TSUDIA MATHEMATICA, T. LXIII. (1978)

On Toeplitz operators associated with strongly pseudoconvex domains

by

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Abstract. Let Ω be a strongly pseudoconvex domain with smooth boundary in C^n, $n > 2$, and let $H^2(\delta\Omega) \subset L^2(\delta\Omega)$ be the closure of the space of boundary values of holomorphic functions which extend smoothly to $\overline{\Omega}$. We show that the C^*-algebra of all Toeplitz operators on $H^2(\delta\Omega)$ with continuous symbol contains the ideal of compact operators, and modulo this ideal is isomorphic to $C(\delta\Omega)$. We also improve a result of Janas on Toeplitz operators acting in the interior of Ω.

Introduction. Let Ω be a strongly pseudoconvex domain with smooth boundary in C^n. Toeplitz operators associated with Ω were first considered by Venugopalkrishna [9], who concerned himself with operators with continuous symbol on the space $H^2(\Omega)$ of square integrable functions holomorphic in Ω. Coburn [2] considered the special case where Ω is the unit ball in C^n; he worked with the C^*-algebras $\mathscr{T}(\Omega)$ and $\mathscr{T}(\delta\Omega)$ generated by Toeplitz operators with continuous symbol acting respectively on $H^2(\Omega)$ and on $H^2(\delta\Omega)$, the Hardy space of square integrable boundary values of holomorphic functions. He showed that both $\mathscr{T}(\Omega)$ and $\mathscr{T}(\delta\Omega)$ contain the ideal $\mathscr{K}$ of compact operators, and that both $\mathscr{T}(\Omega)/\mathscr{K}$ and $\mathscr{T}(\delta\Omega)/\mathscr{K}$ are isomorphic to $C(\delta\Omega)$. Janas [7] has for general Ω identified modulo the compact operators the C^*-algebra of $B(H^2(\Omega))$ generated by Venugopalkrishna's operators.

In this note we consider Toeplitz operators on $H^2(\delta\Omega)$, the closure in $L^2(\delta\Omega)$ of the boundary values of holomorphic functions in Ω which extend smoothly to $\overline{\Omega}$. In §1 we consider the tangential Cauchy–Riemann operator $\bar{\delta}_b$, and observe that Venugopalkrishna's arguments apply in our setting. In §2 we discuss the C^*-algebra $\mathscr{T}(\delta\Omega)$ generated by the Toeplitz operators on $H^2(\delta\Omega)$ with continuous symbol; we show that $\mathscr{T}(\delta\Omega)$ contains the ideal $\mathscr{K}$ of compact operators and that the symbol map induces an isomorphism: $\mathscr{T}(\delta\Omega)/\mathscr{K} \to C(\delta\Omega)$. Finally we answer (Theorem 3.1) a problem of Janas [7]; we show the same result holds for Toeplitz operators acting in the interior of Ω.

We would like to thank Morris Kalka for some helpful conversations and Ron Douglas for showing us Lemma 2.1; this considerably shortened our proof of Theorem 2.2.

§1. Let Ω be a bounded domain with smooth boundary in C^n, $n>1$, and let ϱ be a defining function for Ω. Let $A^q(\overline{\Omega})$ denote the space of $(0,q)$ forms on $\overline{\Omega}$ whose coefficients can be extended smoothly to a neighbourhood of $\overline{\Omega}$, and let $C^q(\overline{\Omega})$ denote the set of $\varphi\in A^q(\overline{\Omega})$ satisfying $\bar\partial\varrho\wedge\varphi=0$ on $\partial\Omega$. Let $\tilde{A}^q$ (respectively $\tilde{C}^q$) denote the sheaf of germs of $A^q(\overline{\Omega})$ $(C^q(\overline{\Omega}))$, and let $\tilde{B}^q$ denote the quotient sheaf $\tilde{A}^q/\tilde{C}^q$, which is a locally free sheaf supported on $\partial\Omega$. Let B^q denote the space of sections of $\tilde{B}^q$. Now $\bar\partial\colon C^q\to C^{q+1}$ and so induces a quotient map $\bar\partial_b\colon B^q\to B^{q+1}$. For further details see Folland and Kohn [4]. The usual inner product on $L^2(\partial\Omega)$ induces an inner product on B^q for $q\geqslant 0$; let $\mathscr{L}^q$ denote the completion of B^q in this inner product. By $\bar\partial_b^q$ (or simply $\bar\partial_b$) we shall mean the Hilbert space closure of the operator $\bar\partial_b\colon B^q\to B^{q+1}$. By ϑ_b^q, or ϑ_b, we shall mean the formal adjoint of $\bar\partial_b^q$, which since $\partial\Omega$ is a compact manifold coincides with its Hilbert space adjoint. If T is an operator on a Hilbert space, we shall denote its domain by $\mathscr{D}(T)$. If $q\geqslant 1$, let

