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L_p-approximation by the method of integral Meyer-König and Zeller operators

by

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Dedicated to Professor W. Meyer-König on the occasion of his sixty-fifth birthday.

Abstract. The well-known linear approximation method $(M_n)_{n\in N}$ of W. Meyer-König and K. Zeller on the normed space $(C(I), \|\cdot\|_\infty)$ is extended to a method $(\hat{M}_n)_{n\in N}$ for the L_p-approximation of functions $f\in L_p(I)$, $I=[0,1]$, $1\leqslant p<\infty$ with respect to the L_p-norm on I. Approximation properties and especially the degree of approximation by this method are studied in detail.

1. Preliminaries. It is well known that the nth operator M_n, $n\in N$, of Meyer-König and Zeller is associating with a bounded function $f\colon I=[0,1]\to \boldsymbol{R}$ the so-called nth Bernstein power series

$$M_nf(x)=\sum_{k=0}^{\infty} m_{nk}(x) f\left(\frac{k}{k+n}\right),\qquad m_{nk}(x)=\binom{k+n}{k}(1-x)^{n+1}x^k \tag{1.1}$$

converging for $0\leqslant x<1$. If f is continuous to the left at the point $t=1$, then M_nf can be continuously extended to I by putting

$$M_nf(1):=\lim_{x\to 1-} M_nf(x)=f(1) \tag{1.2}$$

(see [9]). Thus the operators M_n are mapping especially the space $C(I)$ of real-valued continuous functions on I into itself and M_nf can be regarded as an approximation to $f\in C(I)$ on I for each $n\in N$. W. Meyer-König and K. Zeller [9] proved that the sequence $(M_n)_{n\in N}$ gives a linear approximation method on the normed space $(C(I), \|\cdot\|_\infty)$ (with $\|\cdot\|_\infty$ the usual sup-norm on I), i.e. $\lim_{n\to\infty}\|f-M_nf\|_\infty=0$ for all $f\in C(I)$. Its degree of approximation can be estimated by [8]

$$\|f-M_nf\|_\infty\leqslant \tfrac{31}{27}\,\omega_{1,\infty}\left(f,\frac{1}{\sqrt{n}}\right)\qquad (n\in N),$$

where $\omega_{1,\infty}(f,\cdot)$ is the ordinary modulus of continuity of f with respect to the sup-norm.

The aim of this paper is to develop a comparable method for the L_p-approximation of functions $f \in L_p(I)$, $1 \leqslant p < \infty$, the space of real-valued pth power integrable functions on I, with $\|\cdot\|_p$ the usual L_p-norm on I. The corresponding operators $\hat{M}_n$ will roughly speaking be constructed replacing the point evaluations of f in (1.1) at discrete nodes by integral means of f over suitable small and disjoint intervals containing these nodes. For this formal reason we shall refer to them as *integral Meyer-König and Zeller operators*. Their explicit construction follows a method given by G. G. Lorentz [6] when changing Bernstein polynomials into Kantorovič polynomials: Applying the first derivative operator D to $M_n f$, we obtain [7]

$$(1.3)\quad DM_nf(x) = (n+1)(1-x)^n \sum_{k=0}^{\infty} \binom{k+n+1}{k} x^k \left[f\left(\frac{k+1}{k+n+1}\right) - f\left(\frac{k}{k+n}\right)\right],$$

$0 \leqslant x < 1$. If $f \in L_p(I)$, consider the indefinite integral $F(x) = \int_0^x f(t)\,dt$. (1.3) applied to the (absolutely continuous) function F gives for $0 \leqslant x < 1$ and $n \in N$

$$(1.4)\qquad \hat{M}_n f(x) := DM_nF(x) = \sum_{k=0}^{\infty} \hat{m}_{nk}(x) \int_{I_k} f(t)\,dt,$$

with $I_k = \left[\frac{k}{k+n}, \frac{k+1}{k+n+1}\right]$ $(k \in N_0)$ and

$$(1.5)\qquad \hat{m}_{nk}(x) = (n+1)\binom{k+n+1}{k}(1-x)^n x^k.$$

The operators $\hat{M}_n$ are linear, positive and preserve the identity. (Since $\hat{M}_n f$ will be considered as an approximation to a function integrable in the sense of Lebesgue on I, we can define $\hat{M}_n f(1)$ arbitrarily, e.g. $\hat{M}_n f(1) = 0$.) For later reference we list the useful relations:

$$(1.6)\qquad \hat{m}_{nk}(x) \int_{I_k} dt = m_{n-1,k}(x)$$

(where we put formally $m_{0k}(x) = (1-x)x^k$) and

$$(1.7)\qquad \int_0^1 \hat{m}_{nk}(x)\,dx = (n+1)\binom{k+n+1}{k} B(k+1, n+1) = 1.$$

