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On local ergodic theorems for positive semigroups

by

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Abstract. For a one-parameter semigroup $\Gamma = \{T_t;\ t > 0\}$ of positive linear operators on L_1 of a σ-finite measure space which is assumed to be strongly integrable over every finite interval, the following local ergodic theorem is proved: *If* $0 \leqslant f \in L_1$ *and* $\|f\|_1 > 0$ *imply* $\sup_{t>0} \|T_t f\|_1 > 0$, *then for any* $f \in L_1$ *the limit*

$$\lim_{b\to+0} \frac{1}{b} \int_0^b T_t f\, dt$$

exists and is finte almost everywhere. Under the assumption that Γ is locally bounded, i.e., $\sup_{0<t<1} \|T_t\|_1 < \infty$, a necessary and sufficient condition is given for the possibility of completing Γ to a strongly continuous semigroup on $[0, \infty)$. A local ergodic theorem for the adjoint semigroup $\Gamma^* = \{T_t^*;\ t > 0\}$ of Γ is also considered.

1. Introduction and theorems. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space with positive measure μ, and let $L_p(X) = L_p(X, \mathscr{F}, \mu)$, $1 \leqslant p \leqslant \infty$, denote the (complex) Banach spaces defined as usual with respect to $(X, \mathscr{F}, \mu)$. All sets and functions introduced below are assumed to be measurable; all relations are assumed to hold modulo sets of measure zero. If A is a subset of X, then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish on $X - A$. Also, $L_p^+(A)$ denotes the positive cone of $L_p(A)$ consisting of nonnegative $L_p(A)$-functions. A linear operator T on $L_p(X)$ is called *positive* if $T(L_p^+(X)) \subset L_p^+(X)$, and a *contraction* if $\|T\|_p \leqslant 1$. It is well known that if T is positive, then $\|T\|_p < \infty$. The adjoint of T is denoted by T^*.

Let $\Gamma = \{T_t;\ t > 0\}$ be a one-parameter semigroup of positive linear operators on $L_1(X)$, i.e., all the T_t are positive linear operators on $L_1(X)$ and $T_t T_{t'} = T_{t+t'}$ for all $t, t' > 0$. In this paper we assume that Γ is strongly integrable over every finite interval. This means that for each $f \in L_1(X)$ the vector-valued function $t \to T_t f$ is integrable with respect to Lebesgue measure on every finite interval. It then follows from Lemma VIII.1.3 of [4] that Γ is strongly continuous on $(0, \infty)$, i.e., for each $f \in L_1(X)$ and each $s > 0$ we have $\lim_{t\to s} \|T_t f - T_s f\|_1 = 0$. Hence, by an approximation argument (cf. [13], Section 4), we observe that for each $f \in L_1(X)$ there exists a scalar function $T_t f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0,\infty)$ and $\mathcal{F}$, such that, for each fixed $t>0$, $T_t f(x)$, as a function of x, belongs to the equivalence class of $T_t f$. From this and Fubini's theorem it now follows that there exists a subset $E(f)$ of X with $\mu(E(f))=0$, dependent on f but independent of t, such that if $x\notin E(f)$, then the function $t\to T_t f(x)$ is integrable with respect to Lebesgue measure on every finite interval $(a,b)\subset(0,\infty)$ and the integral $\int_a^b T_t f(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f\,dt$. Γ is called *locally bounded* if $\sup_{0<t<1}\|T_t\|_1<\infty$, and *bounded* if $\sup_{0<t<\infty}\|T_t\|_1<\infty$.

