STUDIA MATHEMATICA, T. LXIII. (1978)

On the dual of weighted H^1 of the half-space*

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Abstract. We show that the dual of weighted H^1 of the half-space can be identified with a weighted analogue of the space of functions of bounded mean oscillation. A dense subset of weighted H^1 consisting of "smooth" functions is constructed, and several characterizations of functions of weighted bounded mean oscillation are obtained.

§1. Introduction. The purpose of this paper is to extend the results of [11] to functions of more than one variable, the central problem being the identification of the dual of a weighted version of the Hardy–Stein–Weiss space H^1.

Given a non-negative weight $w(x)$, x denoting a point of n-dimensional Euclidean space E^n, L^1_w denotes the collection of real-valued f such that

$$\|f\|_{L^1_w} = \int_{E^n} |f(x)|\, w(x)\, dx < +\infty.$$

By H^1_w we mean all $(n+1)$-tuples $F(x,t) = \big(u(x,t), v_1(x,t), \ldots, v_n(x,t)\big)$, $(x,t) \in E^{n+1}_+ = \{(x,t)\colon x\in E^n, t>0\}$ such that

(a) F satisfies the Cauchy–Riemann equations in

$$E^{n+1}_+\colon \ \frac{\partial u}{\partial t} = -\sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}; \quad \frac{\partial v_j}{\partial x_k} = \frac{\partial v_k}{\partial x_j},\ j\neq k, \quad \text{and} \quad \frac{\partial v_j}{\partial t} = \frac{\partial u}{\partial x_j};$$

(b) $|||F||| = \sup_{t>0} \|F(\cdot,t)\|_{L^1_w} < +\infty$.

In (b), $\|F(\cdot,t)\|_{L^1_w}$ just means

$$\int_{E^n} \Big[u^2(x,t) + \sum_{1}^{n} v_j^2(x,t)\Big]^{1/2} w(x)\, dx,$$

* Supported in part by NSF grant MPS 75-07596.

so that $|||F|||$ is comparable to

$$\sup_{t>0}\|u(\cdot,t)\|_{L^1_w}+\sum_1^n \sup_{t>0}\|v_j(\cdot,t)\|_{L^1_w}.$$

We assume that w satisfies some A_p condition, i.e., that there is a constant c such that for every "cube" I in E^n

$$(A_p)\qquad \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1}\leqslant c,\quad 1<p<\infty,$$

$$(A_1)\qquad \frac{1}{|I|}\int_I w(x)\,dx\leqslant c\,\mathrm{ess}_I\inf w.$$

The condition that $w\in A_1$ is equivalent to $w^*(x)\leqslant cw(x)$ a.e., where w^* is the Hardy–Littlewood maximal function of w.

The closer p becomes to 1, the stronger the condition that $w\in A_p$ becomes. In different theorems, we need different assumptions on the value of p. For example, if $w\in A_{n/(n-1)}$, any $F(x,t)$ in H^1_w has boundary values $F(x,0)$ pointwise a.e. and in L^1_w norm as $t\to 0$, and

$$|||F|||\approx\|F(\cdot,0)\|_{L^1_w}$$

(see [10], Theorem 2). We can then think of H^1_w as a subset of $L^1_w\oplus\ldots\oplus L^1_w$ by identifying an $F\in H^1_w$ with its boundary values $F(x,0)=(f,f_1,\ldots,f_n)$. Viewed in this way, H^1_w is a closed subspace of $L^1_w\oplus\ldots\oplus L^1_w$ (see the proof of Theorem 3 below).

For our purposes, it will be convenient to have a characterization of H^1_w in terms of Riesz transforms. To obtain this, we strengthen the assumption on w to A_1. For $w\in A_1$, it is proved in [10] that a Cauchy–Riemann system F, $F=(u,v_1,\ldots,v_n)$, is in H^1_w if and only if u and $v_1,\ldots,v_n$ are respectively the Poisson and conjugate Poisson integrals of an $f\in L^1_w$ each of whose Riesz transforms $R_1f,\ldots,R_nf$ is also in L^1_w. By R_jf, $j=1,\ldots,n$, we mean simply

$$R_jf(x)=\lim_{\varepsilon\to 0+}c_n\int_{|y|>\varepsilon}f(x-y)\frac{y_j}{|y|^{n+1}}\,dy,\qquad c_n=\Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2},$$

$y=(y_1,\ldots,y_n)$, where the limit is taken in the usual pointwise sense; R_jf exists a.e. if $f\in L^1_w$, $w\in A_1$ (see [1]). The boundary value $F(x,0)$ of F is $(f,R_1f,\ldots,R_nf)$, and we can think of H^1_w, $w\in A_1$, as the subspace of the direct sum of $n+1$ copies of L^1_w consisting of all $(f,R_1f,\ldots,R_nf)$ with

$$\|(f,R_1f,\ldots,R_nf)\|_{H^1_w}=\|f\|_{L^1_w}+\sum_1^n\|R_jf\|_{L^1_w}<+\infty.$$

It may be of interest to point out here that unlike unweighted H^1, H^1_w may contain some $(f,R_1f,\ldots,R_nf)$ with $f\geqslant 0$. For example, let $n=1$, $w(x)=|x|^a$, $-1<a<0$, and let f be the characteristic function of an interval (a,b). Then up to a multiplicative constant, the Hilbert transform $\tilde f$ of f equals $\log|(a-x)/(b-x)|$. Since this has only logarithmic singularities locally and is of the order of magnitude $O(|x|^{-1})$ as $|x|\to\infty$, it follows that $(f,\tilde f)\in H^1_w$.

To simplify notation, we will write $f\in H^1_w$ if $(f,R_1f,\ldots,R_nf)$ is the boundary value of an F in H^1_w, $w\in A_{n/(n-1)}$. When this is the case, we shall also write $\|f\|_{H^1_w}$ for $|||F|||$.

The class BMO_w of functions of weighted bounded mean oscillation, defined in analogy with the class of John and Nirenberg [6], consists of those locally integrable b such that

$$\int_I|b(x)-b_I|\,dx\leqslant Bm_w(I),$$

where I is a cube in E^n, $b_I=|I|^{-1}\int_I b(x)\,dx$, $m_w(I)=\int_I w(x)\,dx$, and B is a constant independent of I (see [8], [2]). The least such B will be denoted by $\|b\|_*$. Two functions differing by a constant will be identified. Any function bounded by a multiple of w clearly belongs to BMO_w.

Our main result is that the dual of H^1_w can be identified with BMO_w. The proof is based on C. Fefferman's proof for the case $w=1$. Various aspects of the proof require different assumptions on w. The strongest is A_1, and this is used in the most technical part of the proof—namely, that dealing with the existence of a dense subset of H^1_w consisting of smooth functions.

In the case $w=1$, a class of smooth functions dense in H^1 is constructed in [9], p. 230. The same class turns out to be dense in H^1_w, $w\in A_1$. To describe the class, let $\mathcal{S}$ denote the Schwartz space of rapidly decreasing functions, and let $\hat f$ denote the Fourier transform of f. Let

$$\mathscr{C}=\{f\colon f\in\mathcal{S},\ \hat f\text{ has compact support not containing }0\}.$$

We shall see later that if $f\in\mathscr{C}$, then each $R_jf\in\mathscr{C}$. Therefore, if $f\in\mathscr{C}$, f and the R_jf belong to L^1. Moreover, since

$$\int_{E^n}\frac{w(x)}{1+|x|^{np}}\,dx<+\infty$$

for any $w\in A_p$, $1<p<\infty$ (see, e.g., the discussion concerning condition B_p below), it follows that f and the R_jf belong to L^1_w for any $w\in A_p$. By the results of [10] (see Theorem 3 there and the remarks after it), $f\in H^1_w$ for any $w\in A_{n/(n-1)}$. Hence, $\mathscr{C}\subset H^1_w$ if $w\in A_{n/(n-1)}$.

THEOREM 1. *If $w \in A_1$, then $\mathscr{C}$ is a dense subset of H^1_w.*

This will be proved in Section 4, after the proof of the duality results.

The next two theorems state the main results of the paper.

