Beweis. Seien X der Banachraum $\{(x_n)_{n\in N}: x_n \in X_n,\ \sup\|x_n\| < \infty\}$ und $S_m \in B(X)$ mit $S_m(x_n)_n = (\delta_{mn}T_n x_n)_n$. Die S_m sind kompakt und hermitesch mit $S_m^2 \notin H_m$ und wegen 1.3.(a) kann $\|T_m\| = \|S_m\| = 2^{-m}$ angenommen werden. Weiter kann für die Elemente a_n aus 3.2 $\|a_n\| = 1$ angenommen werden. Dann gilt mit 1.3.(b), (c): $T = \sum_{m=1}^{\infty} S_m$ ist kompakt und hermitesch. Aber:

$$\sup\{\|\exp(i\eta T^2)(a_n)_n\|:\ \eta \in \boldsymbol{R}\}$$
$$= \sup\{\|\exp(i\eta T_n^2)a_n\|:\ \eta \in \boldsymbol{R},\ n \in \boldsymbol{N}\} > \sup\{n\|a_n\|:\ n \in \boldsymbol{N}\} = \infty,$$

d.h., $T^2 \notin H_\infty$.

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Duality of linear operators in locally convex spaces

by

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Abstract. Let $T: E\to F$ be a linear non-continuous operator between two locally convex spaces. Facts valid for continuous operators are applicable to T by either strengthening the topology of E or weakening the topology of F. The same point of view for the dual operator leads to a diagram which topological properties are examined for various topologies of the dual spaces.

1. Let $T: E\to F$ be a closable linear operator between two locally convex spaces with domain $D(T)$ dense in E. In order to apply methods available for continuous operators to the treatment of T, there are two obvious possibilities: either to strenghten the topology of $D(T)$ by the graph topology or to weaken the topology of F by the finest locally convex topology that makes T continuous. Let E_T and F^T be these new spaces. They are isomorphic to $G(T)$, the graph of T, and to $(D(T)\times F)/G(-T)$, respectively. Let i_T and i^T be the corresponding continuous injections:

$$(1)\qquad E_T \xrightarrow{i_T} E \xrightarrow{T} F \xrightarrow{i^T} F^T.$$

For a locally convex space X, let $\mathcal{U}_X$ always denote a neighborhood base of 0. Then $\{i_T^{-1}[U\cap T^{-1}(V)]|\ U\in\mathcal{U}_E,\ V\in\mathcal{U}_F\}$ is a neighborhood base of 0 for E_T and $\{i^T[T(U)+V]|\ U\in\mathcal{U}_E,\ V\in\mathcal{U}_F\}$ and $\{i^T(\Gamma[T(U)\cup V])|\ U\in\mathcal{U}_E,\ V\in\mathcal{U}_F\}$ are neighborhood bases of 0 for F^T. Here ΓM is the absolutely convex hull of the set M. For a linear operator S we always write $N(S)$ for its kernel and $R(S)$ for its range.

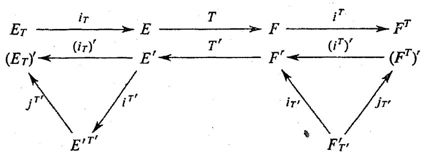
Browder [1] implicitly uses the space E_T to examine closable operators.

Köthe [6] characterises openness and nearly-openness by relations of the equicontinuous sets of the dual spaces and in doing so implicitly uses the space F^T.

In Section 2 we consider questions arising from the dual line of (1). The results are applicable to the examination of state diagrams like those of Loustaunau [7] as well as to the examination of relatively continuous operators (Chilana [2], Förster [3]). This will be carried out in Sections 3 and 4.

2. If in (1) we pass to the duals and use for T' the notations just introduced for T, we get the diagram

(2)



$j_{T'}$, is the bijective mapping arising canonically from the algebraic isomorphy of the spaces $F'_{T'}$, $G(T')$, $G(-T)^0$, $\big((D(T)\times F)/G(-T)\big)'$, and $(F^T)'$. The mapping $j^{T'}$ is defined by $j^{T'} = (i_T)'\circ(i^{T'})^{-1}$ and in general it is only injective because $(i_T)'$ is not surjective.

