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Lévy's probability measures on Banach spaces

by

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Abstract. The purpose of this paper is to discuss limit laws of certain normed sums of independent random variables with values in a real separable Banach space. We characterize these limit laws in terms of their decomposability semigroups and obtain a representation theorem for the characteristic functionals.

1. Notation and preliminaries. Let X denote a real separable Banach space with the norm $\|\cdot\|$ and with the dual space X^*. By $\langle\cdot,\cdot\rangle$ we shall denote the dual pairing between X and X^*. Further $\mathscr{B}(X)$ will denote the algebra of continuous linear operators on X with the norm topology. Given a subset F of $\mathscr{B}(X)$, by $\mathrm{Sem}(F)$ we shall denote the closed multiplicative semigroup of operators spanned by F. The unit and the zero operators will be denoted by I and 0, respectively. By a *probability measure* μ on X we shall understand a countably additive non-negative set function μ on the class of Borel subsets of X with the property that $\mu(X) = 1$. A sequence $\{\mu_n\}$ of probability measures on X is said to *converge to a probability measure* μ on X if for every bounded continuous real-valued function f on X $\int_X f\,d\mu_n \to \int_X f\,d\mu$. The *characteristic functional* of μ is defined on X^* by the formula

$$\hat{\mu}(y) = \int_X e^{i\langle y,x\rangle}\,\mu(dx) \qquad (y \in X^*).$$

Given an operator A from $\mathscr{B}(X)$ and a probability measure μ on X, we denote by $A\mu$ the probability measure defined by the formula $A\mu(E) = \mu(A^{-1}(E))$ for all Borel subsets E of X. Evidently, $\widehat{A\mu}(y) = \hat{\mu}(A^*y)$ $(y \in X^*)$ and $A_n\mu_n \to A\mu$ whenever $A_n \to A$ and $\mu_n \to \mu$. We call a probability measure on X *full* if its support is not contained in any proper hyperplane of X. Further, by δ_x $(x \in X)$ we shall denote the probability measure concentrated at the point x.

In the study of limit probability distributions [14] the author introduced the concept of decomposability semigroup $\mathscr{D}(\mu)$ of linear operators associated with the probability measure μ. Namely, $\mathscr{D}(\mu)$ consists of all operators A from $\mathscr{B}(X)$ for which the equality

$$\mu = A\mu * \nu$$

holds for a certain probability measure ν. The asterisk denotes here the convolution of measures. It is clear that $\mathscr{D}(\mu)$ is a semigroup under multiplication of operators and $\mathscr{D}(\mu)$ always contains the operators 0 and I. Moreover, $\mathscr{D}(\mu)$ is closed in the weak* operator topology. It has been shown in [14], [15], [16] and [17] that some purely probabilistic properties of μ are equivalent with some algebraic and topological properties of its decomposability semigroup $\mathscr{D}(\mu)$. The main aim of this paper is to characterize a class of limit probability distributions by a certain property of their decomposability semigroups. We note that $A \in \mathscr{D}(\mu)$ if and only if

$$\hat{\mu}(y) = \hat{\mu}(A^* y)\hat{\nu}(y) \qquad (y \in X^*)$$

for a certain probability measure ν on X.

By a projector P on X we mean an operator from $\mathscr{B}(X)$ with the property that $P^2 = P$. The following propositions will be used repeatedly, and for further reference we state they here.

PROPOSITION 1.1. *Let P be a projector belonging to $\mathscr{D}(\mu)$. Then $I-P \in \mathscr{D}(\mu)$ and*

$$\mu = P\mu * (I-P)\mu.$$

For the proof see [17], Lemma 1.

PROPOSITION 1.2. *Let $P_1, P_2, \ldots, P_n$ be commuting projectors belonging to $\mathscr{D}(\mu)$ with the property $P_iP_j = 0$ for all indices $i \neq j$. Then for every collection $A_1, A_2, \ldots, A_n$ from $\mathscr{D}(\mu)$ satisfying the condition $A_jP_j = P_jA_j$ $(j = 1, 2, \ldots, n)$ we have $\sum_{j=1}^{n} P_jA_j \in \mathscr{D}(\mu)$.*

Proof. First we prove the formula

$$\mu = P_1\mu * P_2\mu * \ldots * P_n\mu * \Big(I - \sum_{j=1}^{n} P_j\Big)\mu \tag{1.1}$$

by induction with respect to n. The case $n = 1$ follows from Proposition 1.1. Suppose that for an index k $(k < n)$ we have the formula

$$\mu = P_1\mu * P_2\mu * \ldots * P_k\mu * \Big(I - \sum_{j=1}^{k} P_j\Big)\mu. \tag{1.2}$$

By Proposition 1.1 we have also

$$\mu = P_{k+1}\mu * (1 - P_{k+1})\mu.$$

Applying the operator $I - \sum_{j=1}^{k} P_j$ to both sides of the last equation, we

get

$$\Big(I - \sum_{j=1}^{k} P_j\Big)\mu = P_{k+1}\mu * \Big(I - \sum_{j=1}^{k+1} P_j\Big)\mu$$

which, by virtue of (1.2), gives formula (1.1) in the case $n = k+1$. This completes the proof of (1.1). Further, we have the decompositions

$$\mu = A_j\mu * \nu_j \qquad (j = 1, 2, \ldots, n).$$

Thus

$$P_j\mu = P_jA_j\mu * P_j\nu_j \qquad (j = 1, 2, \ldots, n)$$

and, by (1.1),

$$\mu = P_1A_1\mu * P_2A_2\mu * \ldots * P_nA_n\mu * \nu, \tag{1.3}$$

where $\nu = P_1\nu_1 * P_2\nu_2 * \ldots * P_n\nu_n * (I - \sum_{j=1}^{n} P_j)\mu$. Put $B = \sum_{j=1}^{n} P_jA_j$. Then, by (1.1)

$$B\mu = BP_1\mu * BP_2\mu * \ldots * BP_n\mu * B\Big(I - \sum_{j=1}^{n} P_j\Big)\mu$$
$$= P_1A_1\mu * P_2A_2\mu * \ldots * P_nA_n\mu$$

and, consequently, by (1.3)

$$\mu = B\mu * \nu$$

which completes the proof.

The following consequence of Numakura Theorem ([10], Theorem 3.1.1) will be widely exploited.

PROPOSITION 1.3. *Let $A \in \mathscr{B}(X)$. If the monothetic semigroup* Sem$(\{A\})$ *is compact, then the cluster points of the sequence $\{A^n\}$ form a group $\mathscr{G}$. Moreover, $\mathscr{G}$ is the minimal ideal of* Sem$(\{A\})$ *and* Sem$(\{A\})$ *contains exactly one projector P, namely the unit of $\mathscr{G}$.*

2. Statement of the problem. A triangular collection of probability measures μ_{nj} $(j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots, n)$ on X is called *uniformly infinitesimal* if for every neighborhood U of 0 in X

$$\lim_{n\to\infty} \min_{1\leqslant j\leqslant k_n} \mu_{nj}(U) = 1.$$

It is easy to check that the collection μ_{nj} $(j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ is uniformly infinitesimal if and only if $\mu_{nj_n} \to \delta_0$ for each choice of j_n, $1 \leqslant j_n \leqslant k_n$.

In terms of random variables, the problem we study can be formulated as follows: suppose that $\{\xi_n\}$ is a sequence of independent X-valued random variables with the probability distributions $\{\mu_n\}$ and assume that $\{A_n\}$ and $\{x_n\}$ are sequences from $\mathscr{B}(X)$ and X, respectively such that

(*) A_n are invertible,

(**) $\mathrm{Sem}(\{A_m A_n^{-1}\colon n = 1, 2, \ldots, m;\ m = 1, 2, \ldots\})$ is compact (in the norm topology of $\mathscr{B}(X)$),

(***) the probability measures $A_n\mu_j$ $(j = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ form a uniformly infinitesimal collection and the distribution of

$$A_n \sum_{j=1}^{n} \xi_j + x_n$$

converges to a probability measure μ; what can be said about the limit measure μ? In the one-dimensional case this problem has been solved by P. Lévy: the class of all limit measures in question coincides with the class of all self-decomposable probability measures ([8], p. 195, [9], p. 319). Therefore the limit measures μ will be called *Lévy's measures*. This paper is an outgrowth of my work [14] concerning Lévy's measures on finite-dimensional spaces. All that has been done so far for infinite-dimensional spaces is to describe the limit measures when all operators A_n are multiples of the unit operator. In this case A. Kumar and B. M Schreiber proved in [7] an analogue of the Lévy characterization theorem and obtained a representation of the characteristic functional for some Orlicz spaces.

We note that for full Lévy's measures on finite-dimensional spaces the compactness condition (**) can be omitted ([14], Proposition 3.3). The same is true for non-degenerate measures on a Banach space when A_n are multiples of I.

3. Norming sequences. We say that a sequence $\{A_n\}$ of operators from $\mathscr{B}(X)$ with properties (*) and (**) is a *norming sequence* corresponding to a Lévy's measure μ if there exist sequences $\{\mu_n\}$ and $\{x_n\}$ of probability measures on X with property (***) and elements of X, respectively, such that $A_n(\mu_1 * \mu_2 * \ldots * \mu_n) * \delta_{x_n}$ converges to μ.