THEOREM 2. *Let $w \in A_{(n+1)/n}$. Then there is a constant c such that*

$$\Big|\int_{E^n} f(x)b(x)\,dx\Big| \leqslant c\|f\|_{H^1_w}\|b\|_*$$

for all $f \in \mathscr{C}$ and $b \in \mathrm{BMO}_w$.

If $w \in A_{(n+1)/n}$, it also belongs to $A_{n/(n-1)}$, so that as mentioned before, any $F(x,t) \in H^1_w$ can be identified with its boundary values $F(x,0) = (f(x), f_1(x), \ldots, f_n(x))$ and $|||F||| \approx \|F(x,0)\|_{L^1_w}$. A continuous linear functional l on H^1_w can therefore be thought of as a linear functional on these boundary values which satisfies

$$|l(f, f_1, \ldots, f_n)| \leqslant c\Big[\|f\|_{L^1_w} + \sum_1^n \|f_j\|_{L^1_w}\Big].$$

THEOREM 3. *Let $w \in A_{(n+1)/n}$. If l is a continuous linear functional on H^1_w, there exists a unique element b of class BMO_w such that for every $f \in \mathscr{C}$,*

$$l(f, R_1 f, \ldots, R_n f) = \int_{E^n} f(x)b(x)\,dx.$$

As corollaries, we obtain in Theorem 4 below several characterizations of BMO_w. To describe these, let $Pf(x,t)$ denote the Poisson integral of f:

$$Pf(x,t) = \int_{E^n} f(y)P(x-y,t)\,dy,$$

where $P(x,t) = c_n t/(t^2+|x|^2)^{(n+1)/2}$ is the Poisson kernel for E^{n+1}_+. We will also use the modified Riesz transforms

$$R_j^{\#}f(x) = \lim_{\varepsilon\to 0+} c_n \int_{|y|>\varepsilon} f(y)\Big\{\frac{x_j-y_j}{|x-y|^{n+1}} + \chi(|y|>1)\frac{y_j}{|y|^{n+1}}\Big\}dy,$$

where $\chi(|y|>1)$ is the characteristic function of $\{y\colon |y|>1\}$. Of course, $R_j^{\#}f$ may exist at points where $R_j f$ may not, but where $R_j f$ exists, it differs from $R_j^{\#}f$ by a constant (depending on f). An important fact is that $R_j^{\#}f \in \mathrm{BMO}_w$ for any f which is bounded in absolute value by a multiple of w (see Lemma 2).

THEOREM 4. *Let $w \in A_1$.*

(i) *The general element of BMO_w has the form*

$$b = g + \sum_1^n R_j^{\#} g_j,$$

where g and the g_j are bounded in absolute value by multiples of w.

(ii) *If $b(x)(1+|x|)^{-n-1} \in L^1(E^n)$, then $b \in \mathrm{BMO}_w$ if and only if there is a constant c such that for every cube I,*

$$\iint_{B(I)} t\Big|\frac{\partial}{\partial t}Pb\Big|^2 e^{-P\log w}\,dy\,dt \leqslant c m_w(I),$$

where $B(I) = I\times(0, \operatorname{diam} I)$. Moreover, the same is true if we replace $\frac{\partial}{\partial t}Pb$ by ∇Pb in the integral.

We note in passing that the expression $\exp\{-(P\log w)(y,t)\}$ in the last integral is equivalent to $t^{-n}\int_{|x-y|<t} w(x)^{-1}\,dx$ (see Lemma 6).

Theorems 1–3 will be proved in the reverse order. Theorem 4 will be proved at the end of the paper.

Finally, we list a few useful facts about weight functions. If $w \in A_p$, $1 \leqslant p < \infty$, then given $\varepsilon > 0$, there is a $\delta > 0$ such that if E is a measurable subset of a cube I and $m_w(E) < \delta m_w(I)$, then $|E| < \varepsilon|I|$. Any w which satisfies this last condition will be said to belong to A_∞. Moreover, if $w \in A_p$, $1 < p < \infty$, then

$$(B_p) \qquad \int_{E^n} w(x)\frac{t^{np-n}}{t^{np}+|x-x_I|^{np}}\,dx \leqslant c\frac{m_w(I)}{|I|}, \qquad 1<p<\infty,$$

where t and x_I denote the edge-length and center of I, respectively. We shall be primarily concerned with the case $p = (n+1)/n$; in this case, the last integral has an obvious similarity to the Poisson integral of w. It is also known that if $w \in A_p$, $1 < p < \infty$, then there exists $q < p$ such that $w \in A_q$ (and hence also $w \in B_q$). These basic facts are proved in [1], [5], and [7].

If I is a cube and $\alpha > 0$, then αI denotes the cube with the same center as I and edge-length α times that of I. If $w \in B_p$, then by extending the integration there only over $2I$, it follows that there is a constant c independent of I such that

$$m_w(2I) \leqslant c m_w(I).$$

This condition is referred to as the doubling condition.

We will often use the same letter c for different constants which may depend on w and n, but not on f, b or I. We also use L^p to denote ordinary unweighted L^p.

§2. Theorem 3. The following lemma will be useful.

LEMMA 1. *Let w satisfy $B_{(n+1)/n}$ and suppose that $b \in \mathrm{BMO}_w$. Then there is a constant c independent of b and I such that*

$$\int_{E^n} |b(x)-b_I|\frac{t}{t^{n+1}+|x-x_I|^{n+1}}\,dx \leqslant c\|b\|_*\frac{m_w(I)}{|I|},$$

where t and x_I denote the edge-length and center of I.

Proof. We first observe that

(1) $$|b_{2I}-b_I| \leqslant c\|b\|_* \frac{m_w(2I)}{|2I|};$$

in fact,

$$|b_{2I}-b_I| = \left|\frac{1}{|I|}\int_I \{b(x)-b_{2I}\}\,dx\right| \leqslant \frac{1}{|I|}\int_{2I} |b(x)-b_{2I}|\,dx \leqslant 2^n\|b\|_* \frac{m_w(2I)}{|2I|}.$$

To prove the lemma, note that the left side of the conclusion is bounded by c times

(2) $$\sum_{k=0}^{\infty} \int_{2^kI} |b(x)-b_I| \frac{t}{(2^kt)^{n+1}}\,dx.$$

We have

$$\int_{2^kI} |b(x)-b_I|\,dx \leqslant \int_{2^kI} |b(x)-b_{2^kI}|\,dx + \sum_{j=1}^{k} |2^kI||b_{2^jI}-b_{2^{j-1}I}| \leqslant \|b\|_*\left[m_w(2^kI)+\sum_{j=1}^{k}|2^kI|\cdot c\frac{m_w(2^jI)}{|2^jI|}\right].$$

Using this estimate shows that (2) is bounded by

$$c\|b\|_* \sum_{k=0}^{\infty} \frac{t}{(2^kt)^{n+1}} \sum_{j=1}^{k} 2^{(k-j)n} m_w(2^jI).$$

Interchanging the order of summation gives a bound for (2) of

$$c\|b\|_* \sum_{j=1}^{\infty} \frac{t}{(2^jt)^{n+1}} m_w(2^jI).$$

Since w satisfies the doubling condition, $m_w(2^jI) \leqslant cm_w(2^jI-2^{j-1}I)$, and the last sum is at most

$$c\|b\|_* \sum_{j=1}^{\infty} \int_{2^jI-2^{j-1}I} w(x) \frac{t}{t^{n+1}+|x-x_I|^{n+1}}\,dx.$$

The lemma follows immediately from the fact that w satisfies $B_{(n+1)/n}$.

We also need a result about the modified Riesz transforms $R_j^{\#}f$ defined in the introduction.

LEMMA 2. *Let $w \in A_{(n+1)/n}$ and let g/w be bounded. Then $R_j^{\#}g$ is in BMO_w for each j, and*

$$\|R_j^{\#}g\|_* \leqslant c\|g/w\|_\infty, \quad j = 1, 2, \ldots, n,$$

with c independent of g.

The proof is exactly the same as that in [8] for the case $n = 1$.

