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Quotient spaces of (s) with basis

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Abstract. A complete characterization, in terms of a representing matrix, is given for a nuclear Köthe space to be a quotient space of (s). Using this characterization, we obtain a nuclear Fréchet space which is neither a quotient space of (s) nor of $(s)^N$. Moreover, the only nuclear Fréchet spaces with basis which are both a subspace of (s) and a quotient space of (s) are (up to isomorphism) infinite type power series spaces.

In this paper we investigate the nature of spaces with basis which are quotient spaces of (s), and in our main theorem 2.4 we give a complete characterization. As a consequence we are able to construct a nuclear Fréchet space which is neither a quotient of (s) nor of $(s)^N$. This solves a problem posed by Pełczyński ([12]). In addition, we prove that any Köthe space which is both a quotient space and a subspace of (s) must be isomorphic to an infinite type power series space.

Notation. N will denote the collection of natural numbers. If E is a nuclear Fréchet space with a basis (x_j), we say that the matrix (a_j^k) *represents* (x_j) if there exists a fundamental system of semi-norms (p_k) on E such that $a_j^k = p_k(x_j)$. Then, because of the nuclearity, there are two equivalent systems of semi-norms given by

$$\|x\|_k = \sum_{j=1}^{\infty} |\xi_j|\, a_j^k \quad \text{and} \quad |x|_k = \sup_j |\xi_j|\, a_j^k$$

for all $x = \sum_{j=1}^{\infty} \xi_j x_j \in E$. It is easy to see that the representing matrix (a_j^k) can be altered by replacing finitely many elements of each row by arbitrary non-negative scalars. If E is a nuclear Fréchet space with basis (x_j) and if (x_j) has matrix representation (a_j^k), the associated sequence space is the space

$$K = K(a_j^k) = \Big\{t = (t_j)\colon\ p_k(t) = \sum_{j=1}^{\infty} |t_j|\, a_j^k < +\infty,\ k \in N\Big\}.$$

E is isomorphic to K and E' is isomorphic to

$$K^{\times} = \{t\colon \exists k \text{ and } M > 0 \text{ such that } |t_j| \leqslant M a_j^k,\ j \in N\}.$$

A space $K(a_j^k)$ is called a *Köthe space* if for all k, j, $0 < a_j^k < a_j^{k+1}$. In this case, each p_k is a continuous norm on $K(a_j^k)$.

A Köthe space is an *infinite type power series space* (p.s.s.) $\Lambda_\infty(a)$ if it is defined by a matrix of the form $(a_j^k) = (k^{a_j})$, where $0 < a_j \leqslant a_{j+1}$ and $\sup_j \frac{\log(j)}{a_j} < +\infty$. Of primary importance is the space $(s) = \Lambda_\infty(a)$ where $a_j = \log(j+1)$. A Köthe space is a *finite type* p.s.s., $\Lambda_1(\beta)$, if it is defined by a matrix $(a_j^k) = \left(\left(\frac{k}{k+1}\right)^{\beta_j}\right)$, where $0 < \beta_j \leqslant \beta_{j+1}$ with $\lim_j \frac{\log(j)}{\beta_j} = 0$. The space A_1 of functions analytic on the open unit disk is isomorphic to the Köthe space $\Lambda_1(\beta)$ with $\beta_n = n$ for all n.

There are three classifications of nuclear Fréchet spaces E with basis (x_j) and continuous norms which are of interest here. (x_j) is said to be (d_2) if there is a matrix representation (a_j^k) of (x_j) with the property that for all k there exists p such that for all q

$$\lim_j \frac{(a_j^p)^2}{a_j^k a_j^q} = \infty.$$

Condition (d_2) was introduced by Dragilev in [6]. Because of a theorem in [1], if one basis for E is (d_2), then all bases for E are (d_2). Hence we can refer to E as a (d_2) space.

(x_j) is said to be (d_3) if there is a matrix representation (a_j^k) of (x_j) such that for all k and j,

$$\frac{a_j^{k+2}}{a_j^{k+1}} \geqslant \frac{a_j^{k+1}}{a_j^k}.$$

Condition (d_3) was given in [5] and again is independent of the basis for E, so we can refer to the space E as a (d_3) space. It is shown in [5] that E is (d_3) if and only if E is isomorphic to a subspace of (s).

(x_j) is said to be (d_4) if there is a matrix representation (a_j^k) of (x_j) such that for all k, j

$$\frac{a_j^{k+2}}{a_j^{k+1}} \leqslant \frac{a_j^{k+1}}{a_j^k}.$$

E will be called a (d_4) *space* if E has a basis which is (d_4). It is not hard to check that an infinite type p.s.s. is both (d_3) and (d_4) and that a finite type p.s.s. is (d_4).

Two bases (x_j) and (y_j) are *quasi-equivalent* if there exists a permutation π of the natural numbers and scalars $d_j > 0$ such that $\sum t_j x_j$ converges if and only if $\sum_j t_j d_j y_{\pi(j)}$ converges.