PROPOSITION 3.1. *For every norming sequence $\{A_n\}$ corresponding to a full Lévy's measure we have $A_n \to 0$.*

Proof. Suppose that $A_n\nu_n * \delta_{x_n} \to \mu$ where μ is full and $\nu_n = \mu_1 * \mu_2 * \ldots$ $\ldots * \mu_n$ $(n = 1, 2, \ldots)$. By condition (**), $\mathrm{Sem}(\{A_n\colon n = 1, 2, \ldots\})$ is compact. Let A be an arbitrary cluster point of the sequence $\{A_n\}$ and $A^{n_k} \to A$ where $n_k \to \infty$. Since for each n, $n \leqslant n_k$

$$A_{n_k}\nu_{n_k} * \delta_{x_{n_k}} = A_{n_k}\nu_n * A_{n_k}(\mu_{n+1} * \ldots * \mu_{n_k}) * \delta_{x_{n_k}} \quad \text{and}$$

$$A_{n_k}\mu_j \to \delta_0 \quad \text{for each } j \text{ when } k \to \infty,$$

we have

$$\mu = A\nu_n * \mu \quad (n = 1, 2, \ldots). \tag{3.1}$$

Further, by condition (**), $\mathrm{Sem}(\{AA_{n_k}^{-1}\colon k = 1, 2, \ldots\})$ is compact. Let B be a cluster point of the sequence $\{AA_{n_k}^{-1}\}$. Passing, if necessary, to a subsequence we may assume without loss of generality that $AA_{n_k}^{-1} \to B$. Consequently,

$$A = BA. \tag{3.2}$$

By (3.1) we have the equation

$$\mu = AA_{n_k}^{-1}(A_{n_k}\nu_{n_k} * \delta_{x_{n_k}}) * \mu * \delta_{u_k} \tag{3.3}$$

where $u_k = -Ax_{n_k}$. Since the sequence $\{\delta_{u_k}\}$ is conditionally compact ([11], Chapter III, Theorem 2.1), we may assume without loss of generality that $\delta_{u_k} \to \delta_u$. Then (3.3) implies

$$\mu = B\mu * \mu * \delta_u.$$

Consequently,

$$|\hat{\mu}(y)| = |B\mu(y)| \cdot |\hat{\mu}(y)| \qquad (y \in X^*)$$

which yields $|B\mu(y)| = 1$ in a neighborhood of 0 in X^*. Thus $B\mu = \delta_x$ for a certain $x \in X$ ([5], Proposition 2.3). But this is possible for the full measure μ if $B = 0$ and $x = 0$. Now, by (3.2), we get $A = 0$ which shows that $A_n \to 0$.

LEMMA 3.1. *Let $n_k \leqslant m_k$ $(k = 1, 2, \ldots)$ and $n_k \to \infty$. Then for every norming sequence $\{A_n\}$ corresponding to a Lévy's measure μ all cluster points of the sequence $\{A_{m_k}A_{n_k}^{-1}\}$ belong to $\mathscr{D}(\mu)$.*

Proof. Suppose that $A_n\nu_n * \delta_{x_n} \to \mu$, where $\nu_n = \mu_1 * \mu_2 * \ldots * \mu_n$. Then

$$A_{m_k}\nu_{m_k} * \delta_{x_{m_k}} = A_{m_k}A_{n_k}^{-1}(A_{n_k}\nu_{n_k} * \delta_{x_{n_k}}) * \omega_k, \tag{3.4}$$

where ω_k is a probability measure. Let A be a cluster point of the sequence $\{A_{m_k}A_{n_k}^{-1}\}$. For simplicity of notation we may assume that $\{A_{m_k}A_{n_k}^{-1}\}$ is convergent to A and, moreover, the sequence $\{\omega_k\}$ being conditionally compact ([11], Chapter III, Theorem 2.1) converges to a probability measure ω. Then (3.4) yields the equation

$$\mu = A\mu * \omega$$

which shows that $A \in \mathscr{D}(\mu)$.

Given a probability measure μ on X, by $\mathscr{A}(\mu)$ we shall denote the subset of $\mathscr{D}(\mu)$ consisting of all operators A with the property $\mu = A\mu * \delta_x$ for a certain $x \in X$. It is clear that $\mathscr{A}(\mu)$ is a closed subsemigroup of $\mathscr{D}(\mu)$ and $I \in \mathscr{A}(\mu)$.

LEMMA 3.2. *For every norming sequence $\{A_n\}$ corresponding to a full Lévy's measure μ*

$$\mathscr{A}(\mu) \cap \mathrm{Sem}(\{A_m A_n^{-1}\colon n = 1, 2, \ldots, m;\ m = 1, 2, \ldots\}) \tag{3.5}$$

is a compact group containing all cluster points of the sequence $\{A_{n+1}A_n^{-1}\}$.

Proof. The compactness of set (3.5) is evident. Suppose that A is a cluster point of the sequence $\{A_{n+1}A_n^{-1}\}$ and $A_{n_k+1}A_{n_k}^{-1}\to A$. From the equation

$$A_{n+1}\nu_{n+1}*\delta_{x_{n+1}} = A_{n+1}A_n^{-1}(A_n\nu_n*\delta_{x_n})*A_{n+1}\mu_{n+1}*\delta_{u_n}$$

where $\nu_n = \mu_1*\mu_2*\ldots*\mu_n$ and $u_n = x_{n+1}-A_{n+1}A_n^{-1}x_n$ we get, by virtue of (***), $\mu = A\mu*\delta_x$. Thus $A\in\mathscr{A}(\mu)$ and, consequently, A belongs to set (3.5).

Suppose now that B is an element of set (3.5). Consider the monothetic compact semigroup $\mathrm{Sem}(\{B\})$. By Proposition 1.3, the cluster points of the sequence $\{B^n\}$ form a group $\mathscr{G}$. Moreover, $\mathscr{G}$ is the minimal ideal of $\mathrm{Sem}(\{B\})$ and $\mathrm{Sem}(\{B\})$ contains exactly one idempotent P, namely the unit of $\mathscr{G}$. Hence it follows that $\mathscr{G}$ contains an element C with the property

$$BC = CB = P. \tag{3.6}$$

Of course P and C belong to set (3.5). Thus $\mu = P\mu*\delta_x$ for a certain $x\in X$. Since μ is full and P is an idempotent, the last formula yields $PX = X$. Thus $P = I$ and, by (3.6), $C = B^{-1}$ which completes the proof.

Proposition 3.2. *To every full Lévy's measure there corresponds a norming sequence $\{A_n\}$ with the property*

$$A_{n+1}A_n^{-1}\to I. \tag{3.7}$$

Proof. Let $\{B_n\}$ be an arbitrary norming sequence corresponding to a full Lévy's measure μ,

$$B_n\nu_n*\delta_{x_n}\to\mu, \tag{3.8}$$

$\nu_n = \mu_1*\mu_2*\ldots*\mu_n$ and the collection $B_n\mu_j$ $(j = 1,2,\ldots,n;\ n = 1,2,\ldots)$ is uniformly infinitesimal. Put

$$\mathscr{G} = \mathscr{A}(\mu)\cap\mathrm{Sem}(\{B_mB_n^{-1}\colon n = 1,2,\ldots,m;\ m = 1,2,\ldots\}).$$

By Lemma 3.2 $\mathscr{G}$ is a compact group containing all cluster points of the sequence $\{B_{n+1}B_n^{-1}\}$. Consequently, we can choose a sequence $\{C_n\}$ of elements of $\mathscr{G}$ with the property

$$C_n^{-1}-B_{n+1}B_n^{-1}\to 0. \tag{3.9}$$

Put $A_1 = B_1$ and $A_n = C_1C_2\cdot\ldots\cdot C_{n-1}B_n$ $(n = 2,3,\ldots)$. Obviously, A_n are invertible and $\mathrm{Sem}(\{A_mA_n^{-1}\colon n = 1,2,\ldots,m;\ m = 1,2,\ldots\})$ being a closed subsemigroup of $\mathrm{Sem}(\{B_mB_n^{-1}\colon n = 1,2,\ldots,m;\ m = 1,2,\ldots\})$ is compact. Further, by the assumption, $B_n\mu_{j_n}\to\delta_0$ for each choice of j_n, $1\leqslant j_n\leqslant n$. Since the sequence $\{C_1C_2\cdot\ldots\cdot C_n\}$ of elements of $\mathscr{G}$ is conditionally compact, the last relation yields $A_n\mu_{j_n}\to\delta_0$. Thus condition (***) is fulfilled. Moreover, the conditional compactness of $\{C_1C_2\cdot\ldots\cdot C_n\}$ implies the conditional compactness of the sequence $\{A_n\nu_n*\delta_{u_n}\}$, where $u_n = C_1C_2\cdot\ldots\cdot C_{n-1}x_n$ $(n = 2,3,\ldots)$. By (3.8) each cluster point of $\{A_n\nu_n*\delta_{u_n}\}$ is of the form $C\mu$, where C is a cluster point of the sequence $\{C_1C_2\cdot\ldots\cdot C_n\}$. But $C\in\mathscr{G}$ and, consequently, $\mu = C\mu*\delta_{v_c}$. Hence it follows that we can choose elements v_n in X in such a way that

$$A_n\nu_n*\delta_{v_n}\to\mu.$$

Thus $\{A_n\}$ is a norming sequence corresponding to μ.

To prove condition (3.7) we observe that the norms of elements of the compact set $\mathscr{G}$ are bounded in common, say by a constant b. Thus

$$\|A_{n+1}A_n^{-1}-I\| = \|C_1C_2\cdot\ldots\cdot C_n(B_{n+1}B_n^{-1}-C_n^{-1})C_{n-1}^{-1}C_{n-2}^{-1}\ldots C_1^{-1}\|$$
$$\leqslant b^2\|B_{n+1}B_n^{-1}-C_n^{-1}\|$$

which, by (3.9), implies (3.7). Proposition 3.2 is thus proved.

4. Decomposability semigroups of full Lévy's measures. In this section we shall give a characterization of full Lévy's measures on X in terms of their decomposability semigroups. Let μ be a full Lévy's measure. By Proposition 3.2 we choose a norming sequence $\{A_n\}$ corresponding to μ with the property $A_{n+1}A_n^{-1}\to I$. We fix this norming sequence for the remainder of this section and for simplicity of notation we put

$$\mathscr{S} = \mathscr{D}(\mu)\cap\mathrm{Sem}(\{A_mA_n^{-1}\colon n = 1,2,\ldots,m;\ m = 1,2,\ldots\}).$$

Let P be a projector belonging to $\mathscr{S}$ and

$$\mathscr{S}_P = \mathscr{S}\cap\{A\colon AP = PA = A\}.$$

It is clear that $\mathscr{S}_P$ is a compact subsemigroup of $\mathscr{D}(\mu)$. Further, by $\mathscr{G}_P$ we denote the subset of $\mathscr{S}_P$ consisting of those operators A for which $P\mu = A\mu*\delta_x$ for a certain $x\in X$.

