L_p-approximation by the method of integral Meyer-König and Zeller operators

by

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Dedicated to Professor W. Meyer-König on the occasion of his sixty-fifth birthday.

Abstract. The well-known linear approximation method \( (M_n)_{n \in \mathbb{N}} \) of W. Meyer-König and K. Zeller on the normed space \( C(I), \| \cdot \|_m \) is extended to a method \( (M_n)_{n \in \mathbb{N}} \) for the \( L_p \)-approximation of functions \( f \in L_p(I), I = [0,1], 1 \leq p < \infty \) with respect to the \( L_p \)-norm on \( I \). Approximation properties and especially the degree of approximation by this method are studied in detail.

1. Preliminaries. It is well known that the \( n \)th operator \( M_n \), \( n \in \mathbb{N} \), of Meyer-König and Zeller is associated with a bounded function \( f : I = [0,1] \rightarrow \mathbb{R} \) the so-called \( n \)th Bernstein power series

\[
M_n f(x) = \sum_{k=0}^{n} m_{nk}(x) \left( \frac{k}{n+1} \right), \quad m_{nk}(x) = \binom{n}{k} (1-x)^{n-k} x^k
\]

converging for \( 0 \leq x < 1 \). If \( f \) is continuous to the left at the point \( t = 1 \), then \( M_n f \) can be continuously extended to \( I \) by putting

\[
M_n f(1) = \lim_{x \to 1^-} M_n f(x)
\]

(see [9]). Thus the operators \( M_n \) are mapping especially the space \( C(I) \) of real-valued continuous functions on \( I \) into itself and \( M_n f \) can be regarded as an approximation to \( f \in C(I) \) on \( I \) for each \( n \in \mathbb{N} \). W. Meyer-König and K. Zeller [9] proved that the sequence \( (M_n)_{n \in \mathbb{N}} \) gives a linear approximation method on the normed space \( (C(I), \| \cdot \|_m) \) (with \( \| \cdot \|_m \) the usual sup-norm on \( I \)), i.e., \( \lim_{n \to \infty} \| f - M_n f \|_m = 0 \) for all \( f \in C(I) \). Its degree of approximation can be estimated by [8]

\[
\| f - M_n f \|_m \leq \omega_{1,m}(f, \frac{1}{n}) \quad (n \in \mathbb{N}),
\]

where \( \omega_{1,m}(f, \cdot) \) is the ordinary modulus of continuity of \( f \) with respect to the sup-norm.
The aim of this paper is to develop a comparable method for the $L_p$-approximation of functions $f \in L_p(I), \ 1 \leq p < \infty$, the space of real-valued pth power integrable functions on $I$, with $\| \cdot \|_p$ the usual $L_p$-norm on $I$. The corresponding operators $\hat{M}_n$ will roughly speaking be constructed replacing the point evaluations of $f$ in (1.1) at discrete nodes by integral means of $f$ over suitable small and disjoint intervals containing these nodes. For this formal reason we shall refer to them as *integral Meyer-König and Zeller operators*. Their explicit construction follows a method given by G. G. Lorentz [6] when changing Bernstein polynomials into Kantorovich polynomials: Applying the first derivative operator $D$ to $M_n f$, we obtain [7]

\begin{equation}
D M_n f(x) = (n+1)(1-x)^n \sum_{k=0}^{m} \binom{k+n+1}{k} x^k \int \left[ f \left( \frac{k+1}{k+n+1} \right) - f \left( \frac{k}{k+n} \right) \right] dt.
\end{equation}

0 \leq x < 1. If $f \in L_p(I)$, consider the indefinite integral $F(x) = \int f(t) dt$. (1.3) applied to the (absolutely continuous) function $F$ gives for $0 \leq x < 1$ and $n \in \mathbb{N}$

\begin{equation}
\hat{M}_n f(x) = D M_n F(x) = \sum_{k=0}^{m} \hat{m}_k(x) \int_{I_k} f(t) dt,
\end{equation}

with $I_k = \left[ \frac{k}{k+n}, \frac{k+1}{k+n+1} \right)$ ($k \in \mathbb{N}_0$) and

\begin{equation}
\hat{m}_k(x) = (n+1) \binom{k+n+1}{k} (1-x)^n x^k.
\end{equation}