We are examining the topological properties of $j_{T'}$ and $j^{T'}$: The dual spaces are equipped with polar topologies and we consider the cases in which E', F', $(E_T)'$, and $(F^T)'$ have the weak topologies σ, the Mackey topologies τ, the polar topologies of the precompact sets π, and the strong topologies β. In the sequel we always write ξ for one of these four types of topology. Correspondingly, a ξ-set is a finite, a weakly compact and absolutely convex, a precompact, or a bounded set, respectively.

It is obvious that we cannot expect openness of $j_{T'}$ if we take the topology for E' from E and not from $D(T)$: Let D be a dense subspace of E, $F = D$, and let T be the identity. Then $F^T = E$ and $E' = F'$. For $\xi = \sigma$ or $\xi = \tau$, $\xi(E', E)$ is strictly finer than $\xi(E', D)$, supposed that $D \neq E$. $F'_{T'}$ has $\xi(E', E)$, $(F^T)'$ has $\xi(E', D)$. So $j_{T'}$ is not open.

Counterexamples for π and β are available by specializing E. On the other hand, if $j_{T'}$ is continuous if E' has $\xi\big(E', D(T)\big)$, it is continuous, too, if E' has $\xi(E', E)$, because $F'_{T'}$ has a stronger topology in this case.

To prove that $j_{T'}$ is open we have to show: for ξ-sets $K \subseteq D(T)$, $M \subseteq F$ there exists a ξ-set $N \subseteq F^T$ such that

(3) $$N^0 \subseteq j_{T'}[i_{T'}^{-1}\circ T'^{-1}(K^0)\cap i_{T'}^{-1}(M^0)].$$

For arbitrary operators $S\colon G\to H$ between locally convex spaces and arbitrary sets $Q \subseteq D(S)$ we have

(4) $$S'^{-1}(Q^0) = [S(Q)]^0 \cap D(S'),$$

(5) $$Q^0\cap R(S') = S'\big([S(Q)]^0\big).$$

Since $(F^T)' = D([i^T\circ T]')$, we therefore have for $K \subseteq D(T)$

$$\begin{aligned} j_{T'}[i_{T'}^{-1}\circ T'^{-1}(K^0)\cap i_{T'}^{-1}(M^0)] &= (i^T)'^{-1}[T'^{-1}(K^0)\cap M^0] = (i^T\circ T)'^{-1}(K^0)\cap (i^T)'^{-1}(M^0)\\ &= [i^T\circ T(K)]^0 \cap [i^T(M)]^0 = \big(i^T[T(K)\cup M]\big)^0. \end{aligned}$$

Hence (3) is equivalent to

(7) $$N^0 \subseteq \big(i^T[T(K)\cup M]\big)^0.$$

In view of (5), to prove the continuity of $j_{T'}$, it is sufficient to show that for any ξ-set $N \subseteq F^T$ there exist ξ-sets $K \subseteq D(T)$ and $M \subseteq F$ such that

(8) $$\big(i^T[T(K)\cup M]\big)^0 \subseteq N^0 \quad\text{or}\quad \big(i^T[T(K)+M]\big)^0 \subseteq N^0.$$

THEOREM 1. *For the types σ, τ, π, and β let E', F', and $(F^T)'$ simultaneously have the same type of topology from $D(T)$, E, and F, respectively. Then $j_{T'}$ is open.*

Proof. For $\xi \neq \tau$ set $N = i^T[T(K)\cup M]$ and for $\xi = \tau$ set $N = \Gamma\big(i^T[T(K)\cup M]\big)$; then (7) is valid. It follows from the continuity of $i^T\circ T$ and i^T that N is a ξ-set in both cases.