1. Preliminary results. We fix an infinite matrix (a_n^k) of positive numbers satisfying

$$\text{(1)}\qquad \frac{a_{n+1}^k}{a_n^k} < \frac{a_{n+1}^{k+1}}{a_n^{k+1}} \qquad n, k \in N.$$

Given numbers $t_1, \ldots, t_p$ not all 0 and $k \in N$, we define

$$\varrho^k(t_1, \ldots, t_p) = \min\left\{\varrho\colon \min_{1\leqslant i\leqslant p} \frac{a_i^k}{|t_i|} = \frac{a_\varrho^k}{|t_\varrho|}\right\}.$$

1.1. Lemma. $\varrho^{k+1}(t_1, \ldots, t_p) \leqslant \varrho^k(t_1, \ldots, t_p)$.

Proof. Suppose $\varrho > \varrho^k = \varrho^k(t_1, \ldots, t_p)$. Then, by definition,

$$\frac{a_{\varrho^k}^k}{|t_{\varrho^k}|} \leqslant \frac{a_\varrho^k}{|t_\varrho|}.$$

Hence, applying (1),

$$\frac{|t_\varrho|}{|t_{\varrho^k}|} \leqslant \frac{a_\varrho^k}{a_{\varrho^k}^k} < \frac{a_\varrho^{k+1}}{a_{\varrho^k}^{k+1}}$$

so

$$\frac{a_{\varrho^k}^{k+1}}{|t_{\varrho^k}|} < \frac{a_\varrho^{k+1}}{|t_\varrho|}$$

which implies that $\varrho^{k+1}(t_1, \ldots, t_p) \neq \varrho$ for every $\varrho > \varrho^k(t_1, \ldots, t_p)$. Hence, $\varrho^{k+1}(t_1, \ldots, t_p) \leqslant \varrho^k(t_1, \ldots, t_p)$.

1.2. Lemma. *If $0 < \varrho^m < \varrho^{m-1} < \ldots < \varrho^1 < p$ and $0 = l_0 < l_1 < \ldots \ldots < l_m$ are integers, then we can choose $t_1, \ldots, t_p$ such that $t_j = 0$ for $j \neq \varrho^1, \ldots, \varrho^m$, $t_{\varrho^1} \neq 0$, but otherwise arbitrary and such that*

$$\text{(2)}\qquad \frac{a_{\varrho^i}^{l_i}}{a_{\varrho^{i+1}}^{l_i}} < \frac{|t_{\varrho^i}|}{|t_{\varrho^{i+1}}|} < \frac{a_{\varrho^i}^{l_i+1}}{a_{\varrho^{i+1}}^{l_i+1}}, \qquad i = 1, \ldots, m-1.$$

Moreover, if any such choice is made, then

$$\varrho^k(t_1, \ldots, t_p) = \varrho^i \quad \textit{for} \quad l_{i-1} < k \leqslant l_i,\ i = 1, \ldots, m.$$

Proof. First we observe that the choice of $t_1, \ldots, t_p$ satisfying (2) is possible because of (1).

Fix $i = 1, \ldots, m$ and suppose that $1 \leqslant j < i$. Then if $l_{i-1} < k \leqslant l_i$ it follows from (2) and (1) that

$$\frac{|t_{\varrho^j}|}{|t_{\varrho^{j+1}}|} < \frac{a_{\varrho^j}^{l_j+1}}{a_{\varrho^{j+1}}^{l_j+1}} \leqslant \frac{a_{\varrho^j}^k}{a_{\varrho^{j+1}}^k},$$

so that

$$\frac{a^k_{\varrho^{j+1}}}{|t_{\varrho^{j+1}}|} < \frac{a^k_{\varrho^j}}{|t_{\varrho^j}|} \quad \text{for} \quad 1 \leqslant j < i,\ l_{i-1} < k \leqslant l_i.$$

Applying this successively with j replaced by $j+1, j+2, \ldots, i-1$ yields

$$\frac{a^k_{\varrho^i}}{|t_{\varrho^i}|} < \frac{a^k_{\varrho^j}}{|t_{\varrho^j}|} \quad \text{for} \quad 1 \leqslant j < i,\ l_{i-1} < k \leqslant l_i$$

which implies that $\varrho^k(t_1, \ldots, t_p) \leqslant \varrho^i$ for $l_{i-1} < k \leqslant l_i$.

On the other hand, suppose that $i < j \leqslant m$. Then for $k \leqslant l_i$ it follows from (1) and (2) that

$$\frac{a^k_{\varrho^{j-1}}}{a^k_{\varrho^j}} \leqslant \frac{a^{l_{j-1}}_{\varrho^{j-1}}}{a^{l_{j-1}}_{\varrho^j}} < \frac{|t_{\varrho^{j-1}}|}{|t_{\varrho^j}|},$$

so that

$$\frac{a^k_{\varrho^{j-1}}}{|t_{\varrho^{j-1}}|} < \frac{a^k_{\varrho^j}}{|t_{\varrho^j}|} \quad \text{for} \quad i < j \leqslant m,\ k \leqslant l_i.$$

Applying this successively with j replaced by $j-1, j-2, \ldots, i+1$ yields

$$\frac{a^k_{\varrho^i}}{|t_{\varrho^i}|} < \frac{a^k_{\varrho^j}}{|t_{\varrho^j}|} \quad \text{for} \quad i < j \leqslant m,\ k \leqslant l_i$$

which implies that $\varrho^k(t_1, \ldots, t_p) \geqslant \varrho^i$ for $k \leqslant l_i$. Hence, we have that $\varrho^k(t_1, \ldots, t_p) = \varrho^i$ for $l_{i-1} < k \leqslant l_i$.