Lemma 4.1. *$\mathscr{G}_P$ is a compact group with the unit P.*

Proof. It is easy to check that $\mathscr{G}_P$ is a closed subsemigroup of $\mathscr{S}_P$ which implies the compactness of $\mathscr{G}_P$. By the definition of $\mathscr{G}_P$ the projector P is the unit of $\mathscr{G}_P$. Let $A\in\mathscr{G}_P$. Then the monothetic semigroup $\mathrm{Sem}(\{A\})$ is compact and, by Proposition 1.3 contains a projector Q and an operator B with the property

$$AB = BA = Q. \tag{4.1}$$

Of course, $PQ = QP = Q$ and $P\mu = Q\mu*\delta_x$ for an element $x\in X$. Since μ is full, the last formula yields $PX = QX$. Consequently, $P = Q$ and, in view of (4.1), $\mathscr{G}_P$ is a group.

Lemma 4.2. *If $A\in\mathscr{S}_P$ and $P\in\mathrm{Sem}(\{A\})$, then $A\in\mathscr{G}_P$.*

Proof. Let $A^{k_n} \to P$. Of course, without loss of generality we may assume that $k_n \geqslant 2$ and the sequence $\{A^{k_n-1}\}$ is convergent to an operator B. Then we have $AB = P$ and for some probability measures ν and λ

$$\mu = A\mu * \nu, \tag{4.2}$$

$$\mu = B\mu * \lambda, \tag{4.3}$$

because $A, B \in \mathscr{D}(\mu)$. From (4.3) we get $A\mu = P\mu * A\lambda$. Hence and from (4.2) we obtain the equation $\mu = P\mu * A\lambda * \nu$. Consequently, $P\mu = P\mu * * A\lambda * P\nu$ or in terms of the characteristic functionals

$$\widehat{P\mu}(y) = \widehat{P\mu}(y)\widehat{A\lambda}(y)\widehat{P\nu}(y) \quad (y \in X^*).$$

Thus $|\widehat{P\nu}(y)| = 1$ in a neighborhood of 0 in X^* which implies $P\nu = \delta_x$ for a certain $x \in X$ ([5], Proposition 2.3). Now taking into account (4.2) we have $P\mu = A\mu * \delta_x$ which completes the proof.

LEMMA 4.3. *For every non-zero projector P belonging to $\mathscr{S}$ the semigroup $\mathscr{S}_P$ contains a one-parameter semigroup $P\exp tV$ $\big(t \geqslant 0,\ V \in \mathscr{B}(X)\big)$ with $PV = VP = V$. Moreover, $\mathscr{S}_P$ contains a projector Q with the properties $P \neq Q$, $QV = VQ$ and*

$$\lim_{t\to\infty} (P-Q)\exp tV = 0.$$

Proof. By Lemma 4.1 the group $\mathscr{G}_P$ is compact. Put

$$a_{n,m} = \min\{\|P - HA_mA_n^{-1}P\|:\ H \in \mathscr{G}_P\}.$$

Obviously,

$$a_{n,n} = 0 \quad (n = 1, 2, \ldots), \tag{4.4}$$

and, by Proposition 3.1,

$$\lim_{m\to\infty} a_{n,m} = \|P\| \geqslant 1 \quad (n = 1, 2, \ldots). \tag{4.5}$$

Since the semigroup $\mathscr{S}$ is compact, all its elements have the norm bounded in common by a constant b. Consequently, for $m \geqslant n$

$$\begin{aligned} a_{n,m+1} &\leqslant \min\{\|P - HA_mA_n^{-1}P\| + \|H(A_{m+1}A_m^{-1} - I)A_mA_n^{-1}P\|:\ H \in \mathscr{G}_P\} \\ &\leqslant a_{n,m} + b^2\|A_{m+1}A_m^{-1} - I\| \end{aligned}$$

which implies for $m \geqslant n_m$

$$\limsup_{m\to\infty} (a_{n_m,m+1} - a_{n_m,m}) = 0. \tag{4.6}$$

Given a number c satisfying the condition $0 < c < 1$, we can find, by virtue of (4.4) and (4.5), an index $m_n \geqslant n$ such that $a_{n,m_n} < c$ and

$a_{n,m_n+1} \geqslant c$ $(n = 1, 2, \ldots)$. From (4.6) it follows that $a_{n,m_n} \to c$. By the conditional compactness of the sequence $\{A_{m_n}A_n^{-1}\}$ and the compactness of $\mathscr{G}_P$, we can choose a cluster point A_c of $\{A_{m_n}A_n^{-1}\}$ and $D_c \in \mathscr{G}_P$ such that

$$\|P - D_cA_cP\| = c = \min\{\|P - CA_cP\|:\ C \in \mathscr{G}_P\}.$$

By Lemma 3.1, $A_c \in \mathscr{S}$. Consequently, setting $B_c = D_cA_cP$ we have $B_c \in \mathscr{S}_P$ and

$$\|P - B_c\| = c = \min\{\|P - CB_c\|:\ C \in \mathscr{G}_P\} \tag{4.7}$$

which yields

$$B_c \in \mathscr{G}_P. \tag{4.8}$$

Put

$$b_{n,c} = \min\{\|P - CB_c^n\|:\ C \in \mathscr{G}_P\}.$$

By (4.7), we have

$$b_{1,c} = c. \tag{4.9}$$

Consider the semigroup $\mathrm{Sem}(\{B_c\})$. By Proposition 1.3 it contains a projector P_c. Of course,

$$\limsup_{n\to\infty} b_{n,c} \geqslant \min\{\|P - CP_c\|:\ C \in \mathscr{G}_P\}. \tag{4.10}$$

Since $P_c \in \mathscr{S}_P$, $P - P_c$ is also a projector and, by Lemma 4.2, $P_c \neq P$. Thus

$$\|P - P_c\| \geqslant 1. \tag{4.11}$$

Put

$$a = \inf\{\|P - CP_c\|:\ C \in \mathscr{G}_P,\ 0 < c < 1\}.$$

We shall prove that $a > 0$. Contrary to this let us assume that $a = 0$. Then, by the compactness of $\mathscr{S}_P$ and $\mathscr{G}_P$, we can find an element D of $\mathscr{G}_P$ and a cluster point R of $\{P_c:\ 0 < c < 1\}$ with the property $P = DR$. Since R is also a projector and $R \in \mathscr{S}_P$, we have $R = PR = DR = P$. Consequently, P is a cluster point of $\{P_c:\ 0 < c < 1\}$ which contradicts (4.11). Thus $a > 0$ and, by (4.10),

$$\limsup_{n\to\infty} b_{n,c} \geqslant a > 0 \tag{4.12}$$

for every c $(0 < c < 1)$. Further, taking into account that all elements of the compact semigroup $\mathscr{S}$ have norm bounded by a constant b, we have, in view of (4.7),

$$b_{n+1,c} \leqslant \min\{\|P - CB_c^n\| + \|C(B_c^n - B_c^{n+1})\|:\ C \in \mathscr{G}_P\} \leqslant b_{n,c} + bc.$$

Consequently, for any sequence $\{m_n\}$ and $c_n \to 0$ we obtain

$$\limsup_{n\to\infty}(b_{m_n+1,c_n} - b_{m_n,c_n}) = 0. \tag{4.13}$$

Let $c_n \to 0$. Given a number d satisfying the condition $0 < d < a$, we can find, by virtue of (4.9) and (4.12), an integer m_n such that $b_{m_n,c_n} < d$ and $b_{m_n+1,c_n} \geq d$. From (4.13) it follows that $b_{m_n,c_n} \to d$. The sequence $\{B_{c_n}^{m_n}\}$ of elements of $\mathscr{S}_P$ is conditionally compact. Let E_d be its cluster point. Then

$$\min\{\|P - CE_d\|\colon C \in \mathscr{G}_P\} = d \quad (0 < d < a) \tag{4.14}$$

and, consequently,

$$E_d \in \mathscr{G}_P \quad (0 < d < a). \tag{4.15}$$

The set $\{E_d\colon 0 < d < a\}$ is also conditionally compact. Let E_0 be its cluster point when $d \to 0$. Then, by (4.14) and the compactness of $\mathscr{G}_P$, $P = C_0 E_0$ for a certain element C_0 of the group $\mathscr{G}_P$. Since $E_0 \in \mathscr{S}_P$, this implies $E_0 \in \mathscr{G}_P$. Consequently, by Proposition 1.3, there exists a positive integer q such that

$$\|P - E_0^q\| < \tfrac{1}{4}.$$

Taking a positive number d_0 with the property

$$\|E_0^q - E_{d_0}^q\| < \tfrac{1}{4},$$

we put

$$W = E_{d_0}^q. \tag{4.16}$$

Then

$$\|P - W\| < \tfrac{1}{2} \tag{4.17}$$

and, by the definition of the operators E_d,

$$B_{c_n}^{r_n} \to W \tag{4.18}$$

where $r_n \to \infty$. From (4.7) and (4.17) it follows that the operators B_{c_n} and W can be represented in an exponential form