If the topology for E' is to be taken from E and not from $D(T)$, we have to strengthen the topology of $(F^T)'$:

THEOREM 2. *For the types σ, τ, π, and β let E', F', and $(F^T)'$ simultaneously have the same type of topology from E, F, and $(E\times F)/G(-T)$, respectively. Then $j_{T'}$ is open.*

Proof. F^T is isomorphic to $\big(D(T)\times F\big)/G(-T)$; let $\widetilde{i^T}$ be the mapping that maps F^T isomorphically into $\widetilde{F^T} = (E\times F)/G(-T)$. Then we extend $\widetilde{i^T}\circ i^T\circ T$ continuously to a mapping $\tilde{T}$ defined on all of E: since $\widetilde{i^T}\circ i^T\circ T$ maps $x\in D(T)$ on the coset $[(0, Tx)] = [(x, 0)]$, set $\tilde{T}x = [(x, 0)]$ for $x\in E$. Since $\widetilde{i^T}(F^T)$ is dense in $\widetilde{F^T}$, $(\widetilde{F^T})'$ is isomorphic to $(F^T)'$ and we show that $(\widetilde{i^T})'^{-1}\circ j_{T'}$ is an open mapping from $F'_{T'}$ to $(\widetilde{F^T})'$: As for (7), we see that it is sufficient to show that for ξ-sets $K\subseteq E$, $M\subseteq F$ there exists a ξ-set $N \subseteq \widetilde{F^T}$ with $N^0 \subseteq [\tilde{T}(K)\cup \widetilde{i^T}\circ i^T(M)]^0$. Take $N = \tilde{T}(K)\cup\widetilde{i^T}\circ i^T(M)$ for $\xi\neq\tau$ and $N = \Gamma[\tilde{T}(K)\cup\widetilde{i^T}\circ i^T(M)]$ for $\xi = \tau$.

THEOREM 3. *Let F' and $(F^T)'$ have the topologies $\sigma(F', F)$ and $\sigma\big((F^T)', F^T\big)$ and let E' have any topology. Then $j_{T'}$ is continuous.*

Proof. Set in (8) $K = \{0\}$, $M = (i^T)^{-1}(N)$.

For the strong topologies we generally cannot expect that $j_{T'}$ is continuous.

This is a counterexample:

Let $X = C^{N\setminus\{1\}}$ and let F be the Banach space of those elements of X that have a finite sup-norm. For $x\in X$ we always use the representation $x = (x_i)_{i\in N\setminus\{1\}}$. If $p_1, p_2, \ldots$ is the sequence of prime numbers, we set for $x\in X$ and $n\in N$: $q_n(x) = \sum_{k=1}^{\infty}|x_{kp_n}|$ and $E = \{x \mid x\in F, \bigwedge_{n\in N} q_n(x) < \infty\}$. Let χ denote the metrizable topology of E generated by the q_n, let ϱ denote the norm topology of F, and let T be the identical mapping from E into F. It is easy to verify that T is closed. Let ω denote the topology of the inductive limit of ϱ and χ on F. A neighborhood base of 0 for ω is $\{\varepsilon(U+V_n) \mid \varepsilon > 0,\ n \in N\}$ with $U = \{x \mid x \in F,\ \|x\| \leqslant 1\}$, $V_n = \{x \mid x \in F,\ q_n(x) \leqslant 1\}$. Let $N = \{x^{(j,m)} \mid j \in \mathbf{N},\ m \in \mathbf{N}\}$ with

$$x_i^{(j,m)} = \begin{cases} j & \text{if } \quad i = \prod_{k=j+1}^{l} p_{k-1} \text{ for } j+1 \leqslant l \leqslant j+m, \\ 0 & \text{otherwise}. \end{cases}$$