1.3. Theorem. *Let E be a nuclear Fréchet space with fundamental system of norms $(|\ |_k)$ and a basis (x_i). Let $0 = p_0 < p_{n-1} < p_n$, $n \in N$, be integers and (t_i) a sequence of scalars such that for all $n \in N$ there exists $i(n) \in (p_{n-1}, p_n]$ such that $t_{i(n)} \neq 0$. Set $a^k_i = |x_i|_k$ and corresponding to this matrix set $\varrho^k_n = \varrho^k(t_{p_{n-1}+1}, \ldots, t_{p_n})$. Finally let K be the Köthe space determined by the matrix*

$$\left(\frac{a^k_{\varrho^k_n}}{|t_{\varrho^k_n}|}\right).$$

Then there is a quotient map $T\colon E \to K$ satisfying

$$\text{(3)} \qquad Tx_i = t_i e_n, \quad p_{n-1} < i \leqslant p_n,\ n \in N.$$

Conversely, if $\tilde{K}$ is a sequence space in which the coordinate functionals are continuous and if $T\colon E \to \tilde{K}$ is a quotient map satisfying (3), *then (e_n) is a basis for $\tilde{K}$, $\tilde{K}$ is a Köthe space and $\tilde{K} = K$.*

Proof. We prove the second statement first. Let $y \in \tilde{K}$. Then there exists $\xi \in E$ such that $\xi = \sum_{i=1}^{\infty} \xi_i x_i$ and $y = T\xi$. Thus,

$$y = \sum_{i=1}^{\infty} \xi_i T x_i = \sum_{n=1}^{\infty} \Big(\sum_{i=p_{n-1}+1}^{p_n} \xi_i t_i \Big) e_n.$$

Since the coordinate functionals are continuous, the representation of y is unique, so (e_n) is a basis for $\tilde{K}$. Since T is a quotient map, we have the quotient norms

$$\|y\|_k = \inf\{\|x\|_k\colon x \in T^{-1}(y)\}.$$

For all $n, k \in N$,

$$\begin{aligned}
\|e_n\|_k^2 &= \inf\{\|x\|_k^2\colon x \in T^{-1}(e_n)\} \\
&= \inf\Big\{\sum_m \sum_{i=p_{m-1}+1}^{p_m} (\xi_i a^k_i)^2\colon \sum_i \xi_i x_i \in E \text{ and } \sum_{i=p_{m-1}+1}^{p_m} \xi_i t_i = \delta_{mn}\Big\} \\
&= \inf\Big\{\sum_{i=p_{n-1}+1}^{p_n} (\xi_i a^k_i)^2\colon \sum_{i=p_{n-1}+1}^{p_n} \xi_i t_i = 1\Big\} \\
&= \Big(\sum_{i=p_{n-1}+1}^{p_n} \Big(\frac{t_i}{a^k_i}\Big)^2\Big)^{-1},
\end{aligned}$$

where the last step is, for instance, an application of Lagrange multipliers.

Now for any k we have $\bar{k}$ such that $\sum \left(\frac{a^k_n}{a^{\bar{k}}_n}\right)^2 = M_k < \infty$. Thus,

$$\left(\frac{a^k_{\varrho^k_n}}{t_{\varrho^k_n}}\right)^2 \sum_{i=p_{n-1}+1}^{p_n} \left(\frac{t_i}{a^{\bar{k}}_i}\right)^2 = \left(\frac{a^k_{\varrho^k_n}}{t_{\varrho^k_n}}\right)^2 \sum_{i=p_{n-1}+1}^{p_n} \left(\frac{a^k_i}{a^{\bar{k}}_i}\right)^2 \left(\frac{t_i}{a^k_i}\right)^2 \leqslant M_k,$$

and so

$$\|e^n\|_k \leqslant \frac{a^k_{\varrho^k_n}}{t_{\varrho^k_n}} \leqslant M_k^{1/2} \|e^n\|_{\bar{k}}, \qquad k, n \in N.$$

This proves the second statement.

For the first statement we let $G = \{x = \sum_i \xi_i x_i \in E\colon$ for all $n \in N$ $\sum_{i=p_{n-1}+1}^{p_n} \xi_i t_i = 0\}$. Clearly, G is a closed subspace of E so we have a quotient map $\pi\colon E \to E/G$. Let $y_n = \pi\left(\frac{x_{i(n)}}{t_{i(n)}}\right)$. Then, $t_i \pi(x_{i(n)}) = t_{i(n)} \pi(x_i)$ for $n \in N$,

$p_{n-1} < i \leqslant p_n$. It is easy to construct a sequence in $(E/G)'$ biorthogonal to (y_n). Thus, the second statement is satisfied with T, e_n replaced with π, y_n, so we are finished.

2. Quotients of (s) with a basis. We begin by applying the method of Theorem 1.3 to the construction of quotient spaces of (s).

2.1. THEOREM. *Let $K = K(c)$ be a nuclear Köthe space determined by a matrix $c = (c_j^k)$ which satisfies the following condition: There is a sequence (k_j) of integers such that $\lim_j k_j = \infty$ and such that*

$$j \leqslant \frac{c_j^{k+2}}{c_j^{k+1}} \leqslant \frac{1}{2} \frac{c_j^{k+1}}{c_j^k}, \quad k < k_j, j \in N. \tag{4}$$

Then there is a quotient map $T\colon (s) \to K$ of the form (3).

Proof. First we note that the basis (e_n) for (s) can be arranged into an infinite matrix which we may denote $(e_{j,m})$. Then, as was pointed out in [5], a "matrix" for (s) is given by $(jm)^k$.