$$B_{c_n} = P\exp U_n, \quad W = P\exp V \tag{4.19}$$

where $U_n, V \in \mathscr{B}(X)$, $PV = VP = V$, $PU_n = U_n P = U_n$,

$$WV = VW, \tag{4.20}$$

and, by (4.18),

$$r_n U_n \to V. \tag{4.21}$$

Let t be an arbitrary positive real number. Then, by (4.19) and (4.21),

$$B_{c_n}^{[r_n t]} \to P\exp tV,$$

where square brackets denote the integral part. Since $B_{c_n} \in \mathscr{S}_P$, we infer that the one-parameter semigroup $P\exp tV$ $(t \geq 0)$ is contained in $\mathscr{S}_P$. Consider the semigroup $\mathrm{Sem}(\{W\})$. By Proposition 1.3, it contains a projector Q. By (4.20), Q and V commute with one another. Moreover, by (4.16) $Q \in \mathrm{Sem}(\{E_{d_0}\})$. By (4.15) and Lemma 4.2, we have the inequality $P \neq Q$. Obviously, $Q \in \mathscr{S}_P$ and the set $\{(P-Q)\exp tV\colon t \geq 0\}$ is conditionally compact. Let H be its cluster point when $t \to \infty$. Then for a sequence $\{t_n\}$ tending to ∞ we have

$$(P-Q)\exp t_n V \to H. \tag{4.22}$$

Passing to a subsequence, if necessary, we may assume without loss of generality that both sequences $\{P\exp[t_n]V\}$ and $\{P\exp(t_n - [t_n])V\}$ are convergent to H_1 and H_2, respectively. By (4.19) H_1 is a cluster point of the sequence $\{W^n\}$. Consequently, $QH_1 = H_1 Q = H_1$. Thus $(P-Q)H_1 = 0$, because $H_1 \in \mathscr{S}_P$. Furthermore, by (4.22), $H = (P-Q)H_1 H_2$ which implies $H = 0$. Thus we have proved that

$$\lim_{t\to\infty}(P-Q)\exp tV = 0$$

which completes the proof of the lemma.

LEMMA 4.4. *Suppose that μ is a probability measure on X and $\mathscr{D}(\mu)$ contains a one-parameter semigroup $\exp tV$ $(t \geq 0)$ and $\lim_{t\to\infty}\exp tV = 0$. Then for every positive integer m there exists a probability measure μ_m such that for every $t \geq 0$ $\exp tV \in \mathscr{D}(\mu_m)$ and $\mu = \mu_m^{*m}$, where the power is taken in the sense of convolution.*

Proof. We use arguments similar to that given by A. Kumar and B. M. Schreiber in [7] (Theorem 2.6). Put $T_u = \exp uV$ $(u > 0)$. Then

$$\mu = T_u\mu * \nu_u$$

and, by iteration,

$$\mu = \nu_u * T_u\nu_u * T_{2u}\nu_u * \ldots * T_{(n-1)u}\nu_u * T_{nu}\mu.$$

Setting

$$\nu_{n,u} = \nu_u * T_u\nu_u * T_{2u}\nu_u * \ldots * T_{(n-1)u}\nu_u,$$

we have $\mu = \nu_{n,u} * T_{nu}\mu$. By the assumption $T_{nu} \to 0$ which yields $T_{nu}\mu \to \delta_0$. Consequently,

$$\nu_{n,u} \to \mu. \tag{4.22}$$

Given a positive integer m, we put

$$\lambda_{n,u} = \nu_u * T_{mu}\nu_u * T_{2mu}\nu_u * \ldots * T_{(n-1)mu}\nu_u.$$

Then

$$\lambda_{n,u} * T_u\lambda_{n,u} * T_{2u}\lambda_{n,u} * \ldots * T_{(m-1)u}\lambda_{n,u} = \nu_{nm,u} \tag{4.23}$$

and the right-hand side of the last equation converges to μ as $n\to\infty$. Consequently, the sequence $\{\lambda_{n,u}\}$ is shift compact, i.e. there exists a sequence $\{x_n\}$ of elements of X such that $\{\lambda_{n,u}*\delta_{x_n}\}$ is conditionally compact ([11], Chapter III, Theorem 2.2). Let λ_u be a cluster point of $\{\lambda_{n,u}*\delta_{x_n}\}$. Then for a subsequence $n_1<n_2<\ldots$

$$\lambda_{n_k,u}*\delta_{y_k}\to\lambda_u \tag{4.24}$$

where $y_k=x_{n_k}$. From (4.22) and (4.23) we get the formula

$$\lambda_u*T_u\lambda_u*T_{2u}\lambda_u*\ldots*T_{(m-1)u}\lambda_u*\delta_{z_u}=\mu \tag{4.25}$$

for a certain element z_u of X. Now let r be an arbitrary positive integer. Then for every n we have the formula

$$\nu_u*T_{mu}\nu_u*\ldots*T_{(r-1)mu}\nu_u*T_{rmu}\lambda_{n,u} = \lambda_{n,u}*T_{nmu}\nu_u*T_{(n+1)mu}\nu_u*\ldots*T_{(n+r-1)mu}\nu_u. \tag{4.26}$$

Clearly, the sequence

$$\{T_{nmu}\nu_u*T_{(n+1)mu}\nu_u*\ldots*T_{(n+r-1)mu}\nu_u\}$$

converges to δ_0 as $n\to\infty$. Consequently, by (4.24), the right-hand side of (4.26) is shift compact. Thus the sequence

$$\{\nu_u*T_{mu}\nu_u*\ldots*T_{(r-1)mu}\nu_u\} \tag{4.27}$$

is also shift compact ([11], Chapter III, Theorem 2.2). Hence and from (4.24) and (4.26) we get, as $n=n_k\to\infty$,

$$\lambda_u=T_{rmu}\lambda_u*\varrho_{r,u} \tag{4.28}$$

where the probability measure $\varrho_{r,u}$ is a cluster point of translates of (4.27). Let $\{u_k\}$ be a sequence of positive numbers converging to 0. By (4.25) $\{\lambda_{u_k}\}$ is shift compact. Passing to a subsequence, if necessary, we may assume that for a sequence $\{x_k\}$ of elements of X the sequence $\{\lambda_{u_k}*\delta_{x_k}\}$ converges to a probability measure λ. Moreover, by (4.25) we have $\lambda^{*m}*\delta_x=\mu$ for a certain $x\in X$. Further, let t be a positive number and $r_k=[t/mu_k]$. Then $r_kmu_k\to t$. Set $r=r_k$ and $u=u_k$ into (4.28). We can argue as above to conclude that there is probability measure ϱ_t such that

$$\lambda=T_t\lambda*\varrho_t.$$

Setting $\mu_m=\lambda*\delta_{x/m}$, we get the assertion of the lemma.

The relation $\mu=\mu_m^{*m}$ means that μ is infinitely divisible. Since for every $y\in X^*$ $\hat{\mu}(ty)(-\infty<t<\infty)$ is the characteristic function of an infinitely divisible probability measure on the real line ([9], p. 297) we have the following corollary.

COROLLARY 4.1. *If $\mathscr{D}(\mu)$ contains a one-parameter semigroup $\exp tV$ $(t\geqslant 0)$ and $\lim_{t\to\infty}\exp tV=0$, then $\hat{\mu}(y)\neq 0$ for every $y\in X^*$.*

COROLLARY 4.2. *Suppose that for every $t\geqslant 0$ we have a decomposition*

$$\mu=\exp tV\mu*\nu_t \tag{4.29}$$

where $\lim_{t\to\infty}\exp tV=0$. Then ν_t is infinitely divisible.

Proof. The probability measure μ fulfils the conditions of Lemma 4.4. Consequently, for every positive integer m there exists a probability measure μ_m such that $\mu_m^{*m}=\mu$ and $\mathscr{D}(\mu_m)$ contains all operators $\exp tV$ $(t\geqslant 0)$. Thus μ_m can be written in the form

$$\mu_m=\exp tV\mu_m*\nu_{m,t}, \tag{4.30}$$

where $\nu_{m,t}$ is a probability measure. By Corollary 4.1, $\hat{\mu}(y)\neq 0$ for all $y\in X^*$. Consequently, $\hat{\mu}_m(y)\neq 0$ for all $y\in X^*$. Since $\hat{\mu}_m(y)^m=\hat{\mu}(y)$, we have, by virtue of (4.29) and (4.30), $\hat{\nu}_{m,t}(y)^m=\hat{\nu}_t(y)$ which implies $\nu_t=\nu_{m,t}^{*m}$. Thus ν_t is infinitely divisible.

Now we are ready to prove a characterization theorem for full Lévy's measures on X.

THEOREM 4.1. *A full probability measure on a real separable Banach space X is a Lévy's measure if and only if its decomposability semigroup contains a one-parameter semigroup $\exp tV$ $(t\geqslant 0)$ where $V\in\mathscr{B}(X)$ and*

$$\lim_{t\to\infty}\exp tV=0.$$

Proof. We start by proving the necessity of the assertion. Suppose that μ is a full Lévy's measure. By Proposition 3.2 we choose a norming sequence $\{A_n\}$ corresponding to μ with the property $A_{n+1}A_n^{-1}\to I$. By Lemma 3.1, $I\in\mathscr{S}$. By consecutive application of Lemma 4.3 we get a system of projectors $P_0=I,P_1,\ldots,P_r$ and a system of operators $V_1,V_2,\ldots,V_r$ with the following properties: $\mathscr{S}_{P_j}$ contains the one-parameter semigroup $P_j\exp tV_{j+1}$ $(t\geqslant 0)$, $P_jV_{j+1}=V_{j+1}P_j=V_{j+1}$, $P_{j+1}\in S_{P_j}$, $P_{j+1}V_{j+1}=V_{j+1}P_{j+1}$, $P_j\neq P_{j+1}$ and $\lim_{t\to\infty}(P_j-P_{j+1})\exp tV_{j+1}=0$ $(j=0,1,\ldots,r-1)$. Moreover, we may assume that $P_r=0$ because in the opposite case we would have a sequence $\{P_n\}$ of different commuting projectors belonging to $\mathscr{S}$ and, consequently, satisfying the inequality $\|P_n-P_m\|\geqslant 1$ $(n\neq m;\ n,m=1,2,\ldots)$ which would contradict the compactness of $\mathscr{S}$. Further, the condition $P_{j-1}\in\mathscr{S}_{P_j}$ yields $P_jP_{j-1}=P_{j-1}P_j=P_j$. Thus, by

Proposition 1.1, the projector $Q_j = P_{j-1} - P_j = P_{j-1}(I - P_j)$ belongs to $\mathscr{D}(\mu)$. Moreover, $\sum_{j=1}^{r} Q_j = I$, $Q_j V_j = V_j Q_j$, the one-parameter semigroup $Q_j \exp tV_j$ $(t \geqslant 0)$ is contained in $\mathscr{D}(\mu)$ and $\lim_{t\to\infty} \sum_{j=1}^{r} Q_j \exp tV_j = 0$. Applying Proposition 1.2, we infer that $\sum_{j=1}^{r} Q_j \exp tV_j \in \mathscr{D}(\mu)$. Setting $V = \sum_{j=1}^{r} Q_j V_j$, we have $\exp tV = \sum_{j=1}^{r} Q_j \exp tV_j$ which completes the proof of the necessity.