The operators $\hat{M}_n$ are linear, positive and preserve the identity. (Since $\hat{M}_n f$ will be considered as an approximation to a function integrable in the sense of Lebesgue on $I$, we can define $\hat{M}_n f(1)$ arbitrarily, e.g. $\hat{M}_n f(1) = 0$.) For later reference we list the useful relations:

\begin{equation}
\hat{m}_k(x) \int_{I_k} dt = m_{n-1-k}(x)
\end{equation}

(2.1) and

\begin{equation}
\hat{m}_k(x) \int_{I_k} f(t) dt = (n+1) \binom{k+n+1}{k} B(k+1, n+1) = 1.
\end{equation}

In Section 2 it is shown that the sequence $(\hat{M}_n f)_{n \geq 0}$ gives a linear approximation method on the normed space $(L_p(I), \| \cdot \|_p)$. In Section

3 the degree of approximation by this method is estimated in terms of the first order modulus of continuity $\omega_{1, p}(f, \cdot)$. The main result (Theorem 3) will be that

\begin{equation}
\| f - \hat{M}_n f \|_p = O \left( \omega_{1, p} \left( f, \frac{1}{\sqrt{n}} \right) \right).
\end{equation}

It should be observed that this order is $O(n^{-m})$ if $f$ is belonging to a Lipschitz class $\text{Lip}(a, L_p)$. The method of proof is smoothing, i.e., $f$ is first approximated by a function $g$ with $g' \in L_p(I)$ and then $g$ is approximated by $\hat{M}_n g$. The connection between these two processes is given via the $K$-functional of Peetre.

2. $L_p$-approximation. Given $f \in L_p(I), \ 1 \leq p < \infty$, we write $\hat{M}_n f$ as a singular integral of the type

\begin{equation}
\hat{M}_n f(x) = \frac{1}{c} \int_{I_k} H_n(x, t) f(t) dt
\end{equation}

with the positive kernel

\begin{equation}
H_n(x, t) = \sum_{k=0}^{m} m_k(x) I_k(t),
\end{equation}

where $I_k$ is the characteristic function of the interval $I_k$ with respect to $I$. Utilizing (1.6) and (1.7), we have for all $n$ and $x$ or $t$ respectively

\begin{equation}
\int_{I_k} H_n(x, t) dt = \sum_{k=0}^{m} m_{n-1-k}(x) = 1,
\end{equation}

and thus by a theorem of W. Orlicz [13] follows easily that $\hat{M}_n f$ belongs to $L_p(I)$ and the operator norms $\| \hat{M}_n \|_{L_p(I)}$ are uniformly bounded by 1.

**Theorem 1.** For $f \in L_p(I), \ 1 \leq p < \infty$, there holds

\begin{equation}
\lim_{n \to \infty} \| f - \hat{M}_n f \|_p = 0.
\end{equation}

**Proof.** We show that (2.3) holds for the dense subspace $C(I)$ of $L_p(I)$. Using (1.6), we have for $f \in C(I)$ and an arbitrary $x \in I$

\begin{equation}
\int_{I_k} \hat{m}_k(x) \int_{I_k} f(t) dt = \sum_{k=0}^{m} \hat{m}_k(x) \int_{I_k} f(t) dt = \frac{1}{c} \int_{I_k} f(t) dt = 1.
\end{equation}

In view of $|t - k/(k+n-1)| < 1/(n-1)$ for $t \in I_k$ we obtain from (2.4)
and (1.6)
\[ |\hat{M}_a f(x) - M_{a+1} f(x)| \leq \omega_{1,\infty}(f) \sum_{k=1}^{n} \hat{m}_{ak}(x) f_k(x) dt = \omega_{1,\infty}(f, 1/n) \]
and thus
\[ |\hat{M}_a f - M_{a+1} f|_{\infty} \leq \omega_{1,\infty}(f, 1/n - 1) \]
Now
\[ \|f - M_{a+1} f|_{p} \leq \|f - M_{a+1} f|_{\infty} \|M_{a+1} f - \hat{M}_a f|_p \]
For \( n \to \infty \) each term goes to zero, the first one since \((M_{a+1} f)\) is a linear approximation method on the space \( O(1) \), \( \|\| \sigma \| \) and the second one by (2.5), which proves (2.3) for continuous functions.