We will select a set $\varrho_j^1 > \varrho_j^2 > \ldots > \varrho_j^{k_j} > 0$ of integers for each j satisfying

$$j\varrho_j^{k+1} \leqslant \frac{c_j^{k+1}}{c_j^k} \leqslant j\varrho_j^k, \quad k = 1, \ldots, k_j. \tag{5}$$

To do this, we first choose ϱ_j^1 to be the smallest integer such that

$$\frac{c_j^2}{c_j^1} \leqslant j\varrho_j^1.$$

Assume that ϱ_j^k has been chosen so that the right-hand inequality in (5) holds. We then choose ϱ_j^{k+1} to be the largest integer such that the left-hand inequality in (5) holds. In view of (4) it follows that $\varrho_j^{k+1} \geqslant 1$ and also,

$$\frac{c_j^{k+2}}{c_j^{k+1}} \leqslant \frac{1}{2}\frac{c_j^{k+1}}{c_j^k} < \frac{1}{2} j(\varrho_j^{k+1}+1) \leqslant j\varrho_j^{k+1}$$

so that the right-hand inequality of (5) holds for $k+1$. By induction we may complete the construction of $\varrho_j^1, \ldots, \varrho_j^{k_j}$.

Our next step is to apply Lemma 1.2 with fixed j. The parameters (a_n^k), m, l_i, ϱ^i of that lemma are taken respectively to be $(jm)^k$, k_j, k, ϱ_j^k. Inequality (2) becomes

$$\left(\frac{\varrho_j^k}{\varrho_j^{k+1}}\right)^k < \frac{|t_{\varrho_j^k}|}{|t_{\varrho_j^{k+1}}|} < \left(\frac{\varrho_j^k}{\varrho_j^{k+1}}\right)^{k+1}, \quad k = 1, \ldots, k_j-1,$$

or

$$j\varrho_j^{k+1} < \frac{(j\varrho_j^{k+1})^{k+1}/|t_{\varrho_j^{k+1}}|}{(j\varrho_j^k)^k/|t_{\varrho_j^k}|} < j\varrho_j^k, \quad k = 1, \ldots, k_{j-1}.$$

Comparing this relation with (5) we see that we may choose

$$|t_{\varrho_j^k}| = \frac{(j\varrho_j^k)^k}{c_j^k}, \quad k = 1, \ldots, k_j, \tag{6}$$

and it then follows from Lemma 1.1 that

$$\varrho^k(t_{\varrho^{k_j}}, \ldots, t_{\varrho^1}) = \varrho_j^k \quad \text{for} \quad k = 1, \ldots, k_j. \tag{7}$$

Finally, to apply Lemma 1.2 we consider the quotient space E of (s) generated by the basis elements $(e_{j,m})$, $m = \varrho_j^{k_j}, \ldots, \varrho_j^1$, $j \in N$ and order them lexicographically as follows,

$$\ldots, e_{j-1,\varrho_{j-1}^1}, e_{j,\varrho_j^{k_j}}, e_{j,\varrho_j^{k_j-1}}, \ldots, e_{j,\varrho_j^1}, e_{j+1,\varrho_{j+1}^{k_{j+1}}}, \ldots$$

The indices (p_n) become (j, ϱ_j^1) and the sequence (t_i) becomes $(t_{\varrho_j^k})$. By Lemma 1.2 and (7) we have a quotient map of the form (3) of E onto the Köthe space $K(b)$ where $b = (b_j^k)$ is the matrix given by

$$b_j^k = \frac{(j\varrho_j^k)^k}{|t_{\varrho_j^k}|}, \quad k, j \in N.$$

By (8) and the fact that $\lim_j k_j = \infty$ it follows that $K(b) = K$. Thus, we have a quotient map of the form (3) of E onto K. Composing this with the obvious quotient map of (s) onto E gives the desired result.

In the next theorem, we obtain a necessary condition for a Köthe space to be a quotient space of (s).

2.2. THEOREM. *If K is a Köthe space which is a quotient space of (s), then there is a matrix (a_n^k) for K such that*

$$\frac{a_n^{k+2}}{a_n^{k+1}} \leqslant \frac{a_n^{k+1}}{a_n^k} \quad \textit{for all } n, k \in N. \tag{d_4}$$

Proof. We have a quotient map $J\colon (s) \to K$, so the transpose $J'\colon K' \to (s)'$ has the property that $B \subseteq K'$ is bounded if and only if $J'(B)$ is bounded in $(s)'$. Thus, for any fundamental system of closed, absolutely convex bounded sets (A_k) in $(s)'$, and $B_k = (J')^{-1}(A_k)$, (B_k) is a fundamental system of bounded sets in K', and (B_k^0) is a fundamental system of 0-neighborhoods in K.

Let (y_j) be the basis for K with associated functionals (g_j). Because of the continuous norms on K we may assume that (g_j) is bounded in K'. For any set S, we will use ϱ_S to denote the gauge of S. Let A_k be any fundamental system of closed, absolutely convex sets in $(s)'$ with $A_k \subseteq A_{k+1}$, $B_k = (J')^{-1}(A_k)$.