To prove the sufficiency let us assume that $\mathscr{D}(\mu)$ contains $\exp tV$ for $t \geqslant 0$ and

$$\lim_{t\to\infty} \exp tV = 0.$$

Setting $B_n = \exp(1/n)V$ $(n = 1, 2, \ldots)$, we have the formula

$$\mu = B_n \mu * \lambda_n. \tag{4.31}$$

By Corollary 4.1, $\hat{\mu}(y) \neq 0$ for all $y \in X^*$. Consequently,

$$\hat{\lambda}_n(y) = \frac{\hat{\mu}(y)}{\hat{\mu}(B_n^* y)}. \tag{4.32}$$

From (4.31) and the relation $B_n \mu \to \mu$ it follows that the sequence $\{\lambda_n\}$ is conditionally compact ([11], Chapter III, Theorem 2.1). Since, by (4.32), $\hat{\lambda}_n(y) \to 1$, we infer that $\lambda_n \to \delta_0$. Put

$$A_n = \exp \sum_{j=1}^{n} \frac{1}{j} V \quad (n = 1, 2, \ldots)$$

and

$$\mu_1 = A_1^{-1} \mu, \quad \mu_n = A_n^{-1} \lambda_n \quad (n = 2, 3, \ldots). \tag{4.33}$$

It is easy to check that

$$\{\exp tV \colon t \geqslant 0\} \cup \{0\} = \mathrm{Sem}(\{A_m A_n^{-1} \colon n = 1, 2, \ldots, m;\ m = 1, 2, \ldots\}).$$

Hence it follows that the sequence $\{A_n\}$ fulfils the conditions $(*)$ and $(**)$. We observe that $A_n \to 0$ and, consequently, $A_n \mu_{j_n} \to \delta_0$ whenever the sequence $\{j_n\}$ is bounded. If $j_n \to \infty$ and $j_n \leqslant n$ then, by (4.33) $A_n \mu_{j_n} = A_n A_{j_n}^{-1} \lambda_{j_n}$ $(j_n \geqslant 2)$ and the relation $A_n \mu_{j_n} \to \delta_0$ is a consequence of the conditional compactness of the sequence $A_n A_{j_n}^{-1}$ and the relation $\lambda_n \to \delta_0$. Consequently, the condition $(***)$ is also fulfilled. Setting $\nu_n = \mu_1 * \mu_2 * \ldots * \mu_n$, we have, in view of (4.32) and (4.33)

$$\widehat{A_n \nu_n}(y) = \prod_{j=1}^{n} \hat{\mu}_j(A_n^* y) = \hat{\mu}(y)$$

and, consequently, $A_n \nu_n = \mu$ which shows that μ is a Lévy's measure. The theorem is thus proved.

5. A representation of Lévy's measures. An analogue of the Lévy–Khinchine representation of infinitely divisible probability measures on Banach spaces and even on more general algebraic structures has been studied by A. Tortrat in [12] and [13] and by E. Dettweiler in [3]. Recall that for any bounded non-negative Borel measure F on X vanishing at 0, the Poisson measure $e(F)$ is defined by

$$e(F) = e^{-F(X)} \sum_{k=0}^{\infty} \frac{1}{k!} F^{*k},$$

where $F^{*0} = \delta_0$. The measure F is called a *Poisson exponent* of $e(F)$. Let M be a not necessarily bounded Borel measure on X vanishing at 0. If there exists a representation $M = \sup F_n$, where F_n are bounded and the sequence $\{e(F_n)\}$ of associated Poisson measures is shift compact, then each cluster point of the sequence $\{e(F_n) * \delta_{x_n}\}$ $(x_n \in X)$ is called a *generalized Poisson measure* and denoted by $\tilde{e}(M)$. Clearly, $\tilde{e}(M)$ is uniquely determined up to a translation, i.e. for two cluster points, say μ_1 and μ_2 of $\{e(F_n) * \delta_{x_n}\}$ and $\{e(F_n) * \delta_{y_n}\}$, respectively, we have $\mu_1 = \mu_2 * \delta_x$ for a certain $x \in X$. Further, the measure M is called a *generalized Poisson exponent* of $\tilde{e}(M)$. Clearly, M has a finite mass outside every neighborhood of 0 in X. Let $\mathfrak{M}(X)$ denote the set of all generalized Poisson exponents on X. It is easy to check that $\mathfrak{M}(X)$ is a cone, i.e. for each pair c_1, c_2 of non-negative real numbers and each pair M_1, M_2 from $\mathfrak{M}(X)$ we have $c_1 M_1 + c_2 M_2 \in \mathfrak{M}(X)$. Moreover, for any operator $A \in \mathscr{B}(X)$ and $M \in \mathfrak{M}(X)$ we have $AM \in \mathfrak{M}(X)$ and $A\tilde{e}(M) = \tilde{e}(AM)$. Further, if $M \in \mathfrak{M}(X)$ and $M \geqslant N \geqslant 0$, then $N \in \mathfrak{M}(X)$ and $M - N \in \mathfrak{M}(X)$.

By a *Gaussian measure* on X we mean such a probability measure ϱ on X that for every $y \in X^*$ the induced measure $y\varrho$ on the real line is Gaussian. We refer to X. Fernique [4], J. Kuelbs [6], and N. N. Vakhania [18] for discussions of Gaussian measures on Banach spaces. In this paper we shall consider symmetric Gaussian measures only. For such measures the characteristic functional is of the form

$$\hat{\varrho}(y) = \exp(-\tfrac{1}{2}\langle y, Ry\rangle) \quad (y \in X^*)$$

where R is the covariance operator, i.e. a compact operator from X^* into X with the properties: $\langle y_1, Ry_2\rangle = \langle y_2, Ry_1\rangle$ for all $y_1, y_2 \in X^*$ (symmetry) and $\langle y, Ry\rangle \geqslant 0$ (non-negativity) ([18], p. 136, [2]). By $\mathscr{R}(X)$ we shall denote the set of all covariance operators of Gaussian measures on X. If R_1 is a symmetric non-negative operator from X^* into X and $R_2 - R_1$ is non-negative for a certain operator $R_2 \in \mathscr{R}(X)$, then also $R_1 \in \mathscr{R}(X)$ ([18], p. 151). Clearly, if R is the covariance operator of ϱ and $A \in \mathscr{B}(X)$, then ARA^* is the covariance operator of $A\varrho$.

A. Tortrat established in [12], p. 311 (see also [3], p. 22) the following representation of infinitely divisible laws.

PROPOSITION 5.1. *A probability measure μ on X is infinitely divisible if and only if*

$$\mu = \varrho * \tilde{e}(M), \tag{5.1}$$

where ϱ is a symmetric Gaussian measure and $M \in \mathfrak{M}(X)$. Moreover, the decomposition (5.1) *is unique.*

LEMMA 5.1. *Suppose that $\mu = \varrho * \tilde{e}(M)$ where ϱ is a symmetric Gaussian measure with the covariance operator R and $M \in \mathfrak{M}(X)$. If $A \in \mathscr{D}(\mu)$ and $\mu = A\mu * \nu$, where ν is infinitely divisible, then $A \in \mathscr{D}(\varrho)$ and $A \in \mathscr{D}(\tilde{e}(M))$. Moreover, $R - ARA^* \in \mathscr{R}(X)$ and $M - AM \in \mathfrak{M}(X)$.*

Proof. By Proposition 5.1 the measure ν has the representation $\nu = \varrho_1 * \tilde{e}(M_1)$, where ϱ_1 is a symmetric Gaussian measure with the covariance operator R_1 and $M_1 \in \mathfrak{M}(X)$. Hence it follows that $\mu = A\varrho * \varrho_1 * \tilde{e}(AM + M_1)$. Consequently, by the uniqueness of Tortrat representation, $\varrho = A\varrho * \varrho_1$ and $M = AM + M_1$, which yields $\tilde{e}(M) = A\tilde{e}(M) * \tilde{e}(M_1)$ and $R = ARA^* + R_1$. The assertion of the lemma is a direct consequence of these equations.

THEOREM 5.1. *Let $V \in \mathscr{B}(X)$ and $\lim_{t\to\infty} \exp tV = 0$. Then $\mathscr{D}(\mu)$ contains the one-parameter semigroup $\exp tV$ $(t \geqslant 0)$ if and only if $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure with the covariance operator R and $M \in \mathfrak{M}(X)$ such that the operator $VR + RV^*$ is non-positive, i.e.*

$$\langle y, (VR + RV^*)y\rangle \leqslant 0 \quad \text{for all } y \in X^*$$
$$\text{and} \quad M \geqslant (\exp tV) M \quad \text{for all} \quad t \geqslant 0.$$

Proof. Suppose that $\mathscr{D}(\mu)$ contains the semigroup $\exp tV$ $(t \geqslant 0)$. Then, by Lemma 4.4, μ is infinitely divisible. Moreover, by Corollary 4.2, for each $t \geqslant 0$ $\mu = \exp tV\mu * \nu_t$, where ν_t is also infinitely divisible. Thus, by Proposition 5.1 and Lemma 5.1, μ has a representation $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure with the covariance operator R and $M \in \mathfrak{M}(X)$. Moreover, for each $t \geqslant 0$

$$R - (\exp tV)R(\exp tV^*) \quad \text{and} \quad M - (\exp tV)M$$

belong to $\mathscr{R}(X)$ and $\mathfrak{M}(X)$, respectively. In particular, the measure $M - (\exp tV)M$ is non-negative. The operator $R - (\exp tV)R(\exp tV^*)$ that is covariance operator is non-negative, too. Taking into account the expansion in a neighborhood of 0

$$R - (\exp tV)R(\exp tV^*) = -t(VR + RV^*) + o(t),$$

we infer that the operator $VR + RV^*$ is non-positive which completes the proof of the necessity.