We will now show that if $f \in \mathscr{C}$, then $R_jf \in \mathscr{C}$, $j = 1, 2, \ldots, n$. One way to see this is to begin with the known fact that $(R_jf)\hat{}\,(x) = i\frac{x_j}{|x|}\hat{f}(x)$ a.e. Since $f \in \mathscr{C}$, the right side here is in $\mathscr{S}$, and so equals $\hat{g}$ for some $g \in \mathscr{S}$. Since R_jf is continuous, it follows that $R_jf = g$ everywhere. Hence, $R_jf \in \mathscr{S}$, $(R_jf)\hat{}\,(x) = i\frac{x_j}{|x|}\hat{f}(x)$ everywhere, and the result follows. This allows us to conclude, as mentioned in the introduction, that $\mathscr{C} \subset H_w^1$ if $w \in A_{n/(n-1)}$. It is also useful in deducing the convergence of various integrals which arise below.

LEMMA 3. *Let $w \in A_{(n+1)/n}$, g/w be bounded, and $f \in \mathscr{C}$. Then*

$$\int_{E^n} R_jf(x)g(x)\,dx = -\int_{E^n} f(x)R_j^{\#}g(x)\,dx, \quad j = 1, \ldots, n.$$

Proof. We first claim that both integrals converge absolutely. This is a corollary of Lemma 1 since f, $R_jf \in \mathscr{S}$ and g, $R_j^{\#}g \in \mathrm{BMO}_w$ (for g, this follows from the definition of BMO_w, and for $R_j^{\#}g$, it follows from Lemma 2).

If $g \in L^p(E^n)$, $1 < p < \infty$, we have the well-known formula

$$\int_{E^n} R_jf\cdot g\,dx = -\int_{E^n} f\cdot R_jf\,dx;$$

this is easy to prove by the Fourier transform when $p = 2$, and then follows for $1 < p < \infty$ from the boundedness of R_j on L^p. Now let g satisfy the hypothesis of the lemma, and set $g_k = g\chi_{|x|<k}$, $k = 1, 2, \ldots$ Since w is locally in L^p for some $p > 1$ (see [7]), $g_k \in L^p(E^n)$ for some $p > 1$. Hence, the last formula holds for each g_k. From this and the dominated convergence theorem,

$$\int_{E^n} R_jf\cdot g\,dx = \lim_{k\to\infty}\int_{E^n} R_jf\cdot g_k\,dx = -\lim_{k\to\infty}\int_{E^n} f\cdot R_jg_k\,dx.$$

Since R_jg_k exists a.e., it differs a.e. from $R_j^{\#}g_k$ by a constant. Therefore, since $\int_{E^n} f\,dx = 0$,

$$\int_{E^n} f\cdot R_jg_k\,dx = \int_{E^n} f\cdot R_j^{\#}g_k\,dx.$$

Hence,

$$\int_{E^n} R_j f\cdot g\,dx = -\lim_{k\to\infty}\int_{E^n} f\cdot R_j^{\#} g_k\,dx.$$

To complete the proof, it remains only to show that

$$\int_{E^n} f\cdot R_j^{\#} g_k\,dx \to \int_{E^n} f\cdot R_j^{\#} g\,dx$$

for a sequence of k's tending to ∞. Let N and l be integers greater than 1, and let $k = N^l$. Then

$$\Big|\int_{E^n} f\cdot R_j^{\#} g\,dx - \int_{E^n} f\cdot R_j^{\#} g_k\,dx\Big|$$

is bounded by the sum of

(3) $$\int_{|x|\leqslant N} |f\cdot R_j^{\#}(g-g_k)|\,dx$$

and

(4) $$\int_{|x|>N} |f\cdot R_j^{\#}(g-g_k)|\,dx.$$

If $|x|\leqslant N$,

$$|R_j^{\#}(g-g_k)(x)| = \Big|\int_{|y|\geqslant k}\Big\{\frac{x_j-y_j}{|x-y|^{n+1}}+\frac{y_j}{|y|^{n+1}}\Big\} g(y)\,dy\Big|$$

$$\leqslant c|x|\int_{|y|\geqslant k}\frac{w(y)}{|y|^{n+1}}\,dy \leqslant \frac{cN}{k^{n+1}}\int_{|y|\leqslant k} w(y)\,dy \qquad (|x|\leqslant N),$$

since w satisfies $B_{(n+1)/n}$. However, since $w\in A_{(n+1)/n}$, it also satisfies B_μ for some $\mu<(n+1)/n$. Hence,

$$\frac{1}{k^{n\mu}}\int_{k/2\leqslant|y|\leqslant k} w(y)\,dy \leqslant c\int_{E^n} w(y)\frac{1}{1+|y|^{n\mu}}\,dy \leqslant c,$$

so that by the doubling condition,

(5) $$\int_{|y|\leqslant k} w(y)\,dy \leqslant ck^{n\mu}.$$

Combining estimates and choosing l sufficiently large but fixed, we have

(6) $$|R_j^{\#}(g-g_k)(x)| \leqslant \frac{cN}{k^{n+1}}\,k^{n\mu} \leqslant \frac{c}{N} \qquad (|x|\leqslant N).$$

Therefore, (3) is bounded by $cN^{-1}\int_{E^n}|f|\,dx$, which tends to 0 as $N\to\infty$.

Now consider (4). From the definition of g_k and Lemma 2, $\|R_j^{\#}(g-g_k)\|_*$ is bounded by a constant independent of k. Letting

$$\varepsilon(N) = \sup_{|x|\geqslant N}\Big\{|f(x)|\,\frac{N^{n+1}+|x|^{n+1}}{N}\Big\},$$

we therefore obtain from Lemma 1 that (4) is bounded by

$$c\varepsilon(N)\Big[N^{-n}\int_{|x|\leqslant N}|R_j^{\#}(g-g_k)|\,dx + N^{-n}\int_{|x|\leqslant N} w\,dx\Big].$$

Using (5), (6) and the fact that $N\varepsilon(N)\to 0$ as $N\to\infty$, we see that (4) tends to 0 as $N\to\infty$. This completes the proof.

Proof of Theorem 3. We will first show that H_w^1 is complete if $w\in A_p$ for some p. Let $\{F^{(k)}(x,t)\}_{k=1}^{\infty}$ be a Cauchy sequence with respect to $|||\cdot|||$. By a result of Hardy–Littlewood (see [3], Lemma 2, p. 172), given $s>0$, there exists a constant c such that

$$|F^{(k)}(x,t)-F^{(m)}(x,t)|^s \leqslant \frac{c}{t^{n+1}}\int_{\substack{|\xi-x|<t/2\\ t/2<\eta<3t/2}} |F^{(k)}(\xi,\eta)-F^{(m)}(\xi,\eta)|^s\,d\xi\,d\eta$$

$$\leqslant \frac{c}{t^n}\sup_{\eta>0}\int_{|\xi-x|<t/2}|F^{(k)}(\xi,\eta)-F^{(m)}(\xi,\eta)|^s\,d\xi.$$

Letting $s=1/p$ and applying Hölder's inequality with exponents p and p', $1/p+1/p'=1$, we get

$$|F^{(k)}(x,t)-F^{(m)}(x,t)|^{1/p} \leqslant ct^{-n}|||F^{(k)}-F^{(m)}|||^{1/p}\Big(\int_{|\xi-x|<t/2} w(\xi)^{-1/(p-1)}\,d\xi\Big)^{1/p'}.$$

Since $w^{-1/(p-1)}$ is locally integrable for any $w\in A_p$, it follows that $\{F^{(k)}\}$ converges uniformly on compact subsets of E_+^{n+1} to a limit F. By standard arguments, F is a Cauchy–Riemann system. Also, since given $\varepsilon>0$, we have

$$\int_{E^n}|F^{(k)}(x,t)-F^{(m)}(x,t)|\,w(x)\,dx \leqslant |||F^{(k)}-F^{(m)}||| < \varepsilon$$

if k and m are large enough, Fatou's lemma implies that $|||F-F^{(m)}|||\leqslant\varepsilon$ for large m. This shows that H_w^1 is complete.

Next, letting $w\in A_{n/(n-1)}$ and identifying systems $F(x,t)\in H_w^1$ with their boundary values $F(x,0) = (f(x), f_1(x), \ldots, f_n(x))$, we see from the completeness of H_w^1 and the equivalence of $|||F|||$ and $\|F(x,0)\|_{L_w^1}$ that H_w^1 is a closed subspace of $L_w^1\oplus\ldots\oplus L_w^1$.

