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A remark on Edgar's extremal integral representation theorem

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Abstract. It is proved that if K is a closed, bounded, convex subset of a Banach space with the Radon–Nikodym property, then for every $x \in K$ there is a Borel probability measure μ on K, supported by a "separable extremal set" such that x is the barycenter of μ.

In [4], G. A. Edgar has proved a very nice version of Choquet's theorem [7] for separable, closed, bounded, convex subsets of Banach spaces with the Radon–Nikodym property. Namely, he proved that if K is a separable, closed, bounded, convex subset of a Banach space with the Radon–Nikodym property, then for every $y \in K$ there is a probability measure μ on the universally Borel measurable sets in K such that

$$y = \int_K x \, d\mu(x), \tag{1}$$

and the set of extreme points of K has μ–measure 1. His brilliant proof is based on the Kuratowski–Ryll-Nardzewski selection theorem and Chatterji's theorem on the convergence of bounded martingales in Banach spaces with the Radon–Nikodym property.

In [5], the same author has generalized his previous result to the nonseparable case. He defined, for universally Borel, separable supported probability measures on a fixed closed, bounded, convex subset K of a Banach space with the Radon–Nikodym property, an order relation $\prec$ in such a way that $\mu_1 \prec \mu_2$ means, roughly speaking, that the support of μ_2 is closer to the set of extreme points of K than the support of μ_1. He proved that for any $y \in K$ there is a measure maximal with respect to the order relation and such that (1) holds. In such a setting the result and, what is more important, its proof becomes much more complicated than in the separable case.

Below, we present an equivalent version, and we hope – an easier one, of Edgar's nonseparable theorem on extremal integral representation. But there are delicate points in the problem which are worth mentioning in advance. Namely,

(i) One has to decide upon the meaning of the integral in formulae (1). In view of the definition of the Radon–Nikodym property we believe that the natural notion of integral in Banach spaces with the Radon–Nikodym property is the "Bochner integral". But the identity function $f(x) = x$, for $x \in K$, must be μ-almost separable valued, which implies that μ is a separable supported measure. Therefore, we have to restrict our interest to separable supported measures.

(ii) Also, one has to decide on what σ-algebra of subsets the measure must be defined. Since the Bochner integral corresponds to convergence in the norm topology, it seems reasonable to choose the σ-algebra of Borel sets. Therefore, we shall seek a separable supported Borel measure μ on K such that (1) holds and the support of μ is as close as possible to the extreme points of K.

We recall that a Banach space X is said to have the *Radon–Nikodym property* (RNP) iff, for every probability space (Ω, Σ, P) and every X-valued measure ν defined on the σ-algebra Σ with the finite total variation absolutely continuous with respect to P, there is a Bochner integrable function $f\colon \Omega \to X$ such that

$$\nu(A) = \int_A f\,dP \quad \text{for every } A \in \Sigma.$$

If this is the case, we write $f = \frac{d\nu}{dP}$.

Throughout the paper, X will denote a Banach space with the Radon–Nikodym property and by "measure" and "integral" we shall always mean a separable supported Borel measure and the Bochner integral. As usual, we shall identify an integrable function with its equivalence class. If K is a closed, bounded, convex subset of X and (Ω, Σ, P) is a probability space, then

$$L(\Omega, K) = \{f \in L_1(\Omega, X)\colon f(\omega) \in K \text{ for } \omega \in \Omega\}.$$

If $\Sigma_1 \subset \Sigma$ is a σ-algebra and $f \in L_1(\Omega, X)$, then the *conditional expectation of f with respect to Σ_1* is a Σ_1-measurable function $g\colon \Omega \to X$ with the property

$$\int_A g\,dP = \int_A f\,dP \quad \text{for every } A \in \Sigma_1.$$

The conditional expectation of f with respect to Σ_1 will be denoted by $E(f, \Sigma_1)$. It is known (cf. [8]) that the conditional expectation always exists. In the case of Banach spaces with the RNP, the existence of the conditional expectation follows easily from the definition of the RNP.