The proof of this follows by the density of \( O(1) \) in \( L_p(I) \) with respect to the \( L_p \)-norm since \( \|M_{a+1} f\|_p \leq 1 \) for all \( n \in N \).

As an application of Theorem 1 we obtain the following criterion of compactness for a bounded subset
\[ K = \{ f \in L_p(I) : \|f\|_p \leq M, M \text{ a positive constant} \} \]
of \( L_p(I) \): \( K \) is compact with respect to the \( L_p \)-norm iff \( \|f - \hat{M}_a f\|_p \to 0 \) (\( n \to \infty \)) uniformly for all \( f \in K \).

The method of proof is quite similar to an argument given by G. G. Lorentz [6], p. 33 for Kantorovich polynomials using the fact that by Hausdorff's criterion of compactness in complete metric spaces (see [3], p. 108) \( K \) is compact iff for each \( \epsilon > 0 \) there is a finite \( s \)-net.

3. Degree of \( L_p \)-approximation. Let \( L_p(I) := \{ f \in L_p(I) : f \text{ absolutely continuous}, f' \in L_p(I) \} \) \((1 < p < \infty)\) with the norm \( \|f\|_{L_p}^p = \|f\|_p + \|f'\|_p \). The most efficient technique in deriving estimates for the degree of \( L_p \)-approximation is smoothing (see [3], [11], [12]). This means
(i) approximation of \( f \in L_p(I) \) by a "smooth" function \( g \in L_p(I) \),
(ii) approximation of \( g \in L_p(I) \) by the method \((M_{a+1} f)\),
(iii) combination of steps (i) and (ii) via the \( K \)-functional of Poore.

We start with step (ii). In 1973, D. Leviatan [3] gave an estimate of the desired type for the case \( p = 1 \), which reads in our notation as follows:
\[ \|g - \hat{M}_a g\|_p < \sqrt{\frac{2}{\epsilon}} \frac{1}{\sqrt{n}} \int_{\epsilon}^{1} \|g'(a)\|_p (1 - a) da \]
where \( B_1 \) is the set of all functions \( g \in L_p(I) \) being of bounded variation in every closed subinterval of \((0, 1)\) and for which the right-hand integral exists. From this we conclude easily that
\[ \|g - \hat{M}_a g\|_p < \frac{1}{\sqrt{2n}} \|g'\|_p, \quad g \in L_p(I) \quad (n \in N). \]

Next we will derive the corresponding estimate for \( p > 1 \) by a method which is tailored exactly to this case. For its proof we need the following LEMMA. There exists a positive constant \( A \), independent of \( n \in N \) and \( x \in I \), such that
\[ \hat{M}_a (t - x)^2 (a) < \frac{A}{n} \]
Observing that
\[ \hat{M}_a (t - x)^2 (a) = D (\hat{M}_a t^2 (x) - a \hat{M}_a t^2 (x) + x^2) \]
for an arbitrary \( x \in I \), a proof of (3.2) can be based on a careful analysis of the possibility of differentiating asymptotic expressions for \( M_a t^2 (x) \) and \( \hat{M}_a t^2 (x) \) given by P. C. Sikkema ([15], p. 431-433), noticing a lemma in [10], p. 469. Details are left to the reader.

THEOREM 2. For \( g \in L_p(I), p > 1 \), there holds
\[ \|g - \hat{M}_a g\|_p < \frac{C_p}{\sqrt{n}} \|g'\|_p \quad (n \geq 2) \]
where \( C_p \) is some positive constant, independent of \( g \) and \( n \).