Next, let $\Gamma^*=\{T_t^*;\ t>0\}$ denote the adjoint semigroup of Γ. Therefore, Γ^* acts on $L_\infty(X)$, and $\langle u, T_t^* f\rangle=\langle T_t u, f\rangle$ for all $u\in L_1(X)$, $f\in L_\infty(X)$ and $t>0$. For $0\leqslant a<b<\infty$ and $f\in L_\infty(X)$, we let

$$\int_a^b T_t^* f\,dt=\Big(\int_a^b T_t\,dt\Big)^* f.$$

Therefore,

$$\int_a^b T_t^* f\,dt\in L_\infty(X)\quad\text{and}\quad \Big\langle u,\int_a^b T_t^* f\,dt\Big\rangle=\Big\langle\int_a^b T_t u\,dt, f\Big\rangle$$

for all $u\in L_1(X)$. A slight modification of the proof of Theorem 1.1 of Lin [11] implies that for each $f\in L_\infty(X)$ there exists a scalar function $T_t^* f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0,\infty)$ and $\mathcal{F}$, and a subset $E(f)$ of X with $\mu(E(f))=0$, dependent on f but independent of t, such that if $x\notin E(f)$, then the function $t\to T_t^* f(x)$ is integrable with respect to Lebesgue measure on every finite interval $(a,b)\subset(0,\infty)$ and the integral $\int_a^b T_t^* f(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t^* f\,dt$.

Under the additional assumption that all the T_t are contractions on $L_1(X)$, Akcoglu and Chacon [1] proved that for any $f\in L_1(X)$ the following local erogodic limit

$$\text{(1)}\qquad \lim_{b\to+0}\frac{1}{b}\int_0^b T_t f(x)\,dt$$

exists and is finite a.e. on X. Related results in this direction have been obtained by Krengel [7], Ornstein [13], Fong and Sucheston [5], Kubokawa [8], [9], [10], Baxter and Chacon [2], McGrath [12], the author [15], [16], and by others.

In this paper we apply Akcoglu–Chacon's local ergodic theorem to obtain the following results:

THEOREM 1. *Let $\Gamma=\{T_t;\ t>0\}$ be a one-parameter semigroup of positive linear operators on $L_1(X)$, strongly integrable over every finite interval. Then the space X decomposes into two sets P and N such that*

(i) *for all $f\in L_1(N)$ and $t>0$, we have $\|T_t f\|_1=0$,*

(ii) *for all $f\in L_1^+(P)$, with $\|f\|_1>0$, we have $\sup_{t>0}\|(T_t f)1_P\|_1>0$.*

For any $f\in L_1(X)$ the limit (1) *exists and is finite* a.e. *on P.*

The next theorem shows that, on N, the almost everywhere existence of the limit (1) need not hold in general for all $f\in L_1(X)$.

THEOREM 2. *For any $\varepsilon>0$ there exists a finite measure space $(X,\mathcal{F},\mu)$, and a one-parameter semigroup $\Gamma=\{T_t;\ t\geqslant 0\}$ of positive linear operators on $L_1(X)$, strongly continuous on $[0,\infty)$, such that*

(i) *for all $t\geqslant 0$, we have $T_t 1=1$ and $\|T_t\|_1=1+\varepsilon$,*

(ii) *$\mu(N)>0$, where $X=P+N$ is the decomposition given in Theorem* 1,

(iii) *for some $f\in L_1^+(X)$, the limit* (1) *does not exist* a.e. *on N.*

THEOREM 3. *Let $\Gamma=\{T_t;\ t>0\}$ be a one-parameter semigroup of positive linear operators on $L_1(X)$, strongly integrable over every finite interval. Then the space X decomposes into two sets C and D such that*

(i) *for all $f\in L_1(X)$ and $t>0$, we have $T_t f\in L_1(C)$,*

(ii) *for all $f\in L_1^+(C)$, with $f>0$* a.e. *on C, we have*

$$C=\bigcup_{n=1}^{\infty}\{x\in X:\ T_{1/n}f(x)>0\}.$$

Assume, in addition, that Γ is locally bounded. Then T_t converges strongly as $t\to+0$ if and only if there exists a function $g\in L_1^+(D)$, $g>0$ a.e. *on D, and a decreasing sequence (b_n) of positive reals, with $\lim_n b_n=0$, such that the set*

$$\Big\{\frac{1}{b_n}\int_0^{b_n} T_t g\,dt:\ n\geqslant 1\Big\}$$

is weakly sequentially compact in $L_1(X)$.