In the sequel we shall deal with the fixed probability space (Ω, Σ, P). Namely, let $\mathfrak{c}$ denote the first uncountable ordinal. Then $\Omega = \{0, 1\}^{\mathfrak{c}}$, i.e. the set of all $\mathfrak{c}$-sequences of 0's and 1's. The probability P will be the completed product of measures which assign $\frac{1}{2}$ to each element of $\{0, 1\}$ and Σ will be the P-complete σ-algebra of subsets of Ω generated by sets of the form

$$A_\alpha = \{\omega = \omega(\xi)_{\xi<\mathfrak{c}}\colon \omega(\alpha) = 0\} \quad \text{for} \quad \alpha < \mathfrak{c}.$$

Moreover, by Σ_ξ, for $\xi \leqslant \mathfrak{c}$, we shall denote the P-complete σ-algebra generated by the sets A_α, for $\alpha < \xi$. If $\alpha \leqslant \beta \leqslant \mathfrak{c}$, then $\Sigma_\alpha \subset \Sigma_\beta$. In particular, $\Sigma_0 = \{\emptyset, \Omega\}$, $\Sigma_{\mathfrak{c}} = \Sigma$.

Now, fix an arbitrary closed, bounded, convex set K in X. We write $f < g$, for $f, g \in L(\Omega, K)$, provided that there is an $\alpha < \mathfrak{c}$ such that $f = E(g, \Sigma_\alpha)$. It is easy to see that $<$ is a partial order relation in $L(\Omega, K)$. Note that since every $f \in L(\Omega, K)$ is an a.e. limit of step functions, for every f there is an $\alpha < \mathfrak{c}$ such that f is Σ_α-measurable. On the other hand, every $f \in L(\Omega, K)$ is almost separable valued.

LEMMA. *Let K be a closed, bounded, convex subset of X. Then for every $f \in L(\Omega, K)$ there is a $g \in L(\Omega, K)$ such that $f < g$ and g is maximal with respect to the relation $<$.*

Proof. Assume the contrary and let $f \in L(\Omega, K)$ be a function which does not satisfy the Lemma. We shall define, by transfinite induction, a $\mathfrak{c}$-sequence of functions $(f_\alpha)_{\alpha<\mathfrak{c}}$ in $L(\Omega, K)$ with the following properties:

(i) each f_α is Σ_α-measurable,

(ii) $f_\alpha = E(f_\beta, \Sigma_\alpha)$ for every $\alpha < \beta < \mathfrak{c}$,

(iii) for every $\alpha < \mathfrak{c}$ there exists a $\beta > \alpha$ such that $f_\alpha \neq f_\beta$.

To this end, let α_0 be the smallest ordinal such that f is Σ_{α_0}-measurable. For every $\alpha \leqslant \alpha_0$, we define $f_\alpha = E(f, \Sigma_\alpha)$. Now, assume that the functions f_α, for $\alpha < \beta$, have been defined. If β is not a limit ordinal, then there is γ such that $\gamma + 1 = \beta$. Since there is no maximal function greater than f, we infer that there is a $g \in L(\Omega, K)$ such that $f_\gamma < g$ and $f_\gamma \neq g$. Let α_1 be the smallest ordinal such that g is Σ_{α_1}-measurable. We put $f_\alpha = E(g, \Sigma_\alpha)$ for $\beta \leqslant \alpha \leqslant \alpha_1$. Obviously, $f_\alpha = E(f_{\alpha'}, \Sigma_\alpha)$, for every $\alpha < \alpha' \leqslant \alpha_1$. If β is a limit ordinal, we define $f_\beta = \lim_{\alpha<\beta} f_\alpha$. In order to prove that this limit exists, observe first that for any increasing sequence (α_n) of ordinals converging to β, $\lim_n f_{\alpha_n}$ exists by Chatterji's theorem [3], since (f_{α_n}) is a bounded martingale. The conclusion now follows by routine arguments. To show that $f_\gamma = E(f_\beta, \Sigma_\gamma)$, for $\gamma < \beta$, note that, by the dominated convergence theorem, for every $A \subset \Sigma_\gamma$ we have

$$\int_A f_\gamma\,dP = \lim_{\alpha<\beta} \int_A f_\alpha\,dP = \int_A \lim_{\alpha<\beta} f_\alpha\,dP = \int_A f_\beta\,dP.$$