Let k_0 be such that $g_j \in B_{k_0}$ for all j. For $k \geqslant k_0$, $y \in K$, define

$$b_j^k = [\varrho_{A_k}(J'(g_j))]^{-1}, \qquad \|y\|_k = \sup_j |g_j(y)|\, b_j^k.$$

Then $b_j^k > 0$, and we will show that $(\|\ \|_k)_{k \geqslant k_0}$ is a fundamental system of norms for K.

Clearly if $g \in K'$ and $t > 0$, $g \in tB_k$ if and only if $J'(g) \in tA_k$ so $\varrho_{A_k}(J'(g)) = \varrho_{B_k}(g)$. Thus, for $g \in K'$, $k \geqslant k_0$ and $y \in K$ we have

$$\|y\|_k = \sup_j \frac{|g_j(y)|}{\varrho_{B_k}(g_j)} \leqslant \varrho_{B_k^0}(y).$$

On the other hand, since (y_j) is a basis and K is nuclear, then for $k \geqslant k_0$ we can find indices i, l and $M, N > 0$ such that

$$\sum_j |g_j(y)|\, \varrho_{B_k^0}(y_j) \leqslant M \sup_j |g_j(y)|\, \varrho_{B_i^0}(y_j) \leqslant N \varrho_{B_l^0}(y).$$

Hence,

$$\begin{aligned}\varrho_{B_k^0}(y) &\leqslant M \sup_j |g_j(y)|\, \varrho_{B_i^0}(y_j) = M \sup_j \frac{|g_j(y)|}{\varrho_{B_l}(g_j)}\, \varrho_{B_l}(g_j)\, \varrho_{B_i^0}(y_j) \\ &\leqslant M \sup_j \varrho_{B_l}(g_j)\, \varrho_{B_i^0}(y_j)\, \|y\|_l.\end{aligned}$$

But for each j and $y \in K$,

$$|g_j(y)|\, \varrho_{B_i^0}(y_j) \leqslant \frac{N}{M} \varrho_{B_l^0}(y),$$

so that

$$\varrho_{B_i^0}(y_j) g_j \in \frac{N}{M} B_l^{00} = \frac{N}{M} B_l.$$

Hence $\varrho_{B_l}(g_j)\, \varrho_{B_i^0}(y_j) \leqslant N/M$, and $\varrho_{B_k^0}(y) \leqslant N \|y\|_l$, so the systems of norms are equivalent.

Finally, we choose $A_k = \{a \in (s)' \colon\ |a_n| \leqslant n^k,\ n \in N\}$. Observe that if a is absorbed by A_k, then

$$\varrho_{A_k}(a) = \inf\{\lambda > 0 \colon\ |a_n| \leqslant \lambda n^k,\ n \in N\} = \sup_n \frac{|a_n|}{n^k}.$$

Let q^k be the index at which this sup occurs; i.e.,

$$\varrho_{A_k}(a) = \frac{|a_{q^k}|}{(q^k)^k}.$$

Then

$$\frac{\varrho_{A_{k+1}}(a)}{\varrho_{A_{k+2}}(a)} = \frac{|a_{q^{k+1}}|/(q^{k+1})^{k+1}}{|a_{q^{k+2}}|/(q^{k+2})^{k+2}} \leqslant \frac{|a_{q^{k+1}}|/(q^{k+1})^{k+1}}{|a_{q^{k+1}}|/(q^{k+1})^{k+2}} = q^{k+1},$$

and

$$\frac{\varrho_{A_k}(a)}{\varrho_{A_{k+1}}(a)} = \frac{|a_{q^k}|/(q^k)^k}{|a_{q^{k+1}}|/(q^{k+1})^{k+1}} \geqslant \frac{|a_{q^{k+1}}|/(q^{k+1})^k}{|a_{q^{k+1}}|/(q^{k+1})^{k+1}} = q^{k+1}.$$

Applying this inequality with $a = J'(g^i)$ for all j and $k \geqslant k_0$ we see that

$$\frac{b_j^{k+2}}{b_j^{k+1}} \leqslant \frac{b_j^{k+1}}{b_j^k},$$

which is just (d_4) condition.

The final result before our main theorem is a reconciliation of the apparent differences between condition (d_4) and condition (4) of Theorem 2.1.

2.3. THEOREM. *Let E be a nuclear Fréchet space with a basis (x_n) and a continuous norm. If (x_n) has a matrix representation satisfying condition (4) of Theorem 2.1, then (x_n) has a (d_4) matrix representation.*

Conversely if (x_n) has a (d_4) matrix representation, then there is a basis (y_n) for E which is quasiequivalent to (x_n) and which has a representation satisfying condition (4).

Proof. Suppose (x_n) is a basis for E with a representation satisfying (4). Then E is a quotient space of (s), so by Theorem 2.2, (x_n) has a (d_4) representation.