$$\mathscr{D}(L)=\{f\in\mathscr{D}(\bar\partial_b^q)\cap\mathscr{D}(\vartheta_b^{q-1})\colon\ \bar\partial_b f\in\mathscr{D}(\vartheta_b),\ \vartheta_b f\in\mathscr{D}(\bar\partial_b)\}$$

and define $Lf=(\bar\partial_b\vartheta_b+\vartheta_b\bar\partial_b)f$ for $f\in\mathscr{D}(L)$. If $q=0$, let

$$\mathscr{D}(L)=\{f\in\mathscr{D}(\bar\partial_b^q)\colon\ \bar\partial_b f\in\mathscr{D}(\vartheta_b)\},$$

and define $Lf=\vartheta_b\bar\partial_b f$ for $f\in\mathscr{D}(L)$. For $q\geqslant 0$, let $\mathscr{H}^q=\{f\in\mathscr{D}(L)\colon Lf=0\}$, and let $P^q\colon\mathscr{L}^q\to\mathscr{H}^q$ denote the orthogonal projection.

Our main interest is in $\mathscr{H}^0$, which is just $\ker\bar\partial_b$, and which we shall denote by $H^2(\partial\Omega)$ or just H^2; we shall also let $P=P^0$. Observe that since $\bar\partial_b$ is the closure of $\bar\partial_b\colon B^0\to B^1$, we have $\{f\in B^0\colon\bar\partial_b f=0\}$ dense in H^2. By [6], 2.3.2, this means that the set of smooth functions in $\partial\Omega$ with holomorphic extensions to Ω is dense in H^2. Let σ denote the surface measure on $\partial\Omega$ and $P(x,y)$ denote the Poisson kernel of Ω. Then Stein [8] defines $H^2(\Omega)$ to be the closed subspace of $L^2(\partial\Omega)$ consisting of those f such that

$$P[f](x)=\int_{\partial\Omega}P(x,y)f(y)\,d\sigma(y)\qquad\text{for}\qquad x\in\Omega$$

defines a holomorphic function $P[f]$ in Ω. Since if f is continuous in $\overline{\Omega}$ and holomorphic in Ω we have $P[f]=f$ ([8], p. 2), we have $H^2(\partial\Omega)\subset H^2(\Omega)$. Thus in particular if $f\in H^2(\partial\Omega)$, $P[f]$ is holomorphic in Ω.

Now suppose that $n>2$ and that Ω is strongly pseudoconvex. We shall quote some results which are either contained in [4], §5.4 or are

analogous to those for the operator $\bar\partial$ in [4], §3.1. These results show that L has closed range, and that L is bounded away from 0 on $(\mathscr{H}^q)^\perp$. Define $G_b\colon\mathscr{L}^q\to\mathscr{L}^q$ to be the inverse of L on the range of L and zero on its complement.

THEOREM 1.1. (a) *If $q\geqslant 0$, G_b is bounded; if $q\geqslant 1$, G_b is compact;*

(b) *For $\alpha\in\mathscr{L}^q$, $\alpha=LG_bL+P^q\alpha$, and $LG_b\alpha\perp P^q\alpha$;*

(c) $G_bP^q=P^qG_b=0$;

(d) *G_b commutes with $\bar\partial_b$ and ϑ_b on $\mathscr{D}(\bar\partial_b)$ and $\mathscr{D}(\vartheta_b)$, respectively.*

For $\varphi\in C(\partial\Omega)$, let M_φ denote the bounded operator on $L^2(\partial\Omega)$ defined by $M_\varphi f=\varphi f$. By using Theorem 1.2 in place of Theorem 4.1 of [9], we can follow Venugopalkrishna's proof of [9], Theorem 2.1 to deduce that if s is smooth on Ω, then $(I-P)M_s\colon H^2(\partial\Omega)\to L^2(\partial\Omega)$ is compact. Now we can approximate any $s\in C(\partial\Omega)$ by smooth functions, and so we have