So, we can assume that the sequence $(f_\alpha)_{\alpha<\mathfrak{c}}$, satisfying conditions (i)–(iii) has been defined. Now, put

$$\nu(A) = \int_A f_\alpha\, dP \quad \text{for} \quad A \in \Sigma_\alpha.$$

Since $\Sigma = \bigcup_{\alpha<\mathfrak{c}} \Sigma_\alpha$ and because of (ii), we obtain that ν is a P-absolutely continuous measure defined on Σ. On the other hand, since K is bounded, we infer that ν is of finite total variation. Hence, there is a Bochner integrable function $f_\mathfrak{c} = \frac{d\nu}{dP} \in L(\Omega, K)$. Then $f_\mathfrak{c}$ is Σ_α-measurable for some $\alpha < \mathfrak{c}$. For each $\beta < \mathfrak{c}$ we have $f_\beta = E(f_\mathfrak{c}, \Sigma_\beta)$, while, for $\alpha < \beta < \mathfrak{c}$ it is immediate that $f_\mathfrak{c} = E(f_\mathfrak{c}, \Sigma_\beta)$. Consequently $f_\beta = f_\mathfrak{c}$, for $\alpha < \beta < \mathfrak{c}$ which contradicts (iii) and proves the Lemma.

It is known (cf. [1]) that if K is a separable, closed, bounded, convex subset of X, then the set of its extreme points is μ-measurable for every separable supported, Borel measure μ on X. The set of extreme points of K will be denoted by $\operatorname{ext}(K)$.

A separable supported, complete, Borel probability measure μ on a closed, convex subset K of X will be said to be *supported by a separable extremal set* (or for short: to be *separable extremal*) iff, for every closed separable subspace Y of X including $\operatorname{supp}\mu$, we have $\mu(\operatorname{ext}(K \cap Y)) = 1$.

THEOREM. *Let K be a closed, bounded, convex subset of a Banach space X with the Radon–Nikodym property. Then, for every $y \in K$ there exists a separable extremal, Borel probability measure μ on K such that*

$$y = \int_K x\, d\mu(x).$$

Proof. Put $f(\omega) = y$, for $\omega \in \Omega$. By the Lemma, there is a maximal $g \in L(\Omega, K)$ greater than f. Let α be the smallest ordinal such that g is Σ_α-measurable. Without any loss of generality we can assume that $g(\omega) = g(\omega')$ for every pair $\omega, \omega' \in \Omega$ such that $\omega(\xi) = \omega'(\xi)$ for $\xi > \alpha$. Therefore, in the sequel we shall consider g as a P_α-measurable function defined on $\Omega_\alpha = \{0,1\}^\alpha$, where P_α is a completed product measure on Ω_α. Since α is a countable ordinal, we infer that the product topology on Ω_α is metrizable, Ω_α is compact with respect to this topology and P_α is a Borel measure.

Let μ be a complete, Borel measure on K given by the formulae $\mu(B) = P(g^{-1}(B))$, for Borel $B \subset K$. We shall show that μ is separable extremal. Indeed, if this is not the case, then there is a closed separable subspace Y of X which contains $\operatorname{supp}\mu$, and a Borel set B of positive measure μ, disjoint from $\operatorname{ext}(K \cap Y)$. This implies $g(\omega) \notin \operatorname{ext}(K \cap Y)$ for $\omega \in A = g^{-1}(B) \subset \Omega_\alpha$ and $P(A) > 0$. Let $K_1 = K \cap Y$. Since g is P_α-measurable, there is a compact $C_1 \subset A$ of positive measure P_α such that g restricted to C_1 is continuous. Let $C_2 = g(C_1)$. We have $\mu(C_2) \geqslant P(C_1) > 0$. Obviously $C_2 \subset B$.

Using the Kuratowski–Ryll-Nardzewski selection theorem ([6]) we can find universally Borel measurable functions f_0, f_1 on K_1 to K_1 which satisfy $(f_0(x)+f_1(x))/2 = x$ for every $x \in K_1$ and $f_0(x) = f_1(x) = x$ if and only if $x \in \operatorname{ext}(K_1)$ (see [4]). Since μ is a Borel measure on K_1 and K_1 is separable, we infer that there is a compact $C \subset C_2$ of positive μ-measure such that f_0 and f_1 restricted to C are continuous. Put, for $x \in K_1$,

$$g_i(x) = \begin{cases} f_i(x) & \text{for} \quad x \in C, \\ x & \text{for} \quad x \notin C, \end{cases}$$

for $i = 0, 1$. Finally, for $\omega = \omega(\xi) \in \Omega$, we define

$$h(\omega) = \begin{cases} (g_0 \circ g)(\omega) & \text{for } \omega = \omega(\xi) \in \Omega \text{ such that } \omega(\alpha+1) = 0 \\ (g_1 \circ g)(\omega) & \text{for } \omega = \omega(\xi) \in \Omega \text{ such that } \omega(\alpha+1) = 1. \end{cases}$$