Suppose $(|\ |_k)$ is a fundamental system of norms on E such that

$$\frac{|x_n|_{k+2}}{|x_n|_{k+1}} \leqslant \frac{|x_n|_{k+1}}{|x_n|_k}, \qquad \text{for all } k, n.$$

Using a result of Bessaga and Pełczyński ([2]), we can find a permutation (z_n) of (x_n) such that for all k there exists $j > k$ such that

$$\frac{|z_n|_j}{|z_n|_k} \geqslant n^2 \quad \text{for } n \text{ sufficiently large (depending on } k).$$

Next, we define a decomposition of N into disjoint sets N_ν by

$$N_0 = \left\{n \in N \colon\ \frac{|z_n|_2}{|z_n|_1} < n^2\right\},$$

$$N_\nu = \left\{n \in N \colon\ \frac{|z_n|_{\nu+2}}{|z_n|_{\nu+1}} < n^2 \leqslant \frac{|z_n|_{\nu+1}}{|z_n|_\nu}\right\}, \qquad 1 \leqslant \nu < \infty,$$

$$N_\infty = \left\{n \in N \colon\ n^2 \leqslant \frac{|z_n|_{k+1}}{|z_n|_k},\ \text{for all } k \in N\right\}.$$

Note that each N_ν may be infinite, finite, or empty. Moreover, none of the relations in the definition of the N_ν or on the inequality specifying $|\cdot|_k$ are changed if z_n is replaced by a non-zero scalar multiple, y_n. Choose y_n so that if $n \in N_\nu$, $\nu < +\infty$, $|y_n|_{\nu+1} = 1$, and $y_n = z_n$ if $n \in N_\infty$. Clearly, (y_n) is a basis for E quasiequivalent to (x_n).

We define $(\| \ \|_k)$ by

$$\|y_n\|_k = \begin{cases} n^{2(k-\nu-1)} & \text{if} \quad 0 \leqslant \nu < \infty,\ n \in N_\nu,\ k > \nu+1, \\ |y_n|_k & \text{otherwise.} \end{cases}$$

Observe that if $0 \leqslant \nu < \infty$, $n \in N_\nu$, and $k > \nu+1$, then

$$|y_n|_k = \frac{|y_n|_k}{|y_n|_{k-1}} \cdots \frac{|y_n|_{\nu+2}}{|y_n|_{\nu+1}} \cdot |y_n|_{\nu+1} \leqslant n^{2(k-\nu-1)} = \|y_n\|_k,$$

so that $|y_n|_k \leqslant \|y_n\|_k$ for all n, k.

On the other hand, for each k there exists l and $M > 0$ such that $n^{2k} \leqslant M|y_n|_l/|y_n|_k$. Thus, if $\nu+1 < k$ and $n \in N_\nu$, then

$$\|y_n\|_k = n^{2(k-\nu-1)} \leqslant n^{2k} \leqslant M \frac{|y_n|_l}{|y_n|_k} \leqslant M|y_n|_l.$$

Hence $(\|\cdot\|_k)$ is an equivalent system of norms. Now we want to check that

$$n^2 \leqslant \frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}} \leqslant \frac{\|y_n\|_{k+1}}{\|y_n\|_k} \quad \text{for all } n, k.$$

This is obvious if $n \in N_\nu$ and $k+1 \leqslant \nu \leqslant \infty$. If $k > \nu+1$, both ratios are equal to n^2. If $k = \nu$, we have, for $n \in N_\nu$,

$$\frac{\|y_n\|_{\nu+2}}{\|y_n\|_{\nu+1}} = \frac{n^2}{|y_n|_{\nu+1}} = n^2 \leqslant \frac{|y_n|_{\nu+1}}{|y_n|_\nu} = \frac{\|y_n\|_{\nu+1}}{\|y_n\|_\nu}.$$

If $k = \nu+1$ and $n \in N_\nu$, we have

$$\frac{\|y_n\|_{\nu+3}}{\|y_n\|_{\nu+2}} = n^2 = \frac{n^2}{|y_n|_{\nu+1}} = \frac{\|y_n\|_{\nu+2}}{\|y_n\|_{\nu+1}}.$$

Finally we let $c_j^k = 1/2^{k^2}\, \|y_j\|_k$. For all k, j

$$\frac{c_j^{k+2}}{c_j^{k+1}} = \frac{2^{(k+1)^2}}{2^{(k+2)^2}} \frac{\|y_j\|_{k+2}}{\|y_j\|_{k+1}} \leqslant \frac{1}{2^{k+3}} \frac{\|y_j\|_{k+1}}{\|y_j\|_k} = \frac{1}{4} \frac{c_j^{k+1}}{c_j^k} \leqslant \frac{1}{2} \frac{c_j^{k+1}}{c_j^k}.$$

Now choose k_j to be the largest integer less than or equal to $\frac{1}{2}(-3+\log_2(j))$. $\lim_j k_j = \infty$, and if $k < k_j$,

$$j < \frac{j^2}{2^{2k+3}} \leqslant \frac{1}{2^{2k+3}} \frac{\|y_j\|_{k+2}}{\|y_j\|_{k+1}} = \frac{c_j^{k+2}}{c_j^{k+1}}.$$

Hence (c_j^k) is the desired matrix representation.

We are now ready to prove the main theorem.

2.4. THEOREM. *Let E be a nuclear Fréchet space with a basis. Then E is a quotient space of (s) if and only if E is isomorphic to one of the following:*

(a) *A Köthe space satisfying* (d_4).

(b) ω.

(c) $\omega \times K$, *where K is a Köthe space satisfying* (d_4).

(d) $\prod_n K^{(n)}$, *where each $K^{(n)}$ is a Köthe space satisfying* (d_4).