N is ω-bounded: For $n \in \mathbf{N}$, $m \in \mathbf{N}$, $j \in \mathbf{N}$, and $j > n$ we have $q_n(x^{(j,m)}) = 0$, because $x_i^{(j,m)} = 0$ for those i that are divisible by p_n. For $j \leqslant n$ we have $\|x^{(j,m)}\| = j$. Therefore, $N \subseteq n(U \cup V_n)$. We now show: There exist no two sets $K \subseteq E$, $M \subseteq F$, K χ-bounded, M ϱ-bounded, with $N \subseteq \overline{\Gamma(K \cup M)}^{\omega}$. It is sufficient to take absolutely convex K and M. Then this inclusion implies $N \subseteq K + M + U + V_n$ for all n. Let $q_n(K) \leqslant c_n$, $\|M\| \leqslant c$. Take $n > 2c+4$ and take m such that $2c_n \leqslant nm$. Let $x^{(n,m)} = y+z+u+v$ for $y \in K$, $z \in M$, $u \in U$, $v \in V_n$. There exist exactly m indices i that are divisible by p_n with $x_i^{(n,m)} = n$. Since $q_n(y) \leqslant \frac{1}{2}nm$, there exists one i among them such that $|y_i| \leqslant \frac{1}{2}n$. For this i we must have $|z_i + u_i + v_i| \geqslant \frac{1}{2}n$ and, since $|u_i| \leqslant 1$, $|v_i| \leqslant 1$, also $|z_i| \geqslant \frac{1}{2}n - 2 > c$. This contradicts $\|z\| \leqslant c$. This shows that $j_{T'}$ is not continuous in this case, for, since $E = D(T)$, we would have to fulfil (8) and that is impossible, because passing to polars within $((F^T)', F^T)$ yields $N \subseteq N^{00} \subseteq \overline{\Gamma(i^T[T(K) \cup M])}$ form (8).

Here E is representable as a closed subspace of a countable product of Banach spaces and hence is a Fréchet space. F is a Banach space.

So we can expect $j_{T'}$ to be continuous only if we make restrictions on the spaces E and F or on the operator T.

Theorem 4. *If E', F', and $(F^T)'$ are furnished with $\pi(E', E)$, $\pi(F', F)$, and $\pi((F^T)', F^T)$, respectively, and if E and F are metrizable, then $j_{T'}$ is continuous. This is already valid if E' has $\pi(E', D(T))$.*

Proof. First we show: Let G be a metrizable locally convex space, $H \subseteq G$ a closed subspace, $\nu: G \to G/H$ the canonical homomorphism, $N \subseteq G/H$ precompact; then there exists a precompact set $Q \subseteq G$ such that $N \subseteq \overline{\Gamma\nu(Q)}$. Since G/H is metrizable, too, there exists a null sequence $\{\hat{x}_k\}$ in G/H with $N \subseteq \overline{\Gamma\{\hat{x}_k \mid k \in \mathbf{N}\}}$. Let $\{U_i\}$ be a neighborhood base of zero for E with $U_1 \supseteq U_2 \supseteq \ldots$; then $\{\nu(U_i)\}$ is a neighborhood base of G/H. So for each $i \in \mathbf{N}$ there exists a $k_i \in \mathbf{N}$ such that for $k \geqslant k_i$ we have $\hat{x}_k \in \nu(U_i)$. For $k_i \leqslant k < k_{i+1}$ choose $x_k \in \hat{x}_k$ with $x_k \in U_i$. $\{x_k\}$ is a null sequence so that $Q = \{x_k \mid k \in \mathbf{N}\}$ has the desired properties. We apply this to $G = D(T) \times F$, $H = G(-T)$ so that $k = p(Q)$ and $M = q(Q)$ are precompact if p and q are the projections from $D(T) \times F$ onto $D(T)$ and F, respectively. Now it follows from the construction of the canonical homomorphism μ from $D(T) \times F$ onto F^T that $\mu(Q) \subseteq i^T[T(K) + M]$. This implies (8).

Theorem 5. *If E', F', and $(F^T)'$ are furnished with $\beta(E', E)$, $\beta(F', F)$, and $\beta((F^T)', F^T)$, respectively, and if E and F are metrizable nuclear or metrizable Schwartz spaces, then $j_{T'}$ is continuous. This is already valid if E' has $\beta(E', D(T))$.*

Proof. Subspaces, products, and quotients modulo closed subspaces of nuclear spaces are nuclear, so F^T is nuclear. Since in nuclear spaces the properties precompactness and boundedness coincide, the assertion follows from (4). The same conclusion applies to Schwartz spaces.