Proof. Fix \( x \in [0, 1] \). Then by (1.4) and (1.6)
\[ \|g(x) - \hat{M}_a g(x)\|_p = \sum_{k=1}^{n} \hat{m}_{ak}(x) \int_{\epsilon}^{1} \|g'(a)\|_p (1 - a) da \]
\[ \leq \sum_{k=1}^{n} \hat{m}_{ak}(x) \int_{\epsilon}^{1} |g'(a)| \int_{\epsilon}^{1} |g'(a)| \int_{\epsilon}^{1} (1 - a) da \]
\[ \leq \theta_p (x) \sum_{k=1}^{n} \hat{m}_{ak}(x) \int_{\epsilon}^{1} |t - x| \int_{\epsilon}^{1} (1 - a) da \]
where
\[ \theta_p (x) := \sup_{x \in [0, 1]} \frac{1}{t - x} \int_{\epsilon}^{1} |g'(a)| \int_{\epsilon}^{1} (1 - a) da \]
is the Hardy-Littlewood majorant of \( g' \), \( g' \in L_p(I) \) implies for \( p > 1 \) by a theorem of Hardy and Littlewood (see [16], Theorems 13, 15) \( \theta_p \in L_p(I) \)
with

\[
\int_0^1 \theta_p(x) \, dx \leq \left( \frac{p}{p-1} \right)^{1/p} \int_0^1 \left| g(x) \right|^p \, dx.
\]

Applying Cauchy-Schwarz's inequality and the lemma, we obtain from (3.3)

\[
|g(x) - \bar{M}_a g(x)| \leq \theta_p(x) \left( \sum_{n=1}^{\infty} \frac{\theta_n(x)}{V_n} \right)^{1/2} \left\| \tau \right\|.
\]

and from this by (3.4) for \( p > 1 \)

\[
\|g - \bar{M}_a g\|_p \leq \left( \frac{1}{V_n} \frac{1}{p-1} \frac{p}{p-1} \frac{1}{V_n} \right)^{1/2} \|g\|_p,
\]

which completes the proof.

In what follows we will measure smoothness by using the K-functional of J. Peetre [14]. It is for \( f \in L_p(I) \), \( 1 \leq p < \infty \), defined by

\[
K_p(f, t) = \inf \left\{ \|f - g\|_p + t \left( \int_0^1 \left| g(x) \right|^p \, dx \right)^{1/p} \right\},
\]

for \( t > 0 \).

Roughly speaking the K-functional is a semi-norm on \( L_p(I) \) measuring the degree of approximation of a function \( f \in L_p(I) \) by smoother functions \( g \in L^r_p(I) \) with simultaneous control on the size of \( \|g\|_p \).

The more classical measure for smoothness, the integral modulus of continuity, which for \( f \in L_p(I), 1 \leq p < \infty \), is defined by

\[
\omega_{p, n}(f, t) := \sup_{x \in [0, 1]} |(t + x) - (t + y)|^p \left( \frac{1}{V_n} \sum_{n=1}^{\infty} \frac{\theta_n(x)}{V_n} \right)^{1/2},
\]

(3.3)

for \( t \in [0, 1] \).

(3.4)

where \( \|g\|_p \) is indicating that the \( L_p \)-norm is to be taken over the interval \( I_n = (0, 1 - h) \) is in a certain sense equivalent to the \( K \)-functional.

H. Johnen ([4], Prop. 6.1) proved that there are constants \( c_2 > 0 \) and \( c_1 > 0 \), independent of \( f \) and \( p \), such that

\[
\omega_{p, n}(f, t) \leq K_p(f, t) \leq c_1 \omega_{p, n}(f, t) \quad (0 \leq t \leq 1).
\]

(3.5)

Theorem 3. For \( f \in L_p(I), 1 \leq p < \infty \), there holds

\[
\|f - \bar{M}_a f\|_p \leq \left( \frac{1}{V_n} \right) \left( \frac{1}{p} \right)^{1/p} \left( \frac{1}{V_n} \right)^{1/p} \|g\|_p,
\]

where \( \bar{M}_a \) is some positive constant, independent of \( f \) and \( p \).