THEOREM 4. *Let $\Gamma=\{T_t;\ t>0\}$ be a one-parameter semigroup of positive linear operators on $L_1(X)$, strongly integrable over every finite interval. Assume that $X=C$ in Theorem 3. Then for each $f\in L_\infty(X)$ there exists a scalar function $T_t^* f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0,\infty)$ and $\mathcal{F}$, such that for each fixed $t>0$, $T_t^* f(x)$, as a function of x, belongs to the equivalence class of $T_t^* f$; moreover, the follow-

ing local ergodic limit

(2) $$\lim_{b\to+0}\frac{1}{b}\int_0^b T_t^* f(x)\,dt$$

exists and is finite a.e. *on* X.

Theorem 1 contains a local ergodic theorem of Kubokawa [8] as a special case, since $X = P$ for every one-parameter semigroup $\Gamma = \{T_t;\ t > 0\}$ of positive linear operators on $L_1(X)$ which is strongly continuous on $(0, \infty)$ and satisfies strong-$\lim_{t\to+0} T_t = I$ (the identity operator). Theorem 3 generalizes Theorem 4.1 of Akcoglu and Chacon [1]; their argument is due to the almost everywhere existence of the limit (1) and Lebesgue's convergence theorem, therefore the additional assumption that all the T_t are contractions on $L_1(X)$ cannot be weakened in the argument (cf. Theorem 2). Theorem 4, together with Theorems 1 and 3, generalizes Theorem 1 of Krengel [6], Theorem 7.2 of Lin [11], and a part of Corollary 2 of the author [14].

2. Proof of Theorem 1. Let $\mathscr{A} = \{A \in \mathscr{F}\colon f \in L_1(A) \text{ implies } \|T_t f\|_1 = 0$ for all $t > 0\}$. Since $A, B \in \mathscr{A}$ implies $A \cup B \in \mathscr{A}$, and since $(X, \mathscr{F}, \mu)$ is a σ-finite measure space, there exists a set $N \in \mathscr{A}$ with $A \subset N$ for all $A \in \mathscr{A}$. To prove (ii), let $P = X - N$ and let $f \in L_1^+(P)$ be a nonzero function. By the definition of N, we have $\|T_t f\|_1 > 0$ for some $t > 0$, therefore by (i),

$$\big\|T_{t/2}\big((T_{t/2}f)1_P\big)\big\|_1 = \|T_{t/2}(T_{t/2}f)\|_1 = \|T_t f\|_1 > 0.$$

To prove the remainder of the theorem we may and will assume without loss of generality that $X = P$. Let $a > 0$ be a constant satisfying $e^{-a}\|T_1\|_1 < 1$. Then it follows that for each $f \in L_1(X)$ the vector-valued function $t \to e^{-at}T_t f$ is integrable with respect to Lebesgue measure on $(0, \infty)$. Therefore we can define a positive linear operator R on $L_1(X)$ by the relation

$$Rf = \int_0^\infty e^{-at} T_t f\,dt \qquad (f \in L_1(X)).$$

Let $u = R^* 1 \in L_\infty^+(X)$. Then for any $f \in L_1^+(X)$, with $\|f\|_1 > 0$, we have

$$\langle f, u\rangle = \langle Rf, 1\rangle = \Big\langle \int_0^\infty e^{-at} T_t f\,dt, 1\Big\rangle$$
$$= \int_0^\infty e^{-at}\langle T_t f, 1\rangle\,dt > 0,$$

since $\langle T_t f, 1\rangle$ is a nonnegative nonzero continuous function on $(0, \infty)$. It follows that $u > 0$ a.e. on X. By a similar calculation, we have $T_t^* u \leqslant e^{at} u$ for all $t > 0$. We now define a one-parameter semigroup $\varDelta = \{S_t;\ t > 0\}$ of positive linear contractions on $L_1(X)$ which is strongly continuous on $(0, \infty)$ as follows:

For $t > 0$ and $uf \in L_1(X)$, where $f \in L_1(X)$, let

$$S_t(uf) = e^{-at} u(T_t f).$$

Since

$$\|S_t(uf)\|_1 = \int_X e^{-at} u(T_t f)\,d\mu = \int_X e^{-at}(T_t^* u) f\,d\mu \leqslant \int_X uf\,d\mu$$

for all $f \in L_1^+(X)$, and since $\{uf\colon f \in L_1(X)\}$ is a dense subspace of $L_1(X)$ in the norm topology, S_t may be considered to be a positive linear contraction on $L_1(X)$. By an approximation argument we observe that $S_t S_{t'} = S_{t+t'}$ for all $t, t' > 0$, and that for each fixed $f \in L_1(X)$, the mapping $t \to S_t f$ is strongly continuous on $(0, \infty)$. Therefore we may apply Akcoglu–Chacon's local ergodic theorem to $\varDelta$ to infer that for any $f \in L_1(X)$ the limit

$$\lim_{b\to+0}\frac{1}{b}\int_0^b S_t(uf)(x)\,dt$$

exists and is finite a.e. on X. This proves that for any $f \in L_1(X)$ the limit (1) exists and is finite a.e. on X, since $T_t f = e^{at} S_t(uf)/u$ for all $t > 0$ and $\lim_{t\to+0} e^{at} = 1$.

The proof is complete.

Remark 1. *If all the T_t are contractions on $L_1(X)$, then $T_t(L_1(P)) \subset L_1(P)$.*

To see this, let $g \in L_1^+(P)$, $g > 0$ a.e. on P, and $t > 0$. Define $h_1 = (T_t g)1_P$ and $h_2 = (T_t g)1_N$. Then, since Γ is strongly continuous on $(0, \infty)$, we have

$$h_1 + h_2 = \lim_{s\to+0} T_s(h_1 + h_2) = \lim_{s\to+0} T_s h_1,$$

where the second equality follows from (i) of Theorem 1. Since $\|T_s\|_1 \leqslant 1$ for all $s > 0$, we conclude that $\|h_2\|_1 = 0$, and this completes the proof of Remark 1.

3. Proof of Theorem 2. To prove the theorem, we give the following example:

Let $L_1 = L_1(0, 1]$ (with Lebesgue measure) and let, for each $t \geqslant 0$,

$$S_t f(x) = f([x+t]) \qquad (f \in L_1,\ 0 < x \leqslant 1),$$

where $[x+t] = x+t$ if $x+t \leqslant 1$ and $[x+t] = x+t-n$ if $n < x+t \leqslant n+1$. Then it may be readily seen that $\varDelta = \{S_t;\ t \geqslant 0\}$ is a one-parameter semigroup of positive linear contractions on L_1 and strongly continuous on $[0, \infty)$. Let (a_n) be a sequence of positive reals satisfying

$$a_n > 1 \quad \text{for all } n, \quad \text{and} \quad \sum_{n=1}^{\infty} (1/2^n)\log a_n = \infty.$$

Choose a sequence (β_n) of positive reals satisfying

$$\sum_{n=1}^{\infty} \beta_n < 1, \quad \text{and} \quad (1/\beta_n)\sum_{i>n} \beta_i < 1/a_n \text{ for all } n.$$

We then define

$$d_0 = 0, \quad d_n = \sum_{i=1}^{n} \beta_i, \quad d = \sum_{i=1}^{\infty} \beta_i, \quad a_n = (2^n \beta_n)^{-1},$$

and a function g on $(0, 1]$ by the relation

$$g(x) = \begin{cases} a_n & \text{if} \quad d_{n-1} < x \leqslant d_n, \\ 0 & \text{if} \quad d \leqslant x \leqslant 1. \end{cases}$$

It is obvious that $g \in L_1^+$. But, an elementary calculation shows that if we let

$$g^*(x) = \sup_{0<b<1} \frac{1}{b}\int_0^b S_t g(x)\,dt,$$

then $g^* \notin L_1$. Therefore there exists a sequence (b_n) of positive reals, with $\lim_n b_n = 0$, such that if we let

$$g^*_\infty(x) = \sup_n \frac{1}{b_n}\int_0^{b_n} S_t g(x)\,dt,$$

then $g^*_\infty \notin L_1$. Since

$$\lim_n \frac{1}{b_n}\int_0^{b_n} S_t g(x)\,dt = g(x) \quad \text{a.e. on } (0, 1]$$