Theorem 6. *If E', F', and $(F^T)'$ are furnished with $\beta(E', E)$, $\beta(F', F)$, and $\beta((F^T)', F^T)$, respectively and $D(T)$ and F are (DF)-spaces, then $j_{T'}$ is continuous. This is already valid if E' has $\beta(E', D(T))$.*

Proof. It follows from the permanence properties of (DF)-spaces that F^T is a (DF)-space, too. Each bounded subset of F^T lies in the closed hull of the canonical image of a bounded subset of $D(T) \times F$. As in Theorem 4 this implies the assertion.

Since every normed space is a (DF)-space, $j_{T'}$ is continuous in the normed case.

Theorem 7. *For the types τ, π, and β let E', F', and $(F^T)'$ simultaneously have the same type of topology from $D(T)$ (or E), F, and F^T, respectively. Let T be open and suppose that $N(T)$ and $R(T)$ have topological complements in E and F, respectively. Then $j_{T'}$ is continuous.*

Proof. Let H and L be the topological complements of $N(T)$ and $R(T)$, respectively. Since T is open $\{i^T[T(U) + V_2] \mid U \in \mathscr{U}_E,\ V_2 \in \mathscr{U}_L\}$ is a neighborhood base of zero for F^T. Therefore the mappings $(T|_{D(T) \cap H})^{-1} \circ u \circ (i^T)^{-1}$ and $v \circ (i^T)^{-1}$ are continuous if u and v are the projections from F onto $R(T)$ and L, respectively. So for a ξ-set N the sets $K = (T|_{D(T) \cap H})^{-1} \circ u \circ (i^T)^{-1}(N)$ and $M = v \circ (i^T)^{-1}(N)$ are ξ-sets with $N \subseteq i^T[T(K) + M]$. This implies (8).

Corollary. *$j_{T'}$ is continuous for τ, π, and β if T is a Fredholm operator and if E and F are Fréchet spaces or, more general, are spaces for which the open mapping theorem applies.*

We cannot expect that $j^{T'}$ is open if we take the topologies for E' from E because E_T is bijectively mapped onto $D(T)$ and not onto E. To prove the continuity of $j^{T'}$ we have to show: For each ξ-set $J \subseteq E_T$ there exist ξ-sets $K \subseteq E$, $M \subseteq F$ such that $j^{T'}(\Gamma[i^{T'}(K^0) \cup i^{T'} \circ T'(M^0)]) = (i_T)'(\Gamma[K^0 \cup T'(M^0)]) \subseteq J^0$. In view of (4) we therefore have to show

$$\Gamma[K^0 \cup T'(M^0)] \subseteq [i_T(J)]^0. \tag{9}$$

To prove the openness of $j^{T'}$ we accordingly have to show: For any two ξ-sets $K \subseteq D(T)$, $M \subseteq F$ there exists a ξ-set $J \subseteq E_T$ such that $[i_T(J)]^0 \subseteq \Gamma[K^0 \cup T'(M^0)]$ or

$$[i_T(J)]^0 \subseteq K^0 + T'(M^0). \tag{10}$$

Theorem 8. *For the types σ, τ, π, and β let $(E_T)'$, E', and F' simultaneously have the same type of topology from E_T, $D(T)$ (or E), and F, respectively. Then $j^{T'}$ is continuous.*

Proof. Set $K = i_T(J)$ and $M = T\circ i_T(J)$. Both are ξ-sets and (9) follows from (5): $T'\big([T\circ i_T(J)]^0\big) = [i_T(J)]^0 \cap R(T')$.

Theorem 9. *Let $(E_T)'$ and E' have the topologies $\sigma\big((E_T)', E_T\big)$ and $\sigma\big(E', D(T)\big)$, respectively, and let F' have any topology. Then $j^{T'}$ is continuous.*

Proof. Set $J = i_T^{-1}(K)$.