In Section 2 it is shown that the sequence $(\hat{M}_n)_{n \in N}$ gives a linear approximation method on the normed space $(L_p(I), \|\cdot\|_p)$. In Section 3 the degree of approximation by this method is estimated in terms of the first order modulus of continuity $\omega_{1,p}(f, \cdot)$. The main result (Theorem 3) will be that

$$\|f - \hat{M}_n f\|_p = O\left(\omega_{1,p}\left(f, \frac{1}{\sqrt{n}}\right)\right).$$

It should be observed that this order is $O(n^{-\alpha/2})$ if f is belonging to a Lipschitz class $\mathrm{Lip}(\alpha, L_p)$. The method of proof is smoothing, i.e. f is first approximated by a function g with g' in $L_p(I)$ and then g is approximated by $\hat{M}_n g$. The connection between these two processes is given via the K-functional of Peetre.

2. L_p-approximation. Given $f \in L_p(I)$, $1 \leqslant p < \infty$, we write $\hat{M}_n f$ as a singular integral of the type

$$\hat{M}_n f(x) = \int_0^1 H_n(x, t) f(t)\,dt$$

with the positive kernel

$$H_n(x, t) = \sum_{k=0}^{\infty} \hat{m}_{nk}(x) \mathbf{1}_{I_k}(t),$$

where $\mathbf{1}_{I_k}$ is the characteristic function of the interval I_k with respect to I. Utilizing (1.6) and (1.7), we have for all n and x or t respectively

$$(2.1)\qquad \int_0^1 H_n(x, t)\,dt = \sum_{k=0}^{\infty} m_{n-1,k}(x) = 1,$$

$$(2.1)\qquad \int_0^1 H_n(x, t)\,dx = \sum_{k=0}^{\infty} \mathbf{1}_{I_k}(t) = 1$$

and thus by a theorem of W. Orlicz [13] follows easily that $\hat{M}_n f$ belongs to $L_p(I)$ and the operator norms $\|\hat{M}_n\|_p$ are uniformly bounded by 1.

THEOREM 1. *For $f \in L_p(I)$, $1 \leqslant p < \infty$, there holds*

$$(2.3)\qquad \lim_{n\to\infty} \|f - \hat{M}_n f\|_p = 0.$$

Proof. We show that (2.3) holds for the dense subspace $C(I)$ of $L_p(I)$. Using (1.6), we have for $f \in C(I)$ and an arbitrary $x \in I$

$$(2.4)\ |\hat{M}_n f(x) - M_{n-1} f(x)| \leqslant \sum_{k=0}^{\infty} \hat{m}_{nk}(x) \int_{I_k} \left|f(t) - f\left(\frac{k}{k+n-1}\right)\right| dt \qquad (n \geqslant 2).$$

In view of $|t - k/(k+n-1)| < 1/(n-1)$ for $t \in I_k$ we obtain from (2.4) and (1.6)

$$|\hat{M}_n f(x) - M_{n-1} f(x)| \leqslant \omega_{1,\infty}\left(f, \frac{1}{n-1}\right) \sum_{k=0}^{\infty} \hat{m}_{nk}(x) \int_{I_k} dt = \omega_{1,\infty}\left(f, \frac{1}{n-1}\right)$$

and thus

$$\|\hat{M}_n f - M_{n-1}\|_p \leqslant \|\hat{M}_n f - M_{n-1} f\|_\infty \leqslant \omega_{1,\infty}\left(f, \frac{1}{n-1}\right). \tag{2.5}$$

Now

$$\|f - \hat{M}_n f\|_p \leqslant \|f - M_{n-1} f\|_\infty + \|M_{n-1} f - \hat{M}_n f\|_p.$$

For $n \to \infty$ each term goes to zero, the first one since $(M_n)_{n \in N}$ is a linear approximation method on the space $(C(I), \|\cdot\|_\infty)$ and the second one by (2.5), which proves (2.3) for continuous functions.