To prove the sufficiency let us assume that $M \in \mathfrak{M}(X)$, $M_t = M - (\exp tV)M \geqslant 0$, $R \in \mathscr{R}(X)$ and $VR + RV^*$ is non-positive. Clearly $M_t \in \mathfrak{M}(X)$ and

$$\tilde{e}(M) = \exp tV\tilde{e}(M) * \tilde{e}(M_t). \tag{5.2}$$

Given $y \in X^*$, we put

$$f_y(t) = \langle y, \big(R - (\exp tV)R(\exp tV^*)\big)y\rangle.$$

By a simple calculation we get the formula

$$\frac{d}{dt} f_y(t) = -\langle \exp tV^* y, (VR + RV^*) \exp tV^* y\rangle$$

which implies the inequality $\frac{d}{dt} f_y(t) \geqslant 0$. Taking into account the initial condition $f_y(0) = 0$, we get the inequality $f_y(t) \geqslant 0$ for all $t \geqslant 0$ and all $y \in X^*$. Thus the operator $R_t = R - (\exp tV)R(\exp tV^*)$ is non-negative. Since $R - R_t$ is also non-negative, we have $R_t \in \mathscr{R}(X)$. Let ϱ and ϱ_t be symmetric Gaussian measures with the covariance operators R and R_t, respectively. We may assume that

$$\varrho = \exp tV\varrho * \varrho_t. \tag{5.3}$$

Setting $\mu = \varrho * \tilde{e}(M)$, we have, in view of (5.2) and (5.3),

$$\mu = \exp tV\mu * \nu_t \quad (t \geqslant 0),$$

where $\nu_t = \varrho_t * \tilde{e}(M_t)$. Thus $\mathscr{D}(\mu)$ contains all operators $\exp tV$ $(t \geqslant 0)$ which completes the proof of the theorem.

Our next aim is to give a representation of the characteristic functional for probability measures whose decomposability semigroups contain a one-parameter semigroup $\exp tV$ $(t \geqslant 0)$ where $V \in \mathscr{B}(X)$ and $\lim_{t\to\infty} \exp tV = 0$. We fix this semigroup $\{\exp tV\}$ for the remainder of this section and we put for simplicity of notation $T_t = \exp tV$ $(-\infty < t < \infty)$. It is easy to check that $\|T_t\| \leqslant ae^{-bt} (t \geqslant 0)$ for some positive constants a and b. This fact implies the following lemma.

LEMMA 5.2. *Let f be a complex-valued Borel measurable function on X and $|f(x)| \leqslant g(\|x\|)$ $(x \in X)$ for a real-valued function g satisfying the condition $\int_0^\infty g(ce^{-bt})\,dt < \infty$ for every positive number c. Then $\int_0^\infty f(T_t x)\,dt$ is finite for every $x \in X$.*

A continuous real-valued function Φ on X is said to be a *weight function* on X if the following conditions are fulfilled:

(a) $\Phi(0) = 0$ and $\Phi(x) > 0$ for all $x \neq 0$,

(b) $\Phi(x)$ converges to a positive limit as $\|x\| \to \infty$,

(c) $\Phi(x) \leqslant c\|x\|^2$ for a certain positive constant c and all $x \in X$,

(d) $\int_X \Phi(x)\, M(dx) < \infty$ for every $M \in \mathfrak{M}(X)$,

(e) if $M_n \in \mathfrak{M}(X)$, $\tilde{e}(M_n) \to \mu$ and $\int_X \Phi(x)\, M_n(dx) \to 0$, then $\mu = \delta_x$ for a certain $x \in X$.

The weight functions will play a crucial role in our considerations. It is well known that if X is a Hilbert space, then as a weight function on X we can take $\Phi(x) = \|x\|^2/(1+\|x\|^2)$ ([11], Chapter VI, Theorem 4.10). In this case condition (e) can be strengthened. Namely, $M \in \mathfrak{M}(X)$ if and only if $\int_X \frac{\|x\|^2}{1+\|x\|^2} M(dx) < \infty$ and $M(\{0\}) = 0$.

PROPOSITION 5.2. *For every X there exists a weight function on X.*

Proof. We note that the space X^* is separable in the X-topology. For an arbitrary sequence $\{y_n\}$ dense in the unit ball of X^* in the X-topology and for all x in the unit ball of X we put

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{\langle y_n, x\rangle^2}{2^n}.$$

Clearly, $\varphi(x) \leqslant \|x\|^2$ and $\varphi(x) = 0$ if and only if $x = 0$. Moreover,

$$\langle y_n, x\rangle^2 \leqslant 2^n \varphi(x) \tag{5.3}$$

for all x with $\|x\| \leqslant 1$. Set $\Phi(x) = \varphi(x)$ if $\|x\| \leqslant 1$ and $\Phi(x) = \varphi\left(\frac{x}{\|x\|}\right) \times \frac{1}{\|x\|} + 1 - \frac{1}{\|x\|}$ otherwise. It is obvious that Φ fulfils conditions (a), (b) and (c). In order to prove condition (d) assume that $M \in \mathfrak{M}(X)$. Then

$$|\hat{\tilde{e}}(M)(y)|^2 = \exp \int_X (\cos\langle y, x\rangle - 1)\, M(dx) \tag{5.4}$$

(see [3], p. 25). It is well known that the unit ball in X^* is compact in the X-topology. Since the characteristic functional is continuous in the X-topology of X^* and does not vanish for infinitely divisible probability measures, we infer, by (5.4), that the supremum

$$s = \sup\left\{\int_X (1-\cos\langle y, x\rangle)\, M(dx) \colon \|y\| \leqslant 1\right\}$$

is finite. From the elementary inequality $1-\cos t \geqslant c_1 t^2$ $(-1 \leqslant t \leqslant 1)$ where c_1 is a positive constant we get

$$c_1 \int_{\|x\|\leqslant 1} \langle y_n, x\rangle^2\, M(dx) \leqslant s \quad (n = 1, 2, \ldots).$$

Consequently, $\int_{\|x\|\leqslant 1} \Phi(x)\, M(dx) < \infty$. Since M has a finite mass outside every neighborhood of 0 in X and Φ is bounded, the integral $\int_X \Phi(x)\, M(dx)$ is finite. Condition (d) is thus fulfilled. Suppose now that $M_n \in \mathfrak{M}(X)$, $\tilde{e}(M_n) \to \mu$ and $\int_X \Phi(x)\, M_n(dx) \to 0$. By (5.3) we have the inequality

$$\langle y_k, x\rangle^2 \leqslant 2^k \Phi(x) \quad \text{if} \quad \|x\| \leqslant 1$$

and

$$\langle y_k, x\rangle^2 = \|x\|^2 \left\langle y_k, \frac{x}{\|x\|}\right\rangle^2 \leqslant 2^{k+2} \varphi\left(\frac{x}{\|x\|}\right) \leqslant 2^{k+3}\Phi(x) \quad \text{if} \quad 1 \leqslant \|x\| \leqslant 2.$$

On the other hand, $\Phi(x) \geqslant \frac{1}{2}$ if $\|x\| \geqslant 2$. Taking into account the inequality $1-\cos t \leqslant c_2 t^2$ for all t with a certain positive constant c_2, we get finally the inequality

$$1 - \cos\langle y_k, x\rangle \leqslant a_k \Phi(x)$$

for all $x \in X$, a_k being a positive constant. Hence and from formula (5.4) it follows that $|\hat{\tilde{e}}(M_n)(y_k)|^2 \to 1$ as $n \to \infty$ $(k = 1, 2, \ldots)$. Consequently, $|\hat{\mu}(y_k)| = 1$, and, by the density of $\{y_k\}$ in the unit ball of X^* in the X-topology, $|\hat{\mu}(y)| = 1$ if $\|y\| \leqslant 1$. But this fact yields $\mu = \delta_x$ for a certain $x \in X$ ([5], Proposition 2.3). Thus condition (e) is also satisfied which completes the proof.

Given a subset E of X, we put $\tau(E) = \{T_t x \colon x \in E,\ -\infty < t < \infty\}$. It is clear that for any compact set E with the property $0 \notin E$ and for any pair $r_1 < r_2$ of positive numbers the inequality $r_1 \leqslant \|T_{t_n} x_n\| \leqslant r_2$ $(x_n \in E)$ implies the boundedness of the sequence $\{t_n\}$. This simple fact yields the following lemma.