THEOREM 1.2. *If $s\in C(\partial\Omega)$, then $(I-P)M_s\colon H^2(\partial\Omega)\to L^2(\partial\Omega)$ is a compact operator.*

§2. Let Ω be a strongly pseudoconvex domain in C^n $(n>2)$ with smooth boundary. For $\varphi\in C(\partial\Omega)$, let T_φ denote the Toeplitz operator with symbol φ defined by $T_\varphi f=P(\varphi f)$ for $f\in H^2(\partial\Omega)$. Let $\mathscr{T}(\partial\Omega)$ denote the C^*-subalgebra of $B(H^2)$ generated by $\{T_\varphi\colon\varphi\in C(\partial\Omega)\}$. We shall use the following lemma, which we were shown by Ron Douglas.

LEMMA 2.1. *Let K be a closed subspace of a Hilbert space H, and let $T\in B(H)$ be normal. If K is an invariant subspace for T and the compression of T to K is normal, then K is a reducing subspace for T.*

Proof. Let P denote the orthogonal projection of H onto K. It is enough to show that K is invariant for T^*; in other words, that $PT^*P=T^*P$. Now the compression $T|_K$ of T to K is TP, and $(T|_K)^*=PT^*P$; hence $T|_K$ normal means that $TPT^*P-PT^*PTP=0$. Now since $T(K)\subset K$ and T is normal, this implies $PT(P-I)T^*P=0$, or $PT(P-I)^2T^*P=0$; but this is the same as $PT(P-I)=0$ which was to be proved.

THEOREM 2.2. *The C^*-algebra $\mathscr{T}(\partial\Omega)$ contains $\mathscr{K}(H^2)$, the ideal of all compact operators on $H^2(\partial\Omega)$, and $\mathscr{T}(\partial\Omega)/\mathscr{K}(H^2)$ is isometrically *-isomorphic to $C(\partial\Omega)$.*

Proof. To show that $\mathscr{T}(\partial\Omega)$ contains $\mathscr{K}(H^2)$ it is enough to show that $\mathscr{T}(\partial\Omega)$ is irreducible and that $\mathscr{T}(\partial\Omega)$ contains at least one non-zero compact operator ([3], 5.39). Suppose that Q is a projection in $B(H^2)$ such that $T_\varphi Q=QT_\varphi$ for all $\varphi\in C(\partial\Omega)$, and let $Q(1)=g$. If φ is holomorphic in Ω and C^∞ in $\overline{\Omega}$, then $\varphi g\in H^2(\partial\Omega)$ and so

$$Q(\varphi)=Q\big(T_\varphi(1)\big)=T_\varphi Q(1)=T_\varphi g=P(\varphi g)=\varphi g.$$

For all holomorphic polynomials φ and ψ, we have $(\varphi g,\psi)=(Q\varphi,\psi)=(\varphi,g\psi)$, so that $(g-\bar g,\bar\varphi\psi)=0$. By the Stone–Weierstrass theorem the set $\{\bar{\varphi}\psi: \varphi, \psi$ are holomorphic polynomials$\}$ is dense in $L^2(\delta\Omega)$ and so g is real-valued. Since $g \in H^2(\delta\Omega)$, this implies that $P[g]$ is a holomorphic real-valued function, and so g is constant. If $Q \neq 0$, then there is some non-zero $f \in H^2(\delta\Omega)$ such that $f = Qf$. For every φ holomorphic in Ω and C^∞ in $\bar{\Omega}$ we have $(Qf, \varphi) = (gf, \varphi)$ so that $Qf = gf$; then $gf = f$ and so $g = 1$. It then follows that $Q = I$ and $\mathscr{T}(\delta\Omega)$ is irreducible. If $\varphi, \psi \in C(\delta\Omega)$, then on $H^2(\delta\Omega)$ $T_\varphi T_\psi - T_{\varphi\psi} = PM_\varphi(P-I)M_\psi$, and so is compact by Theorem 1.2; in particular, $T_\varphi T_\varphi^* - T_\varphi^* T_\varphi$ is compact for each $\varphi \in C(\delta\Omega)$. Now suppose there are no non-zero compact operators in $\mathscr{T}(\delta\Omega)$. Then T_{z_1} is normal; but T_{z_1} is the compression of $M_{z_1} \in B(L^2(\delta\Omega))$ to $H^2(\delta\Omega)$, and so by Lemma 2.1 H^2 is invariant for $M_{z_1}^* = M_{\bar{z}_1}$. But this implies $\bar{z}_1 \in H^2(\delta\Omega)$ which is false since $P[\bar{z}_1] = \bar{z}_1$ is not holomorphic in Ω. Thus there is a non-zero compact operator in $\mathscr{T}(\delta\Omega)$ and $\mathscr{K}(H^2) \subset \mathscr{T}(\delta\Omega)$.

