for positive numbers a and β. Then, for every $f \in L'(r^2)$ and for any pair $\{n_k\}$, $\{m_k\}$ of non-decreasing sequences,

$$\lim_{k\to\infty} n_k m_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+u, y+v)\,\Phi(n_k u)\,\Psi(m_k v)\,du\,dv = f(x,y)$$

almost everywhere on R^2.

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Vector measures on the closed subspaces of a Hilbert space

by

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Abstract. The present paper is concerned with vector valued measures defined on the lattice of all orthogonal projectors in a separable Hilbert space H, with values in a Banach space X. Those measures can be extended to bounded linear operators on the space $L(H)$ of all linear operators in H. In particular, we consider the measures taking their values in a Hilbert space $\mathscr{H}$ and in $L(\mathscr{H})$. As a corollary we obtain a description of homomorphisms of a standard Hilbert logic into itself. This is the generalization of the well-known theorem of Wigner.

Introduction. Let H (or $\mathscr{H}$) denote a Hilbert space (real or complex). Throughout we always assume dim $H \geqslant 3$. Let S_H (resp. $S_{\mathscr{H}}$) be the lattice of all orthogonal projectors in H (resp. $\mathscr{H}$) and let $L(H)$ be the space of all bounded linear operators acting in H.

An operator $M \in L(H)$, which is self-adjoint, nonnegative and trace-class will be called the s-*operator*.

For any subspace $H' \subset H$ we shall denote by $S_{H'}$ the lattice of all projective operators acting in H'.

$S_{H'}$ will also be treated as a set of operators from S_H which vanish on $H \ominus H'$.

Let X be a Banach space (real or complex).

DEFINITION 0. The mapping $\xi\colon S_H \to X$ will be called the *vector Gleason measure* (VG-