by Akcoglu–Chacon's local ergodic theorem and since

$$\lim_n \left\| \frac{1}{b_n}\int_0^{b_n} S_t g\,dt - g \right\|_1 = 0,$$

it now follows from Theorem 4.3 of Derriennic and Lin [3] that there exists a sub-σ-field $\mathscr{B}$ of the Lebesgue measurable subsets of $(0, 1]$ such that $\lim_n E\left(\frac{1}{b_n}\int_0^{b_n} S_t g\,dt \,|\, \mathscr{B}\right)(x)$ does not exist a.e. on $(0, 1]$, where for any $f \in L_1$, $E(f\,|\,\mathscr{B})$ denotes the conditional expectation of f with respect to $\mathscr{B}$.

Let $\varepsilon > 0$ be an arbitrary but fixed number. For each $t \geqslant 0$, define an operator T_t on $L_1(0, 1+\varepsilon]$ by the relation

$$T_t f(x) = \begin{cases} S_t(f 1_{(0,1]})(x) & \text{if} \quad 0 < x \leqslant 1, \\ E\big(S_t(f 1_{(0,1]})\,|\,\mathscr{B}\big)\left(\dfrac{x-1}{\varepsilon}\right) & \text{if} \quad 1 < x \leqslant 1+\varepsilon. \end{cases}$$

Then it is easily seen that $\Gamma = \{T_t;\ t \geqslant 0\}$ is a one-parameter semigroup of positive linear operators on $L_1(0, 1+\varepsilon]$ and strongly continuous on $[0, \infty)$; moreover for any $t \geqslant 0$ we have $T_t 1 = 1$ and $\|T_t\|_1 = 1+\varepsilon$. It is clear that $P = (0, 1]$ and $N = (1, 1+\varepsilon]$. Let $h \in L_1^+(0, 1+\varepsilon]$ be such that $h = g$ a.e. on P. Then for each $n \geqslant 1$ we have, a.e. on N,

$$\frac{1}{b_n}\int_0^{b_n} T_t h(x)\,dt = \left(\frac{1}{b_n}\int_0^{b_n} E(S_t g\,|\,\mathscr{B})\,dt\right)\left(\frac{x-1}{\varepsilon}\right)$$
$$= E\left(\frac{1}{b_n}\int_0^{b_n} S_t g\,dt \,\Big|\, \mathscr{B}\right)\left(\frac{x-1}{\varepsilon}\right).$$

Therefore, $\lim_n \frac{1}{b_n}\int_0^{b_n} T_t h(x)\,dt$ (and hence $\lim_{b\to+0} \frac{1}{b}\int_0^b T_t h(x)\,dt$) does not exist a.e. on N.

This completes the proof of Theorem 2.

4. Proof of Theorem 3. Let $h \in L_1^+(X)$, $h > 0$ a.e. on X, and let

$$C = \bigcup_{n=1}^{\infty} \{x \in X:\ T_{1/n} h(x) > 0\}.$$

Then, by an approximation argument, we observe that $T_t h \in L_1(C)$ for all $t > 0$, and that $T_t f \in L_1(C)$ for all $f \in L_1^+(X)$ and $t > 0$. Thus, (i) is proved.

To prove (ii), let $f \in L_1^+(C)$, $f > 0$ a.e. on C. Then, by (i) and an approximation argument, $\{x \in X:\ T_{1/n} h(x) > 0\} = \{x \in X:\ T_{1/2n}(T_{1/2n} h)(x) > 0\} \subset \{x \in X:\ T_{1/2n} f(x) > 0\}$ for all $n \geqslant 1$. Hence (ii) follows.

To prove the remainder of the theorem, we assume that Γ is locally bounded. Then, letting $f_1 = \int_0^1 T_t f\,dt$ for any $f \in L_1^+(C)$, with $f > 0$ a.e. on C, we have that $\lim_{t\to+0} \|T_t f_1 - f_1\|_1 = 0$ and, by (ii), that $f_1 > 0$ a.e. on C.