Proof. *Sufficiency*: (a) follows from Theorems 2.1 and 2.3. The fact that ω is a quotient of (s) follows from the closed graph theorem and the theorem of Borel which asserts that for any sequence $\xi \in \omega$ $\exists \varphi \in C_0^\infty(0,1) \ni \varphi^{(j)}(0) = \xi_j$. It is well known that $(s) \cong C_0^\infty(0,1)$ ([10]). (c) follows since $(s) \cong (s) \times (s)$. Next we observe that it is shown by Grothendieck ([7]) that the tensor product of quotient maps of Fréchet spaces is a quotient map. Thus, we have a quotient map $(s) \cong s \otimes s \to \omega \otimes s \cong \prod_{n=1}^{\infty} (s) = (s)^N$. But each $K^{(n)}$ is a quotient of (s) so $\prod K^{(n)}$ is a quotient of $(s)^N$, hence of (s).

Necessity: Let E be a nuclear Fréchet space with a basis which is a quotient space of (s). By a theorem of ([3]) E is isomorphic to one of the following: a Köthe space, ω, $\omega \times K$, where K is a Köthe space, or a countable product of Köthe spaces. The result then follows from Theorem 2.2.

3. Applications and examples. We begin by noting that condition (d_4) is dependent upon the choice of the system of norms. However, it is possible to give an equivalent condition which is independent of the matrix representation.

3.1. THEOREM. *Let (x_j) be a basis for a nuclear Fréchet space E which has a continuous norm. Then (x_j) has a (d_4) matrix representation if and only if for one (equivalently for any) matrix representation (a_j^k) of x_j there exist strictly increasing sequences (k_m) and (r_m) of positive integers such that for all m there exists $j(m)$ with*

$$(8) \qquad \left(\frac{a_j^{k_{m+1}}}{a_j^{k_m}}\right)^{1/r_m} \leqslant \left(\frac{a_j^{k_m}}{a_j^{k_{m-1}}}\right)^{1/r_{m-1}}, \qquad j \geqslant j(m).$$

Proof. Let (c_j^k) satisfy (d_4) and let (a_j^k) be any equivalent matrix. There exist p_1, k_1 such that

$$\frac{a_j^1}{c_j^{p_1}} \leqslant 1 \quad \text{and} \quad \frac{c^{p_1+1}}{a_j^{k_1}} \leqslant 1 \quad \text{for large } j.$$

By induction we find $k_0 = 1 < k_1 < \ldots$ and $1 < p_1 < p_2 < \ldots$ such that for each m

$$\frac{a_j^{k_{m-1}}}{c_j^{p_m}} \leqslant 1 \quad \text{and} \quad \frac{c_j^{p_m+1}}{a_j^{k_m}} \leqslant 1 \quad \text{for large } j.$$

Thus, for each m and sufficiently large j we have

$$\frac{a_j^{k_m+1}}{a_j^{k_m}} \leqslant \frac{c_j^{p_{m+2}}}{c_j^{p_m+1}} = \frac{c_j^{p_{m+2}}}{c_j^{p_{m+2}-1}} \cdots \frac{c_j^{p_m+1}}{c_j^{p_m}} \leqslant \left(\frac{c_j^{p_m+1}}{c_j^{p_m}}\right)^{p_{m+2}-p_m} \leqslant \left(\frac{a_j^{k_m}}{a_j^{k_{m-1}}}\right)^{p_{m+2}-p_m}.$$

If we set $r_m = \prod_{j=1}^{m} (p_{j+2}-p_j)$, the result follows.

Conversely suppose we have a matrix representation (a_j^k) with (k_m) and (r_m) as specified. We construct a (d_4) matrix by interpolating between the a^{k_m}'s.

For each m and $0 \leqslant \varrho < r_{m-1}$, let

$$c_j^{m,\varrho} = a_j^{k_{m-1}}\left(\frac{a_j^{k_m}}{a_j^{k_{m-1}}}\right)^{\varrho/r_{m-1}}.$$

Then for all m and $0 < \varrho < r_{m-1}$,

$$\frac{c_j^{m,\varrho+1}}{c_j^{m,\varrho}} = \left(\frac{a_j^{k_m}}{a_j^{k_{m-1}}}\right)^{1/r_{m-1}}$$

and this decreases as m increases. Observe that since $c_j^{m,0} = a_j^{k_{m-1}}$, $(c_j^{m,\varrho})_{m,\varrho}$ is equivalent to (a_j^k).

3.2. COROLLARY. *Every Köthe space which is both* (d_3) *and* (d_4) *is isomorphic to an infinite type* p.s.s.

Proof. Without loss of generality we choose a matrix (a_j^k) for the Köthe space K such that $a_j^1 = 1$ and, for each k, j, $(a_j^k)^2 \leqslant a_j^{k+1}$. If there is an equivalent (d_4) matrix, then we may apply Theorem 3.1 and obtain k_1 and for each k and r such that for j sufficiently large

$$\frac{a_j^{k+1}}{a_j^k} \leqslant (a_j^{k_1})^r.$$

To do this, let (k_m), (r_m) be as in Theorem 3.1.

If $k_m \leqslant k < k+1 \leqslant k_{m+1}$, then

$$\left(\frac{a_j^{k+1}}{a_j^k}\right)^{1/r_m} \leqslant \left(\frac{a_j^{k_{m+1}}}{a_j^{k_m}}\right)^{1/r_m} \leqslant \left(\frac{a_j^{k_1}}{a_j^1}\right)^{1/r_1}, \quad \text{for large } j.$$

Hence, for each k $\exists r$ and $M_k > 0$ such that

$$a_j^k \leqslant \frac{a_j^{k+1}}{a_j^k} \leqslant M_k (a_j^k)^r \quad \text{for all } j.$$

By induction we prove that for all ϱ, $(a^{k_1})^{2\varrho} \leqslant a^{k_1+\varrho}$. For $\varrho = 1$, this is just the statement $(a_j^{k_1})^2 \leqslant a_j^{k_1+1}$.