Now let l be any continuous linear functional on H^1_w, $w \in A_{(n+1)/n}$. By the Hahn–Banach theorem, l has an extension $\bar{l}$ to all of $L^1_w \oplus \ldots \oplus L^1_w$. Hence, there exist $\varphi, \varphi_1, \ldots, \varphi_n$ in L^∞ such that

$$\bar{l}(f, f_1, \ldots, f_n) = \int_{E^n} \Big\{f\varphi + \sum_1^n f_j \varphi_j\Big\} w\,dx$$

for any $(f, f_1, \ldots, f_n)$ in $L^1_w \oplus \ldots \oplus L^1_w$. In particular, if $f \in \mathscr{C}$,

$$l(f, R_1 f, \ldots, R_n f) = \int_{E^n} \Big\{f\varphi + \sum_1^n R_j f \cdot \varphi_j\Big\} w\,dx.$$

Letting $g = \varphi w$, $g_j = \varphi_j w$, $j = 1, \ldots, n$, and writing $l(f, R_1 f, \ldots, R_n f) = l(f)$, we obtain by Lemma 3 that

$$l(f) = \int_{E^n} f\Big\{g - \sum_1^n R_j^{\#} g_j\Big\} dx, \quad f \in \mathscr{C}.$$

By Lemma 2, the function $b = g - \sum_1^n R_j^{\#} g_j$ is in BMO_w with $\|b\|_* \leqslant c\{\|\varphi\|_\infty + \sum_1^n \|\varphi_j\|_\infty\}$. Thus, $\|b\|_*$ is bounded by a multiple of the norm of l as a continuous linear functional.

To complete the proof of Theorem 3, we must show that the function b above is unique up to an additive constant. If $\beta \in \mathrm{BMO}_w$ and

$$\int_{E^n} fb\,dx = \int_{E^n} f\beta\,dx$$

for all $f \in \mathscr{C}$, then the Fourier transform of $b - \beta$ (as a tempered distribution) is supported at the origin, since $\hat{f}$ is any element of $\mathscr{S}$ which vanishes near $x = 0$. Thus, $b - \beta$ is a polynomial. Since it is also in BMO_w, it must be constant (Lemma 1).

§ 3. Theorem 2. In the proof, we will use the results stated in the next four lemmas.

LEMMA 4. *If $f \in \mathscr{C}$, then its Poisson integral $Pf(x, t)$ belongs to $\mathscr{S}(\overline{E^{n+1}_+})$.*

This is stated in [9], p. 231; the proof is computational, but straightforward.

LEMMA 5. *Let w satisfy $B_{(n+1)/n}$. If $b(x)(1+|x|)^{-n-1} \in L^1(E^n)$ and $f \in \mathscr{C}$, then*

$$\int_{E^n} f(x) b(x)\,dx = 4 \iint_{E^{n+1}_+} t \frac{\partial}{\partial t} Pf(x, t) \frac{\partial}{\partial t} Pb(x, t)\,dx\,dt.$$

Proof. Note that $Pb(x, t)$ is well-defined since $b(x)(1+|x|)^{-n-1}$ is integrable. The integral on the left side of the conclusion above clearly converges absolutely. To show the same for the one on the right, note that the simple estimate $t\left|\frac{\partial}{\partial t} Pb(x, t)\right| \leqslant cP(|b|)(x, t)$ and Lemma 4 give the bound

$$\iint_{E^{n+1}_+} \left|\frac{\partial}{\partial t} Pf\right| P(|b|)\,dx\,dt$$
$$\leqslant c \int_{E^n} |b(y)| \Big\{ \iint_{E^{n+1}_+} \frac{1}{(1+|x|+t)^N} \frac{t}{t^{n+1} + |x-y|^{n+1}}\,dx\,dt\Big\}\,dy,$$

where N can be taken arbitrarily large, and $c = c(n, N)$. An elementary computation with $N = 2n+2$ shows that the expression in curly brackets is bounded by a multiple of $(1+|y|)^{-n-1}$. The resulting integral therefore converges.

The conclusion of Lemma 5 is standard if $b \in L^2$ (see, e.g., p. 85 of [9]). For any b, choose $\{b_k\}$ with $b_k \to b$ pointwise, $|b_k| \leqslant |b|$, $b_k \in L^2$. Then

$$\int_{E^n} fb_k\,dx = 4 \iint_{E^{n+1}_+} t\left(\frac{\partial}{\partial t} Pf\right)\left(\frac{\partial}{\partial t} Pb_k\right) dx\,dt.$$

Letting $k \to \infty$ and noting that $|fb_k| \leqslant |fb| \in L^1(E^n)$ and

$$t\left|\frac{\partial}{\partial t} Pf\right| \left|\frac{\partial}{\partial t} Pb_k\right| \leqslant c\left|\frac{\partial}{\partial t} Pf\right| P(|b_k|) \leqslant c \left|\frac{\partial}{\partial t} Pf\right| P(|b|) \in L^1(E^{n+1}_+)$$

(as shown above), we obtain the desired formula.

LEMMA 6. *If $w \in A_2$, there exist positive constants c_1 and c_2 depending on w such that*

$$c_1 e^{P(\log w)(x,t)} \leqslant \frac{1}{t^n} \int_{|x-y|<t} w(y)\,dy \leqslant c_2 e^{P(\log w)(x,t)}.$$

Proof. We claim that if $h \in \mathrm{BMO}$ (unweighted), then

$$e^{Ph(x,t)} \leqslant \frac{c}{t^n} \int_{|x-y|<t} e^{h(y)}\,dy. \tag{7}$$

This will imply the lemma, as we now show. If $w \in A_2$, then $\log w \in \mathrm{BMO}$ by [5], Lemma 5, so that (7) gives

$$e^{P(\log w)(x,t)} \leqslant \frac{c}{t^n} \int_{|x-y|<t} w(y)\,dy.$$

Applying this to w^{-1} (which also satisfies A_2), we get

$$e^{-P(\log w)(x,t)} \leqslant \frac{c}{t^n} \int\limits_{|x-y|<t} w(y)^{-1}\,dy \leqslant c\left(\frac{1}{t^n}\int\limits_{|x-y|<t} w(y)\,dy\right)^{-1}$$

by A_2. This is the opposite inequality to that above.

To prove (7), let $h \in \mathrm{BMO}$ (unweighted) and write

$$e^{Ph(x,t)} = \exp\Big\{\int\limits_{E^n} h(y)P(x-y,t)\,dy\Big\},$$

where $P(x,t)$ is the Poisson kernel. If $A_{x,t}$ denotes the integral average of $h(y)$ over $|x-y|<t$, then

$$\int\limits_{E^n} h(y)P(x-y,t)\,dy = A_{x,t} + \int\limits_{E^n} [h(y)-A_{x,t}]P(x-y,t)\,dy,$$

so that

$$e^{Ph(x,t)} = e^{A_{x,t}}\exp\Big\{\int\limits_{E^n} [h(y)-A_{x,t}]P(x-y,t)\,dy\Big\}$$
$$\leqslant e^{A_{x,t}}\exp\Big\{\int\limits_{E^n} |h(y)-A_{x,t}|P(x-y,t)\,dy\Big\}.$$

By Jensen's inequality,

$$e^{A_{x,t}} \leqslant ct^{-n}\int\limits_{|x-y|<t} e^{h(y)}\,dy,$$

and by Lemma 1 (with $w=1$, $b=h$), $\int\limits_{E^n} |h(y)-A_{x,t}|P(x-y,t)\,dy$ is bounded by a constant independent of x, t. This implies (7) and completes the proof.

Lemma 6 is true if $w \in A_p$ for any p; for a proof, just apply (7) to $\log w$ and $\log(w^{-1/(p-1)}) = -\dfrac{1}{p-1}\log w$ (see also [11]). We shall not need this fact.

Finally, we need the following result from [8] about BMO_w.