Suppose there exists a function $g \in L_1^+(D)$, $g > 0$ a.e. on D, and a decreasing sequence (b_n) of positive reals, with $\lim_n b_n = 0$, such that the set

$$\left\{\frac{1}{b_n}\int_0^{b_n} T_t g\,dt:\ n \geqslant 1\right\}$$

is weakly sequentially compact in $L_1(X)$. Write $h = f_1 + g$. Then $h > 0$ a.e. on X, and the set

$$\left\{\frac{1}{b_n}\int_0^{b_n} T_t h\,dt:\ n \geqslant 1\right\}$$

is again weakly sequentially compact in $L_1(X)$. Since the local boundedness of Γ implies

$$\sup_n \left\|\frac{1}{b_n}\int_0^{b_n} T_t\,dt\right\|_1 < \infty,$$

it follows from Lemma II.3.30 of [4] that for all $f \in L_1(X)$ the set

$$\left\{\frac{1}{b_n}\int_0^{b_n} T_t f\,dt:\ n \geqslant 1\right\}$$

is weakly sequentially compact in $L_1(X)$.

To prove the strong convergence of T_t as $t \to +0$, let $f \in L_1(X)$ be an arbitrary but fixed function. Since Γ is strongly continuous on $(0, \infty)$, there exists a closed separable subspace H of $L_1(X)$, with $f \in H$, such that $T_t(H) \subset H$ for all $t > 0$. Let $\{f_n:\ n \geqslant 1\}$ be a countable dense subset of H. Then, by Cantor's diagonal method, we can find a subsequence (c_n) of (b_n) such that $\text{weak-}\lim_n \int_0^{c_n} T_t f_i\,dt$ exists for all the f_i, $i \geqslant 1$. It now follows from an approximation argument that

$$\text{weak-}\lim_n \frac{1}{c_n}\int_0^{c_n} T_t g\,dt = T_0 g$$

exists for all $g \in H$. Since H is a closed subspace of $L_1(X)$, $T_0 g \in H$ for all $g \in H$. It is easily seen that, on H, $T_t T_{t'} = T_{t+t'}$ for all $t, t' \geqslant 0$. Thus $\Delta = \{T_t;\ t \geqslant 0\}$ may be considered to be a one-parameter semigroup of linear operators on H, and it may be readily seen that $T_0 f$ belongs to the norm-closure of the subspace $\bigcup_{t>0} T_t(H)$. Since Δ is strongly continuous on $(0, \infty)$ and locally bounded, it follows that

$$\lim_{t\to+0} \|T_t f - T_0 f\|_1 = \lim_{t\to+0} \|T_t(T_0 f) - T_0 f\|_1 = 0.$$

Hence T_t converges strongly as $t \to +0$.

Conversely, suppose that T_t converges strongly as $t \to +0$. Then for any $g \in L_1(X)$, $\text{strong-}\lim_{b\to+0} \frac{1}{b}\int_0^b T_t g\,dt$ exists, therefore the condition is obviously necessary.

The proof is complete.

5. Proof of Theorem 4. Fix a function $u \in L_1^+(X)$, with $u > 0$ a.e. on X, and let

$$h = \int_0^\infty e^{-at} T_t u\,dt \quad (\in L_1^+(X)),$$

where $a > 0$ is a constant satisfying $e^{-a}\|T_1\|_1 \leqslant 1$. Then, by Theorem 3, we have $h > 0$ a.e. on X. Therefore, for any $t > 0$ and $f \in L_1(X, \mathscr{F}, h d\mu)$, we can define