Theorem 10. *For the types τ, π, and β let $(E_T)'$, E', and F' simultaneously have the same type of topology from E_T, E, and F, respectively. Let T be open and suppose that $N(T)$ and $R(T)$ have topological complements in E and F, respectively. Then $j^{T'}$ is open.*

Proof. Let H and L be the topological complements of $N(T)$ and $R(T)$, respectively. Since T is open, $\{i_T^{-1}(U_1 + [T^{-1}(V)\cap H])|\ U_1 \in \mathscr{U}_{N(T)}, V \in \mathscr{U}_F\}$ is a neighborhood of zero for E_T. Therefore the mappings $i_T^{-1}\circ s$ and $i_T^{-1}\circ(T|_H)^{-1}\circ u$ are continuous if s is the projection from E to $N(T)$ and u is the projection from F to $R(T)$. So $J = \Gamma\big(i_T^{-1}[s(K)\cup(T|_H)^{-1}\circ u(M)]\big)$ is a ξ-set if $K \subseteq E$ and $M \subseteq F$ are ξ-sets. Since T is open, we have $N(T)^0 = R(T')$, so E' and F' are representable as direct sums: $E' = R(T') + H^0$, $F' = R(T)^0 + L^0$. Now it is easy to check that (10) is valid. Note that K really may be taken from E, not only from $D(T)$ here.

Corollary. *$j^{T'}$ is open for τ, π, and β (from E_T, E, and F, respectively) if T is a Fredholm operator and E and F are Fréchet spaces or, more general, are spaces for which the open mapping theorem applies.*

3. Loustaunau [7] presents under very restrictive conditions a method to reduce the study of state diagrams for closed operators to the study of state diagrams for continuous operators. Such results follow from the facts developped above:

The following lemma shows that the operators $T\circ i_T$, T, and $i^T\circ T$ always have the same state.

Lemma. *If one of the operators $T\circ i_T$, T, and $i^T\circ T$ has one of the following properties so do the others, too:*

(i) *injectivity,*

(ii) *surjectivity,*

(iii) *openness,*

(iv) *denseness of the range in the range space.*

Proof. Let $R(i^T\circ T)$ be dense in F^T. Then we show for $y\in F$ and $V\in\mathscr{U}_F$: $(y+V)\cap R(T)\neq\emptyset$. From the assumption follows $i^T[y+T(U)+V] \cap R(i^T\circ T)\neq\emptyset$ for $U\in\mathscr{U}_E$; hence there exist $u\in U$, $v\in V$, $x\in E$ such that $y+Tu+v = Tx$; hence $y+V = T(x-u)\in R(T)$ and $(y+V)\cap \cap R(T)\neq\emptyset$. So $R(T)$ is dense in F.

The other implications are clear immediately.

Since i^T is continuous and defined on all of F, we have $(i^T\circ T)' = T'\circ(i^T)' = T'\circ i_{T'}\circ j_{T'}^{-1}$. The state diagram for continuous operators informs about the possible states of $i^T\circ T$ and $(i^T\circ T)'$. T has the same state as $i^T\circ T$, T' has the same state as $T'\circ i_{T'}$. Hence in those cases in which $j_{T'}$ is a topological isomorphism the state diagram for continuous operators is valid for T, too.

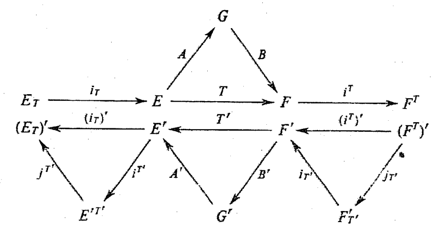
4. Chilana [2] and Förster [3] examine relatively continuous operators for applications in pertubation theory. An operator A: $E\to G$ with $D(T)\subseteq D(A)$ is called *relatively T-continuous* if for any continuous seminorm r on G there exist continuous seminorms p on E and q on F such that for $x\in D(T)$

$$r(Ax) \leqslant p(x) + q(Tx).$$

The operator B: $G\to F$ is called *relatively T-co-continuous* if for any two neighborhoods of zero $U\subseteq E$, $V\subseteq F$ there exists a neighborhood of zero $W\subseteq G$ such that

$$A(W)\subseteq T(U)+V.$$

Using the diagram (2), we equivalently have: A is relatively T-continuous if $A\circ i_T$ is continuous. B is relatively T-co-continuous if $i^T\circ B$ is continuous. If we extend (2) accordingly we easily get the results from [2] and [3] with the following thought:



Under suitable assumptions for the topologies of E, F, and G we have: $i^T\circ B$ is continuous if and only if $(i^T\circ B)'$ is continuous. Since $(i^T\circ B)'\circ j_{T'} = B'\circ(i^T)'\circ j_{T'} = B'\circ i_{T'}$, relative T-co-continuity of B and relative T'-continuity of B' are equivalent in those cases in which $j_{T'}$ is a topological isomorphism.

If $A\circ i_T$ is continuous so does $(i_T)'\circ A'$, since $(i_T)'\circ A'\subseteq (A\circ i_T)'$. Now $(j^{T'})^{-1}\circ(i_T)'\circ A' = i^{T'}\circ A'$, so that relative T-continuity of A implies relative T'-co-continuity of A' in those cases in which $(j^{T'})^{-1}$ is continuous and hence $j^{T'}$ is open.

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On local ergodic theorems for positive semigroups

by

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Abstract. For a one-parameter semigroup $\Gamma = \{T_t;\ t > 0\}$ of positive linear operators on L_1 of a σ-finite measure space which is assumed to be strongly integrable over every finite interval, the following local ergodic theorem is proved: *If* $0 \leqslant f \in L_1$ *and* $\|f\|_1 > 0$ *imply* $\sup_{t>0} \|T_t f\|_1 > 0$, *then for any* $f \in L_1$ *the limit*

$$\lim_{b\to +0} \frac{1}{b} \int_0^b T_t f\,dt$$

exists and is finte almost everywhere. Under the assumption that Γ is locally bounded, i.e., $\sup_{0<t<1} \|T_t\|_1 < \infty$, a necessary and sufficient condition is given for the possibility of completing Γ to a strongly continuous semigroup on $[0, \infty)$. A local ergodic theorem for the adjoint semigroup $\Gamma^* = \{T_t^*;\ t > 0\}$ of Γ is also considered.

1. Introduction and theorems. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space with positive measure μ, and let $L_p(X) = L_p(X, \mathscr{F}, \mu)$, $1 \leqslant p \leqslant \infty$, denote the (complex) Banach spaces defined as usual with respect to $(X, \mathscr{F}, \mu)$. All sets and functions introduced below are assumed to be measurable; all relations are assumed to hold modulo sets of measure zero. If A is a subset of X, then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish on $X - A$. Also, $L_p^+(A)$ denotes the positive cone of $L_p(A)$ consisting of nonnegative $L_p(A)$-functions. A linear operator T on $L_p(X)$ is called *positive* if $T(L_p^+(X)) \subset L_p^+(X)$, and a *contraction* if $\|T\|_p \leqslant 1$. It is well known that if T is positive, then $\|T\|_p < \infty$. The adjoint of T is denoted by T^*.

Let $\Gamma = \{T_t;\ t > 0\}$ be a one-parameter semigroup of positive linear operators on $L_1(X)$, i.e., all the T_t are positive linear operators on $L_1(X)$ and $T_t T_{t'} = T_{t+t'}$ for all $t, t' > 0$. In this paper we assume that Γ is strongly integrable over every finite interval. This means that for each $f \in L_1(X)$ the vector-valued function $t \to T_t f$ is integrable with respect to Lebesgue measure on every finite interval. It then follows from Lemma VIII.1.3 of [4] that Γ is strongly continuous on $(0, \infty)$, i.e., for each $f \in L_1(X)$ and each $s > 0$ we have $\lim_{t\to s} \|T_t f - T_s f\|_1 = 0$. Hence, by an approximation argument (cf. [13], Section 4), we observe that for each $f \in L_1(X)$ there exists