LEMMA 7. *Let $w \in A_2$ and $b \in \mathrm{BMO}_w$. Then there is a constant c such that*

$$\int\limits_I |b(x)-b_I|^2 w(x)^{-1}\,dx \leqslant c\|b\|_*^2\, m_w(I).$$

Proof of Theorem 2. Fix $f \in \mathscr{C}$ and $b \in \mathrm{BMO}_w$, $w \in A_{(n+1)/n}$. Since $b(x)(1+|x|)^{-n-1}$ is integrable by Lemma 1, Lemma 5 gives

$$(8)\qquad \Big|\int\limits_{E^n} fb\,dx\Big| \leqslant 4\iint\limits_{E_+^{n+1}} t\left|\frac{\partial}{\partial t}Pf(x,t)\right|\left|\frac{\partial}{\partial t}Pb(x,t)\right|dx\,dt.$$

Let $\Gamma_h(x)$ denote the 45°-cone with vertex x and height h:

$$\Gamma_h(x) = \{(y,t)\colon\ |x-y|<t,\ 0<t<h\},$$

let $\Gamma(x) = \Gamma_\infty(x)$, and let

$$h(x) = \sup\left\{h\colon \left(\iint\limits_{\Gamma_h(x)} t^{1-n}\left|\frac{\partial}{\partial t}Pb(y,t)\right|^2 dy\,dt\right)^{1/2} \leqslant c_1\|b\|_* w(x)\right\},$$

where c_1 is a constant to be chosen. Note that the integral above is a truncated version of the Lusin area integral

$$S(b)(x) = \left(\iint\limits_{\Gamma(x)} t^{1-n}\left|\frac{\partial}{\partial t}Pb(y,t)\right|^2 dy\,dt\right)^{1/2}.$$

Since $b(y)(1+|y|)^{-n-1}$ is integrable, standard Lebesgue point arguments show that Pb has non-tangential limits a.e. on E^n. Therefore, by [9], p. 206, Theorem 4, any truncation of $S(b)$ is finite a.e.

We will show that the right-hand side of (8) is bounded by

$$(9)\qquad c\int\limits_{E^n}\left(\iint\limits_{\Gamma_{h(x)}(x)} t^{1-n}\left|\frac{\partial}{\partial t}Pf(y,t)\right|\left|\frac{\partial}{\partial t}Pb(y,t)\right|dy\,dt\right)dx.$$

This will imply the theorem, as we now show. By Schwarz's inequality, (9) is at most

$$c\int\limits_{E^n}\left(\iint\limits_{\Gamma_{h(x)}(x)} t^{1-n}\left|\frac{\partial}{\partial t}Pb\right|^2 dy\,dt\right)^{1/2}\left(\iint\limits_{\Gamma_{h(x)}(x)} t^{1-n}\left|\frac{\partial}{\partial t}Pb\right|^2 dy\,dt\right)^{1/2} dx$$
$$\leqslant c\int\limits_{E^n} S(f)(x)\,c_1\|b\|_* w(x)\,dx.$$

However, by the results of [4], $\|S(f)\|_{L_w^1} \leqslant c\|f\|_{H_w^1}$, and the theorem follows.

To show that (9) majorizes the right-hand side of (8), note that (9) equals c times

$$\iint\limits_{E_+^{n+1}} t^{1-n}\left|\frac{\partial}{\partial t}Pf(y,t)\right|\left|\frac{\partial}{\partial t}Pb(y,t)\right||E_{y,t}|\,dy\,dt,$$

where

$$E_{y,t} = \{x\colon\ (y,t)\in\Gamma_{h(x)}(x)\}.$$

It suffices then to prove that $|E_{y,t}| \geqslant ct^n$, $c>0$, or that given $(y,t)\in E_+^{n+1}$, $h(x)\geqslant t$ on a subset of $\{x\colon\ |x-y|<t\}$ whose measure exceeds a fixed

multiple of t^n. In fact, we will show that if λ is large enough, then for any cube I

$$(10)\qquad \left|\left\{x\in I: \left(\iint_{\Gamma_\varrho(x)} t^{1-n}\left|\frac{\partial}{\partial t}Pb(y,t)\right|^2 dy\,dt\right)^{1/2} \leqslant \lambda\|b\|_* w(x)\right\}\right| \geqslant \tfrac{1}{2}|I|,$$

where ϱ and x_I denote the edge-length and center of I.

Fix I and let E be the subset of I complementary to the set on the left of (10). Integrating over E and changing the order of integration gives

$$\lambda^2\|b\|_*^2 m_w(E) \leqslant \int_E\left(\iint_{\Gamma_\varrho(x)} t^{1-n}\left|\frac{\partial}{\partial t}Pb(y,t)\right|^2 dy\,dt\right) w(x)^{-1}dx$$
$$\leqslant \iint_{B(5I)} t^{1-n}\left|\frac{\partial}{\partial t}Pb(y,t)\right|^2\left(\int_{|x-y|<t} w(x)^{-1}dx\right)dy\,dt,$$

where $B(I) = I\times(0,\operatorname{diam} I)$. Consider the larger expression

$$(11)\qquad \iint_{B(5I)} t^{1-n}|\nabla Pb(y,t)|^2\left(\int_{|x-y|<t} w(x)^{-1}dx\right)dy\,dt.$$

We will show that (11) is bounded by $c\|b\|_*^2 m_w(I)$ (with c independent of I and b). This will give $\lambda^2 m_w(E)\leqslant cm_w(I)$. Choosing λ sufficiently large and using the fact that $w\in A_\infty$, we then get $|E|<\frac{1}{2}|I|$, thereby proving (10).

To show that (11) is bounded by $c\|b\|_*^2 m_w(I)$, let $J = 10I$ and write

$$\nabla Pb(y,t) = \int_{E^n}[b(z)-b_J]\nabla P(y-z,t)\,dz = \int_J + \int_{J'} = \beta_1(y,t)+\beta_2(y,t),$$

say. First consider the contribution made to (11) by β_2. For $y\in 5I$,

$$|\beta_2(y,t)|\leqslant c\int_{J'}|b(z)-b_J|\frac{dz}{|y-z|^{n+1}}\leqslant c\|b\|_*\frac{m_w(I)}{\varrho|I|}$$

by Lemma 1 and the doubling condition. The corresponding part of (11) is at most

$$c\|b\|_*^2\frac{m_w^2(I)}{\varrho^2|I|^2}\iint_{B(5I)} t^{1-n}\left(\int_{|x-y|<t} w(x)^{-1}dx\right)dy\,dt.$$

Performing the integration with respect to y shows this is bounded by

$$c\|b\|_*^2\frac{m_w^2(I)}{\varrho^2|I|^2}\int_0^{c\varrho} t\left(\int_J w(x)^{-1}dx\right)dt \leqslant c\|b\|_*^2\frac{m_w^2(I)m_{1/w}(I)}{|I|^2}$$

by the doubling condition for w^{-1}. Since $w\in A_2$ (in fact, $w\in A_{(n+1)/n}$), the last expression is at most $c\|b\|_*^2 m_w(I)$, as desired.

Next, consider the contribution of β_1 to (11). Note that $\beta_1 = \nabla Pb_1$, where $b_1 = (b-b_J)\chi_J$. Hence, by changing the order of integration, we see that the part of (11) in question is at most

$$(12)\qquad \int_{E^n}(Sb_1)^2(x)w(x)^{-1}dx,$$

Sb_1 being the area integral $\left(\iint_{\Gamma(x)} t^{1-n}|\nabla Pb_1(y,t)|^2 dy\,dt\right)^{1/2}$.

Since $w\in A_2$, it follows that $w^{-1}\in A_2$. Therefore, by the results of [4], (12) is bounded by

$$c\int_{E^n} b_1^2(x)w(x)^{-1}dx.$$

Observe that

$$\int_{E^n} b_1^2(x)w(x)^{-1}dx = \int_J|b(x)-b_J|^2 w(x)^{-1}dx \leqslant c\|b\|_*^2 m_w(J)$$

by Lemma 7. Combining estimates and using the doubling condition, we obtain the desired bound. This completes the proof of Theorem 2.