The rest of the proof follows by the density of $C(I)$ in $L_p(I)$ with respect to the L_p-norm since $\|\hat{M}_n\|_p \leqslant 1$ for all $n \in N$.

As an application of Theorem 1 we obtain the following criterion of compactness for a bounded subset

$$K := \{f \in L_p(I) |\ \|f\|_p \leqslant M,\ M \text{ a positive constant}\}$$

of $L_p(I)$: K is compact with respect to the L_p-norm iff $\|f - \hat{M}_n f\|_p \to 0$ $(n \to \infty)$ uniformly for all $f \in K$.

The method of proof is quite similar to an argument given by G. G. Lorentz ([6], p. 33) for Kantorovič polynomials using the fact that by Hausdorff's criterion of compactness in complete metric spaces (see [3], p. 108) K is compact iff for each $\varepsilon > 0$ there is a finite ε-net.

3. Degree of L_p-approximation. Let $L_p^1(I) := \{f \in L_p(I) | f$ absolutely continuous, $f' \in L_p(I)\}$ $(1 \leqslant p < \infty)$ with the norm $\|f\|_p^1 = \|f\|_p + \|f'\|_p$. The most efficient technique in deriving estimates for the degree of L_p-approximation is smoothing (see [2], [11], [12]). This means

(i) approximation of $f \in L_p(I)$ by a "smooth" function $g \in L_p^1(I)$,

(ii) approximation of $g \in L_p^1(I)$ by the method $(\hat{M}_n)_{n \in N}$,

(iii) combination of steps (i) and (ii) via the K-functional of Peetre.

We start with step (ii). In 1972, D. Leviatan [5] gave an estimate of the desired type for the case $p = 1$, which reads in our notation as follows:

$$\|g - \hat{M}_n g\|_1 < \sqrt{\frac{2}{e}}\, \frac{1}{\sqrt{n}} \int_0^1 \sqrt{x(1-x)}\,|dg(x)|, \quad g \in B_1 \ (n \in N),$$

where B_1 is the set of all functions $g \in L_1(I)$ being of bounded variation in every closed subinterval of $(0, 1)$ and for which the right-hand integral exists. From this we conclude easily that

$$\|g - \hat{M}_n g\|_1 < \frac{1}{\sqrt{2en}} \|g'\|_1, \quad g \in L_1^1(I) \ (n \in N). \tag{3.1}$$

Next we will derive the corresponding estimate for $p > 1$ by a method which is tailored exactly to this case. For its proof we need the following

Lemma. *There exists a positive constant A, independent of $n \in N$ and $x \in I$, such that*

$$\hat{M}_n (t-x)^2(x) \leqslant \frac{A}{n}. \tag{3.2}$$

Observing that

$$\hat{M}_n (t-x)^2(x) = D\{\tfrac{1}{3} M_n t^3(x) - x M_n t^2(x)\} + x^2$$

for an arbitrary $x \in I$, a proof of (3.2) can be based on a careful analysis of the possibility of differentiating asymptotic expressions for $M_n t^2(x)$ and $M_n t^3(x)$ given by P. C. Sikkema ([15], p. 431–433), noticing a lemma in [10], p. 402. Details are left to the reader.

Theorem 2. *For $g \in L_p^1(I)$, $p > 1$, there holds*

$$\|g - \hat{M}_n g\|_p \leqslant \frac{C_p}{\sqrt{n}} \|g'\|_p \quad (n \geqslant 2),$$

where C_p is some positive constant, independent of g and n.

Proof. Fix $x \in [0, 1)$. Then by (1.4) and (1.6)

$$\begin{aligned} |g(x) - \hat{M}_n g(x)| &= \Big|\sum_{k=0}^{\infty} \hat{m}_{nk}(x) \int_{I_k} \int_x^t g'(u)\,du\,dt\Big| \\ &\leqslant \sum_{k=0}^{\infty} \hat{m}_{nk}(x) \int_{I_k} \Big|\int_x^t g'(u)\,du\Big|\,dt \\ &\leqslant \theta_{g'}(x) \sum_{k=0}^{\infty} \hat{m}_{nk}(x) \int_{I_k} |t-x|\,dt, \end{aligned} \tag{3.3}$$