Define T: $C(\delta\Omega) \to \mathscr{T}(\delta\Omega)/\mathscr{K}(H^2)$ by $T(\varphi) = T_\varphi + \mathscr{K}(H^2)$; clearly, T is *-linear. We shall prove that T is an isomorphism. We have already seen that $T_\varphi T_\psi - T_{\varphi\psi}$ is compact for every $\varphi, \psi \in C(\delta\Omega)$; hence T is a homomorphism and so is also surjective. It remains to show T is injective, or, equivalently, that T_φ compact implies $\varphi = 0$. Suppose that T_φ is compact and that $\varphi(x_0) \neq 0$ for some $x_0 \in \delta\Omega$. It follows from [5], p. 275 that we can find a function f holomorphic in a neighbourhood of $\bar{\Omega}$ such that $|f(x_0)| = 1$ and $|f(x)| < 1$ for all $x \in \bar{\Omega} \setminus \{x_0\}$. Let $\{U_m\}$ be a neighbourhood base at x_0 in $\delta\Omega$; by adjusting f we can find holomorphic functions g_m such that $|g_m(x_0)| > 1$ and $|g_m| < 1$ in $\delta\Omega \setminus U_m$. Let $n_m \in N$ be such that $|g_m^{n_m}| < 1/m$ in $\delta\Omega \setminus U_m$ and $\|g_m^{n_m}\| \geq 1$, and define $f_m = g_m^{n_m}/\|g_m^{n_m}\|$. On the complement of any open neighbourhood of x_0, $f_m \to 0$ uniformly; hence $\|T_\varphi f_m - \varphi(x_0) f_m\| \to 0$ as $m \to \infty$. Without loss of generality we may assume that there is some g in the unit ball of H^2 with $f_m \to g$ weakly; since T_φ is compact, this implies $T_\varphi f_m \to T_\varphi g$ in norm. Now a standard argument shows that $T_\varphi g = \varphi(x_0) g$, and it follows that $|\varphi(x_0)|\,\|f_m - g\| \to 0$ as $m \to \infty$. Since $\varphi(x_0) \neq 0$, we have $\|f_m - g\| \to 0$ as $m \to \infty$, and so in particular $\|g\| = 1$. But a subsequence of $\{f_m\}$ must converge to g pointwise a.e., and so $g = 0$ a.e. This is nonsense, and so $\varphi = 0$ on $\delta\Omega$ as required.

Corollary 2.3. *Let $\varphi \in C(\delta\Omega)$. Then T_φ is a Fredholm operator if and only if φ is invertible in $C(\delta\Omega)$.*

We observe that the analogous results for Toeplitz operators with symbol in $C(\delta\Omega, M_r(C))$ can be obtained by tensoring the short exact sequence

$$0 \to \mathscr{K}(H^2) \to \mathscr{T}(\delta\Omega) \to C(\delta\Omega) \to 0$$

with the C^*-algebra $M_r(C)$ of complex $r \times r$ matrices.

§3. Again let $\Omega \subset C^n$ $(n > 1)$ be a strongly pseudoconvex domain with smooth boundary. Let $H^2(\Omega)$ denote the closed subspace of $L^2(\Omega)$ consisting of the functions holomorphic in Ω, and let P: $L^2(\Omega) \to H^2(\Omega)$ be the orthogonal projection. If $\varphi \in C(\bar{\Omega})$, define $S_\varphi \in B(H^2(\Omega))$ by $S_\varphi f = P(\varphi f)$, and let $\mathscr{T}(\Omega)$ denote the C^*-algebra of $B(H^2(\Omega))$ generated by $\{S_\varphi: \varphi \in C(\bar{\Omega})\}$.