If $(a_j^k)^{2\varrho} \leqslant a_j^{k_1+\varrho}$, then

$$(a_j^{k_1})^{2(\varrho+1)} \leqslant (a_j^{k_1+\varrho})(a_j^{k_1})^2 \leqslant (a_j^{k_1+\varrho})^2 \leqslant a_j^{k_1+\varrho+1}, \quad j \in N.$$

Hence (a_j^k) is equivalent to $\left((a_j^{k_1})^\varrho\right)_{\varrho=1}^\infty$. Thus, up to a permutation and a diagonal transformation, K is an infinite type p.s.s.

In the other direction it is easy to check that every infinite type p.s.s. is both (d_3) and (d_4).

In [11], Mitiagin and Henkin prove the following theorem: *If an* (s)*-nuclear space* E *is isomorphic to a closed subspace of a finite center of a Hilbert scale* F *and is a quotient space of a finite center of a Hilbert scale* G, *then* E *is a finite center of a Hilbert scale.* If we assume that the finite centers F and G are nuclear, then they are finite type p.s.s. and are (s)-nuclear (for instance, see [13]). Hence, a space E which is a subspace of one finite type p.s.s. and a quotient space of another must be a finite type p.s.s. We give a similar result for infinite type p.s.s.

3.3. THEOREM. *If* $\Lambda_\infty(\alpha)$ *and* $\Lambda_\infty(\beta)$ *are infinite type* p.s.s.'s *and if the Köthe space* K *is a subspace of* $\Lambda_\infty(\alpha)$ *and a quotient space of* $\Lambda_\infty(\beta)$, *then* K *is isomorphic to an infinite type* p.s.s.

In particular, if E *is a subspace of* (s) *with a basis which is also a quotient space of* (s), *then* E *is isomorphic to a complemented subspace of* (s).

Proof. Let E be isomorphic to a subspace of $\Lambda_\infty(\alpha)$ and to a quotient space of $\Lambda_\infty(\beta)$. Using the main theorem of [5] we conclude that E is (d_3), while from Theorem 2.2 E is (d_4). Hence, from Corollary 3.2 E is isomorphic to an infinite type p.s.s.

Martineau ([9]) asked whether every nuclear Fréchet space is a quotient of (s) and Pełczyński ([12]) asked whether every nuclear Fréchet space is a quotient of $(s)^N$. Using a result of Zahariuta ([15], Theorem 3) it is easy to construct a negative answer to Martineau's question. Using this or the above result along with the following which appears in [9] we can also answer the other question in the negative.

3.4. THEOREM. (s) *and* $(s)^N$ *have the same quotient spaces. Hence there exist nuclear Köthe spaces which are not quotient spaces of* $(s)^N$.

Proof. Clearly, (s) is a quotient of $(s)^N$ and, as we observed in the proof of Theorem 2.4, $(s)^N$ is a quotient of (s). It is well known that there exist Köthe spaces which are (d_3) but are not isomorphic to an infinite type p.s.s. ([5]). Any such space is not a quotient space of (s), by Theorem 3.3, hence not a quotient space of $(s)^N$. For instance, the space $K(a_j^k)$ with $a_j^k = e^{(kj)}$ is not a quotient space of (s).

It should be noted that Vogt ([14]) has recently shown that every nuclear Fréchet space is a quotient space of some subspace of (s). He also has announced a solution to the problem of Martineau.

3.5. Theorem. *Every* (d_2) *space is a* (d_4) *space.*

Proof. Without loss of generality we may assume that the matrix representing our (d_2) space is (a_j^k), where for all j,

$$\lim_k a_j^k = 1, \quad \text{and for all } k, \quad \lim_j \frac{a_j^k a_j^{k+2}}{(a_j^{k+1})^2} = 0.$$

Hence, for each k, and j sufficiently large,

$$a_j^k a_j^{k+2} \leqslant (a_j^{k+1})^2.$$

3.6. Example. Λ_1 is both a subspace of $(s)^N$ and a quotient space of $(s)^N$ but Λ_1 is not isomorphic to a complemented subspace of $(s)^N$.

Proof. Since Λ_1 is (d_2), Λ_1 is a quotient space of $(s)^N$. By a result of Kōmura and Kōmura ([8]), Λ_1 is isomorphic to a subspace of $(s)^N$. However, from the Dragilev theory ([1]) Λ_1 cannot be isomorphic to any complemented subspace of $(s)^N$.

This last result shows that $(s)^N$ fails to have the property of (s) which is mentioned in the statement of Theorem 3.3. From the result of Mitiagin and Henkin cited above it follows that any stable finite type p.s.s. has this same property. Moreover, this property is shared by any infinite type p.s.s. $\Lambda_\infty(a)$ where

$$\sup \frac{a_{2n}}{a_n} < +\infty \quad \text{or} \quad \lim_n \frac{a_{n+1}}{a_n} = \infty \quad ([4]).$$

This suggests the following

Problem. Which nuclear Fréchet spaces E have the property that any space isomorphic to a subspace of E and a quotient space of E is isomorphic to a complemented subspace of E?

Added in proof: Similar results, in some cases more general, have been obtained independently by D. Vogt and D. Wagner.

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