LEMMA 5.3. *Let E be a compact subset of X and $0 \notin E$. Then for every pair $r_1 < r_2$ of positive numbers the set $\{x \colon r_1 \leqslant \|x\| \leqslant r_2\} \cap \tau(E)$ is compact.*

LEMMA 5.4. *For every $M \in \mathfrak{M}(X)$ there exists a sequence $\{E_n\}$ of compact subsets of X such that $0 \notin E_n$ $(n = 1, 2, \ldots)$, $\tau(E_n) \cap \tau(E_m) = \emptyset$ if $n \neq m$ $(n, m = 1, 2, \ldots)$ and $M = \sum_{n=1}^{\infty} M_n$, where M_n is the restriction of M to $\tau(E_n)$.*

Proof. Let Φ be a weight function on X. By condition (d), the measure $N(E) = \int_E \Phi(x)\, M(dx)$ is finite and, consequently, tight on X ([11], Chapter II, Section 3). Consequently, there exists a compact subset E_1 of X such that $N(X \setminus E_1) < 1$. Since $N(\{0\}) = 0$, we may assume that $0 \notin E_1$. Of course, $N\big(X \setminus \tau(E_1)\big) < 1$. By Lemma 5.3, the set $X \setminus \big(\tau(E_1) \cup \{0\}\big)$ is open and, consequently, the measure N restricted to this set is tight. We can now find a compact subset E_2 containing in $X \setminus \tau(E_1)$ such that $0 \notin E_2$ and $N\Big(X \setminus \big(\tau(E_1) \cup E_2\big)\Big) < \frac{1}{2}$. Clearly, $\tau(E_1) \cap \tau(E_2) = \emptyset$

and $N\big(X\setminus(\tau(E_1)\cup\tau(E_2))\big)<\frac{1}{2}$. We proceed in this manner step by step and finally we obtain a sequence $\{E_n\}$ of compact subsets of $X\setminus\{0\}$ such that $\tau(E_n)\cap\tau(E_m)=\emptyset$ whenever $n\neq m$ and

$$N\big(X\setminus\bigcup_{k=1}^{n}\tau(E_k)\big)<\frac{1}{n}. \tag{5.5}$$

Let M_n be the restriction of M to $\tau(E_n)$ and $Q_n=M-\sum_{k=1}^{n}M_k$. Then $Q_n\in\mathfrak{M}(X)$ and $\int_X \Phi(x)\,Q_n(dx)=N\big(X\setminus\bigcup_{k=1}^{n}\tau(E_k)\big)$ which implies, by virtue of (5.5), $\int_X \Phi(x)\,Q_n(dx)\to 0$. Moreover, $\tilde{e}(M)=\tilde{e}(Q_n)*\tilde{e}\big(\sum_{k=1}^{n}M_k\big)$. From this we conclude that the sequence $\{\tilde{e}(Q_n)\}$ is shift compact ([11], Chapter III, Theorem 2.2). Since the generalized Poisson measures $\tilde{e}(Q_n)$ are determined up to a translation, we may assume without loss of generality that the sequence $\{\tilde{e}(Q_n)\}$ is convergent to a probability measure, say μ. By condition (e) μ is concentrated at a single point which shows that $M=\sum_{n=1}^{\infty}M_n$. The lemma is thus proved.

Suppose that $M\in\mathfrak{M}(X)$ and $M\geqslant T_tM$ for all $t\geqslant 0$. It is clear that for every T_t-invariant set U $(-\infty<t<\infty)$ the restriction M to U, denoted by $M|U$, belongs to $\mathfrak{M}(X)$ and $M|U\geqslant T_t(M|U)$ for all $t\geqslant 0$. Consequently, from Lemma 5.4 we get the following corollary.

COROLLARY 5.1. *Let $M\in\mathfrak{M}(X)$ and $M\geqslant T_t M$ for all $t\geqslant 0$. Then there exists a decomposition $M=\sum_{n=1}^{\infty}M_n$, where $M_n\in\mathfrak{M}(X)$, $M_n\geqslant T_tM_n$ for all $t\geqslant 0$, M_n are concentrated on disjoint sets $\tau(E_n)$, $0\notin E_n$ and E_n are compact.*

This corollary reduces our problem of examining measures $M\in\mathfrak{M}(X)$ with the property $M\geqslant T_tM$ $(t\geqslant 0)$ to the case of measures concentrated on $\tau(E)$ where E is compact and $0\notin E$. We denote this class of measures by $\mathfrak{L}_E$. Our method of examining consists in finding a suitable compactification of $\tau(E)$ and determining the extreme points of a certain convex set formed by probability measures on this compactification.

Let $[-\infty,\infty]$ be the usual compactification of the real line and E be a compact subset of X such that $0\notin E$. Then $E\times[-\infty,\infty]$ endowed with the product topology becomes a compact space. We define an equivalence relation in $E\times[-\infty,\infty]$ as follows: $(x_1,t_1)\sim(x_2,t_2)$ where $x_1,x_2\in E$ and $t_1,t_2\in[-\infty,\infty]$ if and only if there exists a real number s such that $T_sx_1=x_2$ and $t_2=t_1-s$. In order to prove the continuity of this equivalence relation, suppose that $(x_n,t_n)\sim(x_n',t_n')$ $(n=1,2,\ldots)$ and the sequences $\{(x_n,t_n)\}$ and $\{(x_n',t_n')\}$ converge to (x,t) and (x',t'), respectively. Then for some real numbers s_n we have $T_{s_n}x_n=x_n'$ and $t_n'=t_n-s_n$. By the compactness of E and the assumption $0\notin E$ we infer

that the sequence $\{s_n\}$ is bounded. Clearly, for any its cluster point s we have $T_sx'=x$ and $t'=t-s$ which implies $(x,t)\sim(x',t')$. Thus $\sim$ is continuous. Hence it follows that the quotient space $E\times[-\infty,\infty]/\sim$ denoted by $\bar{\tau}(E)$ is compact ([1], p. 97). The element of $\bar{\tau}(E)$, i.e. the coset containing (x,t) will be denoted by $[x,t]$. Each element of $\tau(E)$ is of the form T_tx, where $x\in E$ and t is a real number. In general this representation is not unique. But $T_{t_1}x_1=T_{t_2}x_2$ if and only if $(x_1,t_1)\sim(x_2,t_2)$. Thus the mapping $T_tx\to[x,t]$ is an embedding of $\tau(E)$ into a dense subset of $\bar{\tau}(E)$. In other words, $\bar{\tau}(E)$ is a compactification of $\tau(E)$. In what follows we shall identify elements T_tx of $\tau(E)$ and corresponding elements $[x,t]$ of $\bar{\tau}(E)$. Further, we extend the functions T_s $(-\infty<s<\infty)$ and $\|\cdot\|$ from $\tau(E)$ onto $\bar{\tau}(E)$ by continuity, i.e. we put $T_s[x,-\infty]=[x,-\infty]$, $T_s[x,\infty]=[x,\infty]$, $\|[x,-\infty]\|=\infty$, $\|[x,\infty]\|=0$ for all $x\in E$. Then we have the formula

$$T_s[x,t]=[x,t+s].$$

Let Φ be a weight function on X. By Lemma 5.3 and condition (b), Φ is bounded from below on every set $\{x:\|x\|\geqslant r\}\cap\tau(E)$ with $r>0$. Further, Φ can be extended to $\bar{\tau}(E)$ by assuming $\Phi([x,\infty])=0$ and $\Phi([x,-\infty])=\lim_{\|z\|\to\infty}\Phi(z)$. Let N be a finite Borel measure on $\bar{\tau}(E)$. Put

$$M_N(U)=\int_U\frac{N(du)}{\Phi(u)} \tag{5.6}$$

for every subset U of $\bar{\tau}(E)$ with the property $\inf\{\|u\|:u\in U\}>0$. This formula defines a σ-finite measure M_N on $\{u:\|u\|>0\}\cap\bar{\tau}(E)$. Let $\mathfrak{H}_E$ denote the class of all finite measures N on $\bar{\tau}(E)$ for which the corresponding measures M_N fulfil the condition $M_N\geqslant T_tM_N$ for all $t\geqslant 0$. It is easy to check that the set $\mathfrak{H}_E$ is closed and convex. Let us consider measures M from $\mathfrak{L}_E$ as measures on $\bar{\tau}(E)$. Set

$$N^M(U)=\int_U\Phi(u)\,M(du) \tag{5.7}$$

for all Borel subsets U of $\bar{\tau}(E)$. It is evident that $M\in\mathfrak{L}_E$ if and only if $N^M\in\mathfrak{H}_E$. By $\mathfrak{I}_E$ we denote the subset of $\mathfrak{H}_E$ consisting of probability measures. Clearly, $\mathfrak{I}_E$ is convex and compact. We shall now find all its extreme points.

By Lemma 5.2 and condition (c) $\int_0^\infty\Phi(T_sx)\,ds<\infty$ for every real number t and $x\in X$. For every $z\in\tau(E)$ we put

$$N_Z(U)=C(z)\int_0^\infty 1_U(T_tz)\,\Phi(T_tz)\,dt, \tag{5.8}$$

where $C^{-1}(z)=\int_0^\infty\Phi(T_tz)\,dt$ and 1_U denotes the indicator of the subset

U of $\bar{\tau}(E)$. Moreover, N_z are probability measures on $\bar{\tau}(E)$ concentrated on $\tau(\{x\})$ and for every subset U of $\bar{\tau}(E)$ with the property $\inf\{\|u\|: u \in U\} > 0$ and $z = T_t x$ we obtain, after some computation,

$$M_{N_z}(U) = C(z)\,|\{a\colon\ [x, a] \in U,\ a \geqslant t\}|$$

and

$$T_s M_{N_z}(U) = C(z)\,|\{a\colon\ [x, a] \in U,\ a \geqslant t+s\}|,$$

where $|W|$ denotes the Lebesgue measure of a subset W of the real line. Hence we conclude that $M_{N_z} \geqslant T_s M_{N_z}$ $(s \geqslant 0)$ and, consequently, $N_z \in \mathfrak{I}_E$ $(z \in \tau(E))$. We extend the definition of N_z to $z \in \bar{\tau}(E) \setminus \tau(E)$ by assuming $N_z = \delta_z$. In this case we have also $N_z \in \mathfrak{I}_E$. Moreover, the mapping $z \to N_z$ from $\bar{\tau}(E)$ into $\mathfrak{I}_E$ is one-to-one and continuous. Consequently, it is a homeomorphism between $\bar{\tau}(E)$ and $\{N_z\colon z \in \bar{\tau}(E)\}$.

