &\leqslant \beta^q + \beta^q + M\beta^q < \tfrac{10}{3}M\beta^q \leqslant g(h).\end{aligned}$$

Therefore in place of $\alpha_0(K, K^*)$ we may take the number $\frac{10}{3}M\beta^q$. This proves the lemma for the case where $p \leqslant 2$. The proof for the case where $q \leqslant 2$ differs from this one only in employing Lemma 1.2 in place of Lemma 1.2*.

In §2 we concluded the proof of Theorem 2.1 by extracting a contradiction, on the basis only of Lemma 1.3 and Lemma 2.2. For the proof of Theorem 3.1 it remains only to repeat word for word the arguments used to complete the proof of Theorem 2.1.

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