§ 4. Theorem 1. Let $w\in A_1$. Note that Theorem 1 is trivial if $w=0$ a.e. Hence, we may assume that w is positive in some set of positive measure. In this case, we claim that for any $f\in L_w^1$,

$$\int_{E^n}\frac{|f(x)|}{(1+|x|)^n}dx < +\infty.$$

In fact, since $w^*(x)\geqslant c(1+|x|)^{-n}$ and $w(x)\geqslant cw^*(x)$ a.e., we have $w(x)\geqslant c(1+|x|)^{-n}$ a.e., and the claim follows immediately.

LEMMA 8. *Let $w\in A_1$ and $f\in L_w^1$. If $\psi\in\mathscr{S}$ and $\psi_\delta(x) = \delta^n\psi(\delta x)$, $\delta>0$, there is a constant c independent of f and δ such that*

$$\|f*\psi_\delta\|_{L_w^1}\leqslant c\|f\|_{L_w^1}.$$

Moreover, if $\int_{E^n}\psi\,dx = 1$, then $\|(f*\psi_\delta)-f\|_{L_w^1}\to 0$ as $\delta\to\infty$.

Proof. By Fubini's theorem,

$$\|f*\psi_\delta\|_{L_w^1}\leqslant\int_{E^n}|f(y)|\left\{\int_{E^n} w(x)|\psi_\delta(x-y)|\,dx\right\}dy.$$

Since the expression in curly brackets is $\leqslant cw^*(y)$, the first part of the lemma follows immediately from the fact that $w\in A_1$.

For the second part, we have (since $\int_{E^n}\psi_\delta(y)\,dy = 1$)

$$\|(f*\psi_\delta)-f\|_{L_w^1}\leqslant\int_{E^n}\left(\int_{E^n}|f(x-y)-f(x)|\,|\psi_\delta(y)|\,dy\right)w(x)\,dx$$
$$=\int_{E^n}\int_{|y|>N} + \int_{|x|>2N}\int_{|y|<N} + \int_{|x|<2N}\int_{|y|<N} = \alpha+\beta+\gamma.$$

It is enough to show that if $\delta, N > 1$, there is a constant c, independent of f, δ, and N, such that

$$\alpha \leqslant c\|f\|_{L^1_w}\Big(\int_{|y|>\delta N}|\psi|\,dy+\delta^{-1}\Big), \qquad \beta \leqslant c\int_{|x|>N}|f(x)|\,w(x)\,dx,$$

and that for any N, $\gamma\to 0$ as $\delta\to\infty$. Clearly,

$$\alpha \leqslant \int_{E^n}\Big(\int_{|y|>N}|f(x-y)|\,|\psi_\delta(y)|\,dy\Big)w(x)\,dx+\int_{E^n}\Big(\int_{|y|>N}|f(x)|\,|\psi_\delta(y)|\,dy\Big)w(x)\,dx$$
$$=\int_{E^n}|f(y)|\Big(\int_{|x-y|>N}w(x)|\psi_\delta(x-y)|\,dx\Big)dy+\|f\|_{L^1_w}\int_{|y|>N}|\psi_\delta(y)|\,dy.$$

The second term is $\|f\|_{L^1_w}\int_{|y|>\delta N}|\psi(y)|\,dy$, and the first is less than c times

$$\int_{E^n}|f(y)|\Big(\int_{|x-y|>1}w(x)\frac{\delta^n}{(\delta|x-y|)^{n+1}}\,dx\Big)\,dy.$$

The inner integral here is at most $\delta^{-1}\cdot cw^*(y)\leqslant c\delta^{-1}w(y)$, and the estimate for α follows. Next, note that

$$\beta\leqslant\int_{|x|>2N}\Big(\int_{|y|<N}|f(x-y)|\,|\psi_\delta(y)|\,dy\Big)\,w(x)\,dx+$$
$$+\int_{|x|>2N}\Big(\int_{|y|<N}|f(x)|\,|\psi_\delta(y)|\,dy\Big)\,w(x)\,dx.$$

The second term is at most $c\int_{|x|>2N}|f(x)|\,w(x)\,dx$, and the first equals

$$\int_{|x|>2N}\Big(\int_{|x-y|<N}|f(y)|\,|\psi_\delta(x-y)|\,dy\Big)\,w(x)\,dx.$$

The inequalities $|x|>2N$ and $|x-y|<N$ give $|y|>N$, so that the last expression is at most

$$\int_{|y|>N}|f(y)|\Big(\int_{E^n}w(x)\,|\psi_\delta(x-y)|\,dx\Big)\,dy\leqslant c\int_{|y|>N}|f(y)|\,w^*(y)\,dy.$$

The estimate for β now follows from the fact that $w\in A_1$.

Finally, to show that $\gamma\to 0$ as $\delta\to\infty$, fix N and ε and choose a continuous $c(x)$ such that

$$\int_{|x|<3N}|f(x)-c(x)|\,w(x)\,dx<\varepsilon.$$

Write $|f(x-y)-f(x)|\leqslant|f(x-y)-c(x-y)|+|c(x-y)-c(x)|+|c(x)-f(x)|$, and consider the corresponding three contributions to γ. The first of these is

$$\int_{|x|<2N}\Big(\int_{|y|<N}|f(x-y)-c(x-y)|\,|\psi_\delta(y)|\,dy\Big)\,w(x)\,dx$$
$$\leqslant\int_{|y|<3N}|f(y)-c(y)|\Big(\int_{E^n}w(x)|\psi_\delta(x-y)|\,dx\Big)\,dy$$
$$\leqslant\int_{|y|<3N}|f(y)-c(y)|\,w^*(y)\,dy<c\varepsilon.$$

The third is at most

$$\int_{|x|<2N}|f(x)-c(x)|\Big(\int_{E^n}|\psi_\delta(y)|\,dy\Big)\,w(x)\,dx=c\int_{|x|<2N}|f(x)-c(x)|\,w(x)\,dx<c\varepsilon.$$

Finally, for the second, choose η so that $|c(x-y)-c(x)|<\varepsilon$ if $|y|<\eta$ and $|x|<2N$. Then the integral is at most

$$\int_{|x|<2N}\Big(\int_{|y|<\eta}\varepsilon|\psi_\delta(y)|\,dy+\int_{\eta<|y|<N}2\big(\max_{|x|<3N}|c(x)|\big)|\psi_\delta(y)|\,dy\Big)\,w(x)\,dx$$
$$\leqslant\Big(\int_{|x|<2N}w(x)\,dx\Big)\Big(c\varepsilon+2\max_{|x|<3N}|c(x)|\int_{|y|>\eta}|\psi_\delta(y)|\,dy\Big).$$

The lemma now follows from the fact that $\int_{|y|>\eta}|\psi_\delta(y)|\,dy$ tends to 0 as $\delta\to\infty$.

For the remainder of the section, φ is a fixed function in $\mathscr{S}$ satisfying $0\leqslant\hat\varphi\leqslant 1$, $\hat\varphi=1$ near 0, $\hat\varphi=0$ in $|x|\geqslant 1$, and $\int_{E^n}\hat\varphi\,dx=1$. Of course, $\hat\varphi\in\mathscr{S}$ too, and the properties of $\hat\varphi$ imply that $\int_{E^n}\varphi\,dx=1$ and $\varphi(0)=1$. Consider the convolution

$$(T_\delta f)(x)=(f*\varphi_\delta)(x)=\delta^n\int_{E^n}f(x-y)\varphi(\delta y)\,dy,\qquad\delta>0. \tag{13}$$

LEMMA 9. *Let $w\in A_1$. If $f\in L^1_w\cap L^1$ and $\int_{E^n}f(x)\,dx=0$, then $T_N(1-T_\varepsilon)f$ converges to f in L^1_w norm as $N\to\infty$, $\varepsilon\to 0$.*

Proof. We have

$$\|T_N(1-T_\varepsilon)f-f\|_{L^1_w}=\|T_Nf-T_NT_\varepsilon f-f\|_{L^1_w}\leqslant\|T_Nf-f\|_{L^1_w}+\|T_NT_\varepsilon f\|_{L^1_w}.$$