LEMMA 5.5. *The set $\{N_z\colon z \in \bar{\tau}(E)\}$ is identical with the set of extreme points of $\mathfrak{I}_E$.*

Proof. For any Borel subset E_1 of E the sets $\tau(E_1)$, $\{[x, -\infty]\colon x \in E_1\}$ and $\{[x, \infty]\colon x \in E_1\}$ are invariant under all transformations T_s $(-\infty < s < \infty)$. Hence if $N \in \mathfrak{H}_E$, the restriction of N to any of these sets is again in $\mathfrak{H}_E$. This implies that every extreme point of $\mathfrak{I}_E$ must be concentrated on orbits of elements of $\bar{\tau}(E)$, i.e. on one of the following sets $\tau(\{x\})$, $\{[x, -\infty]\}$ and $\{[x, \infty]\}$ where $x \in E$. Obviously, all measures N_z $\big(z \in \bar{\tau}(E) \setminus \tau(E)\big)$ are extreme points of $\mathfrak{I}_E$. It remains to determine extreme points concentrated on sets $\tau(\{x\})$ $(x \in E)$.

Let N be an arbitrary probability measure concentrated on $\tau(\{x\})$. It is clear, that $N \in \mathfrak{I}_E$ if and only if $M_N(U) \geqslant T_s M_N(U)$ for all $s \geqslant 0$ and all sets U of the form $U = \{[x, t]\colon a \leqslant t < b\}$ $(-\infty < a < b < \infty)$. Setting $h_N(b) = M_N(\{[x, t]\colon t < b\})$, we infer that $N \in \mathfrak{I}_E$ if and only if

$$h_N(b) - h_N(a) - h_N(b-s) + h_N(a-s) \geqslant 0 \tag{5.9}$$

for every triplet a, b, s of real numbers satisfying the conditions $a < b$ and $s \geqslant 0$. Substituting $b = a+s$ into (5.9), we get the inequality

$$h_N(a) \leqslant \tfrac{1}{2}\big(h_N(a+s) + h_N(a-s)\big)$$

for every real number a and $s \geqslant 0$. Thus the function h_N satisfying (5.9) is convex. Since it is always monotone non-decreasing and vanishes at $-\infty$, we have an integral representation

$$h_N(t) = \int_{-\infty}^{t} g_N(s)\, ds,$$

where the function g_N is non-negative and monotone non-decreasing. Of course, we may assume that g_N is continuous from the left. In this

case g_N is uniquely determined by N. Moreover, by a simple computation, we get the formula

$$N(\{[x, t]\colon a \leqslant t < b\}) = \int_a^b \Phi([x, t])\, g_N(t)\, dt \tag{5.10}$$

which yields

$$\int_{-\infty}^{\infty} \Phi([x, t])\, g_N(t)\, dt = 1. \tag{5.11}$$

Conversely, every non-negative monotone non-decreasing continuous from the left function g_N with property (5.11) determines by formula (5.10) a probability measure N concentrated on $\tau(\{x\})$. Moreover, the corresponding function h_N fulfils inequality (5.9) which shows that $N \in \mathfrak{I}_E$. Hence we conclude that a measure N from $\mathfrak{I}_E$ is an extreme point of $\mathfrak{I}_E$ if and only if the corresponding function g_N cannot be decomposed into a non-trivial convex combination of two functions g_{N_1} and g_{N_2} $(N_1, N_2 \in \mathfrak{I}_E)$. But this is possible only in the case $g_N(t) = 0$ if $t \leqslant t_0$ and $g_N(t) = c$ if $t > t_0$ for some constants t_0 and c. By (5.11), $c^{-1} = \int_{t_0}^{\infty} \Phi([x, t])\, dt$. Taking into account (5.10) and the definition of measures N_z, we conclude that the set of extreme points of $\mathfrak{I}_E$ concentrated on $\tau(\{x\})$ consists of all measures N_z with $z \in \tau(\{x\})$. Consequently, the set of extreme points $\mathfrak{I}_E$ coincides with the set $\{N_z\colon z \in \bar{\tau}(E)\}$ which completes the proof of the lemma.

Once the extreme points of $\mathfrak{I}_E$ are found we can apply a well-known Krein–Milman–Choquet theorem ([12], Chapter 3). Since each element N of $\mathfrak{H}_E$ is of the form cN_1 where $N_1 \in \mathfrak{I}_E$, we get the following proposition.

PROPOSITION 5.3. *A measure N belongs to $\mathfrak{H}_E$ if and only if there exists a finite Borel measure m on $\bar{\tau}(E)$ such that*

$$\int_{\bar{\tau}(E)} f(x)\, N(dx) = \int_{\bar{\tau}(E)} \int_{\bar{\tau}(E)} f(u)\, N_z(du)\, m(dz)$$

for every continuous function f on $\bar{\tau}(E)$. If N is concentrated on $\tau(E)$, then m does the same.

From this proposition, by virtue of (5.7) and (5.8), we get after some computation the following corollary.

COROLLARY 5.2. *Let M be a measure from $\mathfrak{M}(X)$ concentrated on $\tau(E)$. Then $M \in L_E$ if and only if there exists a finite measure m on $\tau(E)$ such that*

$$\int_{\tau(E)} f(x)\, M(dx) = \int_{\tau(E)} C(z) \int_0^{\infty} f(T_t z)\, dt\, m(dz)$$

for every M-integrable function f on $\tau(E)$. The function C is given by the formula

$$C^{-1}(z) = \int_0^\infty \Phi(T_t z)\, dt. \tag{5.12}$$

We now turn to the consideration of arbitrary measures M belonging to $\mathfrak{M}(X)$ and satisfying the condition $M \geqslant T_t M$ for $t \geqslant 0$. By Corollary 5.1, there exists a decomposition $M = \sum_{n=1}^{\infty} M_n$, where $M_n \in \mathfrak{M}(X)$, $M_n \geqslant T_t M_n$ for $t \geqslant 0$, M_n are concentrated on disjoint sets $\tau(E_n)$, $0 \notin E_n$ and E_n are compact. Let m_n denote a finite measure on $\tau(E_n)$ corresponding to M_n in the representation given by Corollary 5.2. Then

$$\int_X f(x)\, M(dx) = \sum_{n=1}^{\infty} \int_{\tau(E_n)} C(z) \int_0^\infty f(T_t z)\, dt\, m_n(dz)$$

for every M-integrable function f. Substituting $f = \Phi$ into this formula, we get the equation

$$\int_X f(x)\, M(dx) = \sum_{n=1}^{\infty} m_n\big(\tau(E_n)\big).$$

Consequently, setting $m = \sum_{n=1}^{\infty} m_n$, we get a finite measure on X satisfying the equation

$$\int_X \Phi(x)\, M(dx) = \int_X C(z) \int_0^\infty f(T_t z)\, dt\, m(dz) \tag{5.13}$$

for every M-integrable function f on X. Moreover, $m(\{0\}) = 0$.

The Lévy-Khinchine representation for the characteristic functional of infinitely divisible probability measures on complete locally convex spaces has been studied by E. Dettweiler in [3] (Theorem 2.6). From these results we conclude that

$$\hat{\tilde{e}}(M)(y) = \exp\Big(i\langle y, x_0\rangle + \int_X K(x, y)\, M(dx)\Big)$$

for a certain element $x_0 \in X$. The kernel K is defined by the formula

$$K(x, y) = e^{i\langle y, x\rangle} - 1 - i\langle y, x\rangle 1_B(x),$$

where 1_B denotes the indicator of the unit ball in X. Given a weight function Φ on X and $V \in \mathscr{B}(X)$ with $\lim_{t\to\infty} \exp tV = 0$, we put

$$K_{\Phi,V}(x, y) = C(x) \int_0^\infty K(T_t x, y)\, dt \tag{5.14}$$

where the function C is defined by formula (5.12) and $T_t = \exp tV$. By Lemma 5.2 the kernel $K_{\Phi,V}$ is finite for $x \neq 0$. Moreover, by (5.13),

$$\int_X K(x, y)\, M(dx) = \int_X K_{\Phi,V}(x, y) m(dx),$$

which, by Theorem 5.1, yields the following theorem.

THEOREM 5.2. *Let Φ be a weight function on X, $V \in \mathscr{B}(X)$ and $\lim_{t\to\infty} \exp tV = 0$. Then $\mathscr{D}(\mu)$ contains the one-parameter semigroup $T_t = \exp tV$ $(t \geqslant 0)$ if and only if there exist an element $x_0 \in X$, an operator $R \in \mathscr{R}(X)$ for which the operator $VR + RV^*$ is non-positive, and a finite measure m on X vanishing at 0 such that*

$$\hat{\mu}(y) = \exp\Big(i\langle y, x_0\rangle - \tfrac{1}{2}\langle y, Ry\rangle + \int_X K_{\Phi,V}(x, y)\, m(dx)\Big)$$

for all $y \in X^*$. The kernel $K_{\Phi,V}$ is defined by formula (5.14).

Suppose that X admits a weight function Φ with the following property: $M \in M(X)$ if and only if $\int_X \Phi(x)\, M(dx) < \infty$ and $M(\{0\}) = 0$. A Hilbert space X with the weight function $\Phi(x) = \|x\|^2/(1+\|x\|^2)$ is an example of such situation. Then, by virtue of (5.13), we can easily check that each finite measure m on X vanishing at 0 is a representing measure in Theorem 5.2.

Combining Theorems 4.1 and 5.2, we get a representation theorem for full Lévy's measures on a real separable Banach space X.

THEOREM 5.3. *Let Φ be a weight function on X. A full probability measure on X is a Lévy's measure if and only if there exist an operator $V \in \mathscr{B}(X)$ with $\lim_{t\to\infty} \exp tV = 0$, an element $x_0 \in X$, an operator $R \in \mathscr{R}(X)$ for which the operator $VR + RV^*$ is non-positive and a finite measure m on X vanishing at 0 such that*

$$\hat{\mu}(y) = \exp\Big(i\langle y, x_0\rangle - \tfrac{1}{2}\langle y, Ry\rangle + \int_X K_{\Phi,V}(x, y)\, m(dx)\Big)$$

for all $y \in X^*$.

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