It easily follows from the definition of h that $h \in L(\Omega, K_1) \subset L(\Omega, K)$ $g < h$, h differs from g on the set $g^{-1}(C)$ and $P(g^{-1}(C)) = \mu(C) > 0$, which contradicts the fact that g is maximal. On the other hand

$$y = \int_\Omega f\, dP = \int_\Omega g\, dP = \int_K x\, d\mu(x),$$

which completes the proof.

Remark. Using a similar argument and the technique developed in [2], one can prove that every separable supported, Borel probability measure on K is dominated by a separable extremal one.

It could be interesting to know whether the theorem above and Edgar's theorem on integral representation are equivalent. A positive answer on this question is given by the proposition below.

By $\mathscr{P}(K)$ we shall denote the space of all tight Borel probability measures on K endowed with a suitable topology (see [5]). We recall that $T\colon K \to \mathscr{P}(K)$ is said to be a *dilation* iff T restricted to every separable subset of K is Borel and for every $y \in K$ we have

$$y = \int_K x\, dT(y)(x).$$

If $\mu, \nu \in \mathscr{P}(K)$, then we write $\mu \prec \nu$ iff there is a dilation T such that $\nu = T\mu$, where $T\mu$ is defined by the formulae

$$(T\mu)(A) = \int_K (T(x))(A)\, d\mu(x).$$

It can be proved ([5]) that $\prec$ is a partial order relation and that $T\mu = \mu$ if and only if $T(x) = \delta(x)$ for μ-almost all $x \in K$ (here and below, $\delta(x)$ denotes the Dirac measure concentrated in x). To show that the Theorem and Edgar's theorem on [illegible]gral representation coincide, it is enough to prove the following

PROPOSITION. *A measure $\mu \in \mathscr{P}(K)$, where K is a closed, bounded, convex subset of X, is maximal (with respect to $\prec$) if and only if it is separable extremal.*

Proof. Let $\mu \in \mathscr{P}(K)$ be maximal. Assume that μ is not separable extremal. Then (in the notation of the proof of Theorem) we put $T(x) = \big(\delta(f_0(x)) + \delta(f_1(x))\big)/2$, for $x \in C$, and $T(x) = \delta(x)$, otherwise. Obviously, $\mu \neq T\mu$ and $\mu \prec T\mu$. Hence μ is not maximal; a contradiction.

Conversely, let μ be separable extremal. Assume that μ is not maximal. Then there is a dilation T such that $T\mu \neq \mu$. Since the measure $\nu_x = T(x)$ is separable supported for each $x \in K$ and T is separable Borel, one can show, using the same argument as in [5], that the closed linear subspace

$$Y = \overline{\operatorname{span}} \bigcup_{x \in S} \operatorname{supp} \nu_x,$$

where $S = \operatorname{supp}\mu$, is separable. Obviously $S \subset Y$. Set $A = \{x \in S\colon T(x) \neq \delta(x)\}$. We have $\mu(A) > 0$ and $A \subset (Y \cap K)$. Fix an arbitrary $y \in A$. Since $\nu_y \neq \delta(y)$ and since Y is separable, there is a closed ball B in Y such that $\nu_y(B) > 0$ and $y \notin B$. Let $K_1 = Y \cap K$ and $C = K_1 \setminus B$. We have

$$y = \int_{K_1} x\, d\nu_y(x) = \alpha \int_B \frac{x}{\alpha}\, d\nu_y(x) + \beta \int_C \frac{x}{\beta}\, d\nu_y(x) = \alpha x_1 + \beta x_2,$$

where $\alpha = \nu_y(B)$, $\beta = \nu_y(C)$, $\alpha + \beta = 1$. Obviously, $x_1 \in B$ and $x_2 \in K_1$. Thus $x_1 \neq y$. Hence y is not an extreme point of K_1. This shows that A is disjoint from $\operatorname{ext}(Y \cap K)$; a contradiction. This completes the proof.

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