$$S_t f = (1/h) T_t(fh).$$

Since the mapping $f \to fh$ is a linear isometry from $L_1(X, \mathscr{F}, h d\mu)$ onto $L_1(X, \mathscr{F}, \mu)$, it follows that $\Delta = \{S_t;\ t > 0\}$ is a one-parameter semigroup of positive linear operators on $L_1(X, \mathscr{F}, h d\mu)$ and strongly integrable over every finite interval. It is easily seen that $S_t^* = T_t^*$ for all $t > 0$ and that $S_t 1 \leqslant e^{at}$ for all $t > 0$. Therefore we may and will assume from the first, without loss of generality, that $(X, \mathscr{F}, \mu)$ is a finite measure space and that $\|T_t\|_\infty \leqslant e^{at}$ for all $t > 0$. Then, by the Riesz convexity theorem and an approximation argument, Γ may be considered to be a one-parameter semigroup of bounded linear operators on $L_2(X)$ which is strongly continuous on $(0, \infty)$. Since $L_2(X)$ is a Hilbert space and the adjoint semigroup Γ^* on $L_2(X)$ is weakly continuous on $(0, \infty)$, it follows that Γ^* on $L_2(X)$ is strongly continuous on $(0, \infty)$. Hence, again by an approximation argument, we observe that Γ^* is a one-parameter semigroup of positive linear operators on $L_1(X)$ and strongly continuous on $(0, \infty)$. Moreover, $\|T_t^*\|_1 = \|T_t\|_\infty \leqslant e^{at}$ for all $t > 0$ implies that Γ^* on $L_1(X)$ is strongly integrable over every finite interval. Therefore, for each $f \in L_\infty(X)$ $(\subset L_1(X))$, there exists a scalar function $T_t^* f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathscr{F}$, such that for each fixed $t > 0$, $T_t^* f(x)$, as a function of x, belongs to the equivalence class of $T_t^* f$.

To prove the remainder of the theorem, we consider the one-parameter semigroup $\{e^{-at} T_t^*;\ t > 0\}$ of positive linear contractions on $L_1(X)$. Since this semigroup is strongly continuous on $(0, \infty)$, we can apply Akcoglu–Chacon's local ergodic theorem to infer that for any $f \in L_\infty(X)$ the limit

$$\lim_{b\to+0} \frac{1}{b}\int_0^b e^{-at} T_t^* f(x)\,dt$$

exists and is finite a.e. on X, and hence the limit (2) exists and is finite a.e. on X for all $f \in L_\infty(X)$.

The proof is complete.

6. An extension of Theorem 1. In this section we assume that $\Gamma = \{T_t;\ t > 0\}$ is a one-parameter semigroup of positive linear operators on $L_p(X)$ for some fixed p, with $1 \leqslant p < \infty$, and strongly integrable over every finite interval. Then, as in Section 1, for each $f \in L_p(X)$ there exists a scalar function $T_t f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathscr{F}$, such that for each fixed $t > 0$, $T_t f(x)$, as a function of x, belongs to the equivalence class of $T_t f$. And there exists a subset $E(f)$ of X with $\mu(E(f)) = 0$, dependent on f but independent of t, such that if $x \notin E(f)$, then the function $t \to T_t f(x)$ is integrable with respect to Lebesgue measure on every finite interval $(a, b) \subset (0, \infty)$ and the integral $\int_a^b T_t f(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f\,dt$.

Slightly modifying the proof of Theorem 1, we can prove the following extension of Theorem 1:

THEOREM 5. *Let $\Gamma = \{T_t;\ t > 0\}$ be a one-parameter semigroup of positive linear operators on $L_p(X)$ for some fixed p, with $1 \leqslant p < \infty$, and strongly integrable over every finite interval. Then the space X decomposes into two sets P and N such that*

(i) *for all $f \in L_p(N)$ and $t > 0$, we have $\|T_t f\|_p = 0$,*

(ii) *for all $f \in L_p^+(P)$, with $\|f\|_p > 0$, we have $\sup_{t>0} \|(T_t f)1_P\|_p > 0$.*

For any $f \in L_p(X)$ the limit (1) *exists and is finite* a.e. *on P.*

We note that this theorem generalizes Kubokawa's local ergodic theorem ([9], Theorem 1).

Remark 2. *Let $1 < p < \infty$, and let $\Gamma = \{T_t; t > 0\}$ be a one-parameter semigroup of bounded linear operators on $L_p(X)$. If Γ is strongly continuous on $(0, \infty)$ and locally bounded, then T_t converges strongly as $t \to +0$.*

This follows from Lemma 1 of the author [15], since $L_p(X)$, with $1 < p < \infty$, is a reflexive Banach space.

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