where

$$\theta_{g'}(x) := \sup_{\substack{0 \leqslant t \leqslant 1 \\ t \neq x}} \frac{1}{t-x} \int_x^t |g'(u)|\,du$$

is the Hardy–Littlewood majorant of g'. $g' \in L_p(I)$ implies for $p > 1$ by a theorem of Hardy and Littlewood (see [16], Theorems 13, 15) $\theta_{g'} \in L_p(I)$

with

$$\int_0^1 \theta_{g'}^p(x)\,dx \leqslant 2\left(\frac{p}{p-1}\right)^p \int_0^1 |g'(x)|^p\,dx. \tag{3.4}$$

Applying Cauchy–Schwarz's inequality and the lemma, we obtain from (3.3)

$$|g(x)-\hat{M}_n g(x)| \leqslant \theta_{g'}(x)\left\{\sum_{k=0}^{\infty} \hat{m}_{nk}(x)\int_{I_k}(t-x)^2\,dx\right\}^{1/2}$$

$$= \theta_{g'}(x)\,\{\hat{M}_n(t-x)^2(x)\}^{1/2} \leqslant \sqrt{A}\,\theta_{g'}(x)\,\frac{1}{\sqrt{n}}$$

and from this by (3.4) for $p > 1$

$$\|g-\hat{M}_n g\|_p \leqslant \sqrt{A}\,\frac{1}{\sqrt{n}}\left(\int_0^1 \theta_{g'}^p(x)\,dx\right)^{1/p} \leqslant \sqrt{A}\,\frac{p}{p-1}\sqrt[p]{2}\,\frac{1}{\sqrt{n}}\,\|g'\|_p,$$

which completes the proof.

In what follows we will measure smoothness by using the K-functional of J. Peetre [14]. It is for $f \in L_p(I)$, $1 \leqslant p < \infty$, defined by

$$K_p(t,f) = \inf_{g\in L_p^1}(\|f-g\|_p + t\|g'\|_p) \qquad (0 \leqslant t \leqslant 1). \tag{3.5}$$

Roughly speaking the K-functional is a semi-norm on $L_p(I)$ measuring the degree of approximation of a function $f \in L_p(I)$ by smoother functions $g \in L_p^1(I)$ with simultaneous control on the size of $\|g'\|_p$.

The more classical measure for smoothness, the integral modulus of continuity, which for $f \in L_p(I)$, $1 \leqslant p < \infty$, is defined by

$$\omega_{1,p}(f,t) := \sup_{0<h\leqslant t}\|f(\cdot+h)-f(\cdot)\|_p(I_h) \tag{3.6}$$

(where $\|\cdot\|(I_h)$ is indicating that the L_p-norm is to be taken over the interval $I_h = [0, 1-h]$) is in a certain sense equivalent to the K-functional. H. Johnen ([4], Prop. 6.1) proved that there are constants $c_1 > 0$ and $c_2 > 0$, independent of f and p, such that

$$c_1\omega_{1,p}(f,t) \leqslant K_p(t,f) \leqslant c_2\omega_{1,p}(f,t) \qquad (0 \leqslant t \leqslant 1). \tag{3.7}$$

THEOREM 3. *For $f \in L_p(I)$, $1 \leqslant p < \infty$, there holds*

$$\|f-\hat{M}_n f\| \leqslant M\omega_{1,p}\left(f, \frac{1}{\sqrt{n}}\right) \qquad (n \geqslant 2),$$

where M is some positive constant, independant of f and p.

Proof. Let $D_p = \max(C_p, 1/\sqrt{2e})$. In view of (3.1), Theorem 2 and $\|\hat{M}_n\|_p \leqslant 1$ $(n \in N,\ 1 \leqslant p < \infty)$, we have

$$\|h-\hat{M}_n h\|_p \leqslant \begin{cases} 2\|h\|_p, & h \in L_p(I) \\ \dfrac{D_p}{\sqrt{n}}\|h'\|_p, & h \in L_p^1(I) \end{cases} \qquad (1 \leqslant p < \infty,\ n \geqslant 2).$$