Theorem 3.1. *The C^*-algebra $\mathscr{T}(\Omega)$ contains the ideal $\mathscr{K}(H^2(\Omega))$ of compact operators and $\mathscr{T}(\Omega)/\mathscr{K}(H^2(\Omega))$ is isometrically *-isomorphic with $C(\delta\Omega)$.*

Proof. That $\mathscr{T}(\Omega)$ contains all the compact operators has been shown by Janas [7], Theorem 1.1 (a). Define S: $\mathscr{T}(\Omega)/\mathscr{K}(H^2(\Omega)) \to C(\delta\Omega)$ by $S(S_\varphi + K) = \varphi|_{\delta\Omega}$. To show that S is well-defined it is enough to show that if S_φ is compact, then $\varphi|_{\delta\Omega} = 0$; this can be done by using peak functions exactly as in the proof of Theorem 2.2. Using [9], Theorem 2.1 in place of our 1.2, we have as before that $S_\varphi S_\psi - S_{\varphi\psi}$ is compact for $\varphi, \psi \in C(\bar{\Omega})$, so that S is a homomorphism (Janas observed in [7] that the condition that $S|_{\delta\Omega}$ be non-vanishing in [9], Theorem 2.1 is unnecessary). Clearly S is onto, and it is injective by a simple extension of [9], Theorem 2.3, which completes the proof.

We observe that by tensoring this result we obtain the corresponding result for operators with matrix-valued symbols. This result was originally obtained by Janas ([7], Theorem 2.1) under the additional hypothesis that the analytic polynomials are dense in the algebra $A(\Omega)$ of functions continuous in $\bar{\Omega}$ and holomorphic in Ω.

In the language of Brown, Douglas and Fillmore [1], we have shown that $\mathscr{T}(\delta\Omega)$ and $\mathscr{T}(\Omega)$ both define extensions of $C(\delta\Omega)$ by the ideal of compact operators on a separable Hilbert space. This raises an obvious question: do we get equivalent extensions? The answer is yes when Ω is the unit ball in C^n, as is shown by Coburn [2]. Whether the result is true in general we do not know; computations for some specific examples (in the same spirit as Coburn's) give the same answer, and we conjecture that the result is true in general.

Theorem 3.1 was obtained independently dy K. Yabuta, *A remark to a paper of Janas "Toeplitz operators related to certain domains in C^n"*, Studia Math. 62 (1978). pp. 73–74

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Received July 28, 1976 (1139)

STUDIA MATHEMATICA, T. LXIII. (1978)

A remark on Edgar's extremal integral representation theorem

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Abstract. It is proved that if K is a closed, bounded, convex subset of a Banach space with the Radon–Nikodym property, then for every $x \in K$ there is a Borel probability measure μ on K, supported by a "separable extremal set" such that x is the barycenter of μ.

In [4], G. A. Edgar has proved a very nice version of Choquet's theorem [7] for separable, closed, bounded, convex subsets of Banach spaces with the Radon–Nikodym property. Namely, he proved that if K is a separable, closed, bounded, convex subset of a Banach space with the Radon–Nikodym property, then for every $y \in K$ there is a probability measure μ on the universally Borel measurable sets in K such that

$$y = \int_K x \, d\mu(x), \tag{1}$$

and the set of extreme points of K has μ–measure 1. His brilliant proof is based on the Kuratowski–Ryll-Nardzewski selection theorem and Chatterji's theorem on the convergence of bounded martingales in Banach spaces with the Radon–Nikodym property.

In [5], the same author has generalized his previous result to the nonseparable case. He defined, for universally Borel, separable supported probability measures on a fixed closed, bounded, convex subset K of a Banach space with the Radon–Nikodym property, an order relation $\prec$ in such a way that $\mu_1 \prec \mu_2$ means, roughly speaking, that the support of μ_2 is closer to the set of extreme points of K than the support of μ_1. He proved that for any $y \in K$ there is a measure maximal with respect to the order relation and such that (1) holds. In such a setting the result and, what is more important, its proof becomes much more complicated than in the separable case.

Below, we present an equivalent version, and we hope – an easier one, of Edgar's nonseparable theorem on extremal integral representation. But there are delicate points in the problem which are worth mentioning in advance. Namely,