By Lemma 8, $\|T_Nf-f\|_{L^1_w}\to 0$ as $N\to\infty$, and $\|T_NT_\varepsilon f\|_{L^1_w}\leqslant c\|T_\varepsilon f\|_{L^1_w}$. Hence, it is enough to show that $\|T_\varepsilon f\|_{L^1_w}\to 0$ as $\varepsilon\to 0$. Since $\int_{E^n}f(y)\,dy=0$,

$$(T_\varepsilon f)(x)=\varepsilon^n\int_{E^n}f(y)\,[\varphi(\varepsilon(x-y))-\varphi(\varepsilon x)]\,dy.$$

Therefore,

$$\|T_\varepsilon f\|_{L^1_w} \leqslant \int_{E^n} |f(y)| I(\varepsilon, y)\,dy,$$

$$\text{where}\quad I(\varepsilon, y) = \varepsilon^n \int_{E^n} |\varphi(\varepsilon(x-y)) - \varphi(\varepsilon x)| w(x)\,dx.$$

Since $f \in L^1_w \cap L^1$, the dominated convergence theorem will imply that $\|T_\varepsilon f\|_{L^1_w} \to 0$ if we show that $I(\varepsilon, y) \to 0$ as $\varepsilon \to 0$ and $I(\varepsilon, y) \leqslant c\{w(y) + \int_{|x|<1} w(x)\,dx\}$ a.e. uniformly in ε, $0 < \varepsilon < 1$. Since $\varphi \in \mathscr{S}$,

$$|\varphi(\varepsilon x - \varepsilon y) - \varphi(\varepsilon x)| \leqslant c \frac{\varepsilon |y|}{1 + (\varepsilon |x|)^{n+1}}, \qquad |y| < 1/\varepsilon.$$

Hence if $|y| < 1/\varepsilon$,

$$I(\varepsilon, y) \leqslant c|y| \int_{E^n} w(x) \frac{dx}{\varepsilon^{-n-1} + |x|^{n+1}}.$$

Since $w \in B_{(n+1)/n}$, the last expression is at most

$$c|y|\varepsilon^{n+1} \int_{|x|<\varepsilon^{-1}} w(x)\,dx.$$

If I and J are cubes, or balls, with $I \subset J$, it follows from the fact that $(1/|I|)\int_I w\,dx$ is equivalent to $\operatorname{ess}_I \inf w$ that $\int_J w\,dx \leqslant c(|J|/|I|)\int_I w\,dx$. Applying this to the integral above, we get

$$I(\varepsilon, y) \leqslant c|y|\varepsilon \int_{|x|<1} w\,dx.$$

Thus, $I(\varepsilon, y) \to 0$ as $\varepsilon \to 0$. Next, observe that

$$I(\varepsilon, y) \leqslant c\varepsilon^n \int_{E^n} \left[\frac{1}{1 + (\varepsilon|x-y|)^{n+1}} + \frac{1}{1 + (\varepsilon|x|)^{n+1}}\right] w(x)\,dx,$$

so that as above

$$I(\varepsilon, y) \leqslant c\varepsilon^n \Big[\int_{|x-y|<\varepsilon^{-1}} w(x)\,dx + \int_{|x|<\varepsilon^{-1}} w(x)\,dx\Big]$$

$$\leqslant c\Big[\int_{|x-y|<1} w(x)\,dx + \int_{|x|<1} w(x)\,dx\Big].$$

This completes the proof since $\int_{|x-y|<1} w(x)\,dx \leqslant w^*(y) \leqslant cw(y)$.

Lemma 10. *Let $f \in L^1_w$, $w \in A_1$, and $M^{-n} \int_{|x|<M} w(x)\,dx \to 0$ as $M \to \infty$. Then $T_N(1 - T_\varepsilon)f$ converges to f in L^1_w norm as $N \to \infty$, $\varepsilon \to 0$.*

Proof. By Lemmas 8 and 9, it is enough to show that the class of f in $L^1_w \cap L^1$ with $\int_{E^n} f(x)\,dx = 0$ is dense in L^1_w. Given $f \in L^1_w$ and $\varepsilon > 0$, choose M_0 so large that $\int_{|x|>M_0} |f| w\,dx < \varepsilon$. Since $M^{-n}\int_{|x|<M} w\,dx \to 0$, there is a sequence $\{x_M\}$ with $|x_M| \to \infty$ and $\int_{|x-x_M|<1} w\,dx \to 0$. Choose M_1 so large that $\{x\colon |x - x_{M_1}| < 1\}$ lies in $\{x\colon |x| > M_0\}$ and

$$\int_{|x-x_{M_1}|<1} w\,dx < \varepsilon\Big(\int_{|x|<M_0} |f|\,dx\Big)^{-1}.$$

Define

$$g = f\chi_{|x|<M_0} - k\Big(\int_{|x|<M_0} f\,dx\Big)\chi_{|x-x_{M_1}|<1},$$

$$\text{where}\quad k = 1/|\{x\colon |x - x_{M_1}| < 1\}|.$$

Then $g \in L^1_w \cap L^1$ (f is locally integrable since $f \in L^1_w$ and ω is locally bounded below by a positive constant), and $\int_{E^n} g\,dx = 0$. Moreover,

$$\int_{E^n} |f-g| w\,dx = \int_{|x|>M_0} |f-g| w\,dx \leqslant \int_{|x|>M_0} |f| w\,dx + \int_{|x|>M_0} |g| w\,dx$$

$$< \varepsilon + k\int_{|x|<M_0} |f|\,dx \int_{|x-x_{M_1}|<1} w\,dx < \varepsilon + k\varepsilon.$$

This completes the proof.

Lemma 11. *If $f \in H^1_w$, $w \in A_1$, then $T_N(1-T_\varepsilon)f \to f$ and $T_N(1-T_\varepsilon)R_j f \to R_j f$, $j = 1, \ldots, n$, in L^1_w norm as $N \to \infty$, $\varepsilon \to 0$.*

Proof. If the averages $M^{-n}\int_{|x|<M} w\,dx \to 0$ as $M \to \infty$, the result follows from Lemma 10. If the averages do not tend to 0, there exist $\{M_k\} \to \infty$ and $\delta > 0$ such that $M_k^{-n}\int_{|x|<M_k} w\,dx \geqslant \delta$. Therefore, $w \geqslant \delta$ a.e., and the hypothesis that f, $R_j f \in L^1_w$ implies that f, $R_j f \in L^1$. Hence,

$$\int_{E^n} f\,dx = \int_{E^n} R_j f\,dx = 0$$

(see [10], for example), and the conclusion follows from Lemma 9.

Lemma 12. *If $f \in H^1_w$, $w \in A_1$, then $R_j T_N(1-T_\varepsilon)f = T_N(1-T_\varepsilon)R_j f$, $j = 1, \ldots, n$.*

Proof. Fix ε and N, and write $T_N(1-T_\varepsilon)f = k*f$. Then $k \in \mathscr{S}$, and we must show that $R_j(k*f) = k*R_j f$. For $y \in E^n$, $t > 0$, let $Q_j(y, t)$ denote the jth-conjugate Poisson kernel. We claim that

$$(14)\qquad Q_j * (k*f) = k*(Q_j * f).$$

By Fubini's theorem and the estimate $|Q_j(y,t)| \leqslant c(t+|y|)^{-n}$, it is enough to show that

$$\int_{E^n}\int_{E^n} |f(x-z)|\,|k(z-y)|\,\frac{1}{(1+|y|)^n}\,dz\,dy < +\infty .$$

Performing the integration in y and using $|k(z-y)| \leqslant c(1+|z-y|)^{-2n}$ show that the last integral is bounded by

$$\int_{E^n} |f(x-y)|\,\frac{1}{(1+|z|)^n}\,dz.$$

This is finite since $f\in L^1_w$ and $w(z)\geqslant c(1+|z|)^{-n}$.

To complete the proof, let $t\to 0$ in (14). The left-hand side converges a.e. to $R_j(k*f)$. By [10], Q_j*f converges to R_jf in L^1_w. Hence, by Lemma 8, the right-hand side converges in L^1_w to $k*R_jf$, and the lemma follows.