When $f \in L_p(I)$ and g is an arbitrary function from $L_p^1(I)$, then

$$\|f-\hat{M}_n f\|_p \leqslant \|(f-g)-\hat{M}_n(f-g)\|_p + \|g-\hat{M}_n g\|_p$$

$$\leqslant 2\left(\|f-g\|_p + \frac{D_p}{\sqrt{n}}\|g'\|_p\right).$$

Taking now the infimum over all $g \in L_p^1(I)$ on the right-hand side, using the definition of the K-functional and observing (3.6), we find

$$\|f-\hat{M}_n f\|_p \leqslant 2K\left(\frac{D_p}{\sqrt{n}}, f\right) \leqslant 2c_2\,\omega_{1,p}\left(f, \frac{D_p}{\sqrt{n}}\right)$$

$$\leqslant 2(1+D_p)\,\omega_{1,p}\left(f, \frac{1}{\sqrt{n}}\right),$$

which completes the proof.

COROLLARY: *If $f \in \mathrm{Lip}(\alpha, L_p)$ $(0 < \alpha \leqslant 1)$, then*

$$\|f-\hat{M}_n f\|_p = O(n^{-\alpha/2}) \qquad (n\to\infty).$$

Here the Lipschitz class $\mathrm{Lip}(\alpha, L_p)$ of order α with respect to the L_p-norm is defined as the collection of all functions $f \in L_p(I)$ with the property $\omega_{1,p}(f,t) = O(t^\alpha)$ $(t\to 0+)$.

Remark. The last results can still be made more transparent if one considers the family of intermediate spaces $[L_p^1, L_p]_\alpha$, $0 \leqslant \alpha \leqslant 1$, between $L_p^1(I)$ and $L_p(I)$, constructed from these spaces by means of some modification of the K-functional (see [11], [12]). It can easily be proved that $f \in [L_p^1, L_p]_\alpha$ is equivalent to $f \in \mathrm{Lip}(\alpha, L_p)$, $0 < \alpha \leqslant 1$. Thus the corollary tells that elements of an intermediate space $[L_p^1, L_p]_\alpha$ between $L_p^1(I)$ and $L_p(I)$ are approximated by the method of integral Meyer-König and Zeller operators with respect to the L_p-norm of the order $O(n^{-\alpha/2})$.

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Some characterizations of the n-dimensional Peano derivative

by

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Abstract. A measurable function f is said to have a *Peano derivative* of order k at a point x if there is a polynomial P of degree at most k with the property that $f(x+t) = P(t)+o(|t|)^k$. This work gives a characterization of the Peano derivative for functions of several variables in terms of the behaviour of the expression

$$\sum_{i=1}^{N} A_i f(x+\varrho\sigma(v_i)) - \Big(\sum_{i=1}^{N} A_i\Big) f(x).$$

The A_i are real numbers and the v_i are points on the unit sphere, $\varrho > 0$ and $\sigma \in SO(n)$. Almost everywhere results similar to those of Denjoy–Rademacher–Stepanov, Khintchine, and Stein–Zygmund are obtained in this non-linear setting.

The techniques involve boundary behavior of harmonic functions and analysis on $SO(n)$. When n is greater than 2 the non-commutativity of $SO(n)$ requires special treatment. A technique, introduced by Stein and Zygmund, is developed which allows one to substitute a certain convolution with a central function for a convolution with a zonal function.

Introduction. The purpose of this paper is to present an extension and a unification of several of the characterizations of the n-dimensional Peano derivative. Our characterizations will be stated as a description of the behavior of functions restricted to spheres centered at points of possible differentiability. The action of the rotation group on the sphere will play a significant role.

We say that a function f, defined on a neighborhood of a point x in $\boldsymbol{R}^n$, has a kth *Peano derivative at* x if there is a polynomial P of degree at most k such that $f(x+t) = P(t)+o(|t|^k)$. When $k = 1$, this is the ordinary derivative. When k is greater than 1, f need not be $k-1$ differentiable near x to have a kth Peano derivative at x.

We consider in this paper configurations consisting of a finite number of points on the unit sphere in $\boldsymbol{R}^n$, $v_1, \ldots, v_N$. We assign each point a non-zero weight A_i. The origin is given the weight $B = -\sum_{i=1}^{N} A_i$. To each configuration we associate an integer type m. The integer m is defined as the infimum of the degrees of all polynomials for which $\sum_{i=1}^{N} A_i P(v_i)+$