Proof. Let \( \Delta_n = \max \{ \|g\|_p, 1/\sqrt{2} \} \). In view of (3.1), Theorem 2 and \( \|\bar{M}_a g\|_p \leq 1 \) (\( a \in N, 1 \leq p < \infty \)), we have

\[
\|f - \bar{M}_a f\|_p \leq \left( \frac{1}{V_n} \right) \left( \frac{1}{p} \right)^{1/p} \left( \frac{1}{V_n} \right)^{1/p} \|g\|_p.
\]

Then \( f \in L_p(I) \) and \( g \) is an arbitrary function from \( L^r_p(I) \), then

\[
\|f - \bar{M}_a f\|_p \leq \|f - g\|_p + \|g - \bar{M}_a g\|_p
\]

\[
\leq \left( \frac{1}{V_n} \right) \left( \frac{1}{p} \right)^{1/p} \left( \frac{1}{V_n} \right)^{1/p} \|g\|_p.
\]

Taking now the infimum over all \( g \in L^r_p(I) \) on the right-hand side, using the definition of the K-functional and observing (3.6), we find

\[
\|f - \bar{M}_a f\|_p \leq \left( \frac{1}{V_n} \right) \left( \frac{1}{p} \right)^{1/p} \left( \frac{1}{V_n} \right)^{1/p} \|g\|_p,
\]

which completes the proof.

Corollary: If \( f \in L_p(a, b) \), then

\[
\|f - \bar{M}_a f\|_p = O(n^{-\omega}(n \to \infty)).
\]

References


Some characterizations of the n-dimensional Peano derivative

by

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Abstract. A measurable function $f$ is said to have a Peano derivative of order $k$ at a point $x$ if there is a polynomial $P$ of degree at most $k$ with the property that $f(x + t) = P(t) + o(|t|^k)$. This work gives a characterization of the Peano derivative for functions of several variables in terms of the behaviour of the expression

$$
\sum_{i=1}^{n} A_i f(x + \sigma \cdot y) - (\sum_{i=1}^{n} A_i) f(x).
$$

The $A_i$ are real numbers and the $y_i$ are points on the unit sphere, $\sigma > 0$ and $\sigma \in S_0(n)$. Almost everywhere results similar to those of Denjoy–Rademacher–Stepanov, Khaltofo, and Stein-Zygmund are obtained in this non-linear setting.

The techniques involve boundary behavior of harmonic functions and analysis on $S_0(n)$. When $n$ is greater than 2 the non-commutativity of $S_0(n)$ requires special treatment. A technique, introduced by Stein and Zygmund, is developed which allows one to substitute a certain convolution with a central function for a convolution with a zonal function.

Introduction. The purpose of this paper is to present an extension and a unification of several of the characterizations of the n-dimensional Peano derivative. Our characterizations will be stated as a description of the behavior of functions restricted to spheres centered at points of possible differentiability. The action of the rotation group on the sphere will play a significant role.

We say that a function $f$, defined on a neighborhood of a point $x$ in $\mathbb{R}^n$, has a $k$th Peano derivative at $x$ if there is a polynomial $P$ of degree at most $k$ such that $f(x + t) = P(t) + o(|t|^k)$. When $k = 1$, this is the ordinary derivative. When $k$ is greater than 1, $f$ need not be $k-1$ differentiable near $x$ to have a $k$th Peano derivative at $x$.

We consider in this paper configurations consisting of a finite number of points on the unit sphere in $\mathbb{R}^n$, $v_1, \ldots, v_N$. We assign each point a non-zero weight $A_i$. The origin is given the weight $B = -\sum A_i$. To each configuration we associate an integer type $m$. The integer $m$ is defined as the infimum of the degrees of all polynomials for which $\sum_{i=1}^{N} A_i f(v_i) +$