Note that if $f\in L^1_w$, then $T_N(1-T_\varepsilon)f$ is infinitely differentiable. Moreover, it is slowly increasing, as can be seen by noting that its αth derivative equals $K*f$ for a suitable $K\in\mathscr{S}$ (α, N, ε fixed), and

$$|(K*f)(x)| \leqslant \|f\|_{L^1_w}\{\sup_y w(y)^{-1}|K(x-y)|\}$$
$$\leqslant c\|f\|_{L^1_w}\sup_y \frac{(1+|y|)^n}{(1+|x-y|)^n} \leqslant c\|f\|_{L^1_w}(1+|x|)^n .$$

In order to obtain a function in $\mathscr{C}$, we must modify the growth of $T_N(1-T_\varepsilon)f$ for large $|x|$. Given $f\in L^1_w$, let

$$\text{(15)}\qquad g = g_{\varepsilon,N} = T_N(1-T_\varepsilon)f,\qquad g_k = g_{\varepsilon,N,k} = \varphi\left(\frac{x}{k}\right)g .$$

Lemma 13. *Let $f\in L^1_w$, $w\in A_1$, and let g_k be as in* (15). *If $k>k(\varepsilon,N)$, then $g_k\in\mathscr{C}$; in fact, for such k, $\operatorname{supp}\hat g_k$ is contained in a fixed compact set containing* 0 *which depends only on ε and N.*

Proof. Clearly, $g_k\in\mathscr{S}$ since g is slowly increasing and $\varphi\in\mathscr{S}$. Fix ε and N. If $f\in L^1$, then $g\in L^1$ and

$$\hat g(x) = \hat\varphi\left(\frac{x}{N}\right)\left[1-\hat\varphi\left(\frac{x}{\varepsilon}\right)\right]\hat f(x).$$

Therefore, by the properties of $\hat\varphi$, $\operatorname{supp}\hat g$ is contained in a compact set not containing 0 and depending only on ε and N. From the formula $g_k=\varphi(x/k)g$, we have

$$\hat g_k = k^n\hat\varphi(kx)*\hat g,$$
$$\operatorname{supp}\hat g_k \subset \{x+y:\ x\in\operatorname{supp}\hat\varphi(kx),\ y\in\operatorname{supp}\hat g\}.$$

Since $\operatorname{supp}\hat\varphi(kx)$ is contained in the ball of radius $1/k$ and center 0, it follows that, for sufficiently large k, $\operatorname{supp}\hat g_k$ is a subset of a fixed compact set not containing 0. We stress that this set is independent of f and $k>k(\varepsilon,N)$.

Now assume that $f\in L^1_w$. Choose $\bar f\in L^1_w\cap L^1$ such that $\|f-\bar f\|_{L^1_w}<\eta$. With ε and N fixed, let $\bar g$ and $\bar g_k$ denote the corresponding functions g and g_k. Then for $k>k(\varepsilon,N)$, $\operatorname{supp}\hat{\bar g}_k$ is contained in a fixed compact E, $0\notin E$. However,

$$\|\hat g_k-\hat{\bar g}_k\|_{L^\infty} \leqslant \|g_k-\bar g_k\|_{L^1} = \left\|\varphi\left(\frac{x}{k}\right)[g-\bar g]\right\|_{L^1}$$
$$\leqslant \left\|\varphi\left(\frac{x}{k}\right)w(x)^{-1}\right\|_{L^\infty}\|g-\bar g\|_{L^1_w} \leqslant c_k\|f-\bar f\|_{L^1_w} < c_k\eta .$$

Fixing k, letting $\eta\to 0$, and recalling that E is independent of η, we obtain that $\operatorname{supp}\hat g_k\subset E$ if $k>k(\varepsilon,N)$. This proves the lemma.

Lemma 14. *Let $f\in L^1_w$, $w\in A_1$, and let g and g_k be as in* (15). *Then $g_k\to g$ and $R_jg_k\to R_jg$, $j=1,\ldots,n$, in L^1_w norm as $k\to\infty$.*

Proof. Fix ε and N. By definition, $g=T_N(1-T_\varepsilon)f$ and $g_k=\varphi(x/k)g$. Since $\varphi(0)=1$, $g_k\to g$ in L^1_w norm.

By Lemma 13, there is a compact E depending only on ε and N such that $0\notin E$ and $\operatorname{supp}\hat g_k\subset E$ for large k. Choose $M_j\in\mathscr{S}$ with $\hat M_j(x)=ix_j/|x|$ for $x\in E$. Then $M_j*g_k=R_jg_k$ for large k, which can be seen by checking Fourier transforms (recall that $R_jg_k\in\mathscr{C}$ since $g_k\in\mathscr{C}$). Since $g_k\to g$ in L^1_w norm, Lemma 8 implies that $M_j*g_k\to M_j*g$ in L^1_w norm. To show that $R_jg_k\to R_jg$ in L^1_w norm, it is therefore enough to show that

$$R_jg = M_j*g .$$

If $f\in\mathscr{S}$, then $g\in\mathscr{S}$, and this follows by checking Fourier transforms. For general $f\in L^1_w$, choose $f^{(m)}\in\mathscr{S}$ with $\|f^{(m)}-f\|_{L^1_w}\to 0$. If $g^{(m)}$ denotes the function g corresponding to $f^{(m)}$, Lemma 8 gives $\|g^{(m)}-g\|_{L^1_w}\to 0$, so that by [1], $R_jg^{(m)}$ converges in w-measure to R_jg. But also, $R_jg^{(m)}=M_j*g^{(m)}\to M_j*g$ in L^1_w, and the desired relation follows.

Proof of Theorem 1. Let $f\in H^1_w$, $w\in A_1$. By Lemmas 11 and 12, given $\eta>0$, there exist ε and N such that the function $g=T_N(1-T_\varepsilon)f$ satisfies

$$\|f-g\|_{L^1_w}<\eta,\qquad \|R_jf-R_jg\|_{L^1_w}<\eta,\qquad j=1,\ldots,n.$$

With g_k defined as usual, Lemma 14 then gives

$$\|f-g_k\|_{L^1_w} < \eta, \qquad \|R_j f - R_j g_k\|_{L^1_w} < \eta$$

for large k. Since $g_k \in \mathscr{C}$ for large k (Lemma 13), the proof is complete.

§5. Proof of Theorem 4. To prove part (i) of Theorem 4, note that any function of the form $g+\sum R_j^{\#} g_j$, where $|g|, |g_j| \leqslant cw$, is in BMO_w by Lemma 2. Conversely, if $b \in \mathrm{BMO}_w$, then by Theorems 1 and 2, the linear functional defined by

$$l(f, R_1 f, \ldots, R_n f) = \int_{E^n} f(x) b(x)\, dx, \qquad f \in \mathscr{C}, \tag{16}$$

extends to a continuous linear functional on H^1_w. It then follows from Theorem 3 (or, more precisely, the proof of Theorem 3) that b has such a representation. This proves part (i).

For part (ii), let $b(x)(1+|x|)^{-n-1} \in L(E^n)$. By Lemma 6 (applied to $w(x)^{-1}$), the integral

$$\iint_{B(I)} t\,|\nabla Pb|^2 e^{-P\log w}\, dy\, dt \tag{17}$$

is equivalent to

$$\iint_{B(I)} t^{1-n} |\nabla Pb(y,t)|^2 \Big(\int_{|x-y|<t} w(x)^{-1} dx\Big)\, dy\, dt,$$

which is essentially the integral in (11). In the proof of Theorem 2, we showed (11) to be at most $c\|b\|_*^2 m_w(I)$. Hence, we conclude that (17) satisfies the same estimate. Conversely, if (17) (or its analogue with ∇ replaced by $\partial/\partial t$) is bounded by $cm_w(I)$ for all I, then (11) is also bounded by $cm_w(I)$ for all I. Thus, by retracing the argument of the proof of Theorem 2 between (8) and (11) (with $\|b\|_*$ deleted in all the estimates), we obtain

$$\Big|\int_{E^n} f(x) b(x)\Big|\, dx \leqslant c\|f\|_{H^1_w}, \qquad f \in \mathscr{C}.$$

It then follows from Theorems 1 and 3 that $b \in \mathrm{BMO}_w$. This completes the proof of Theorem 4.

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Received June 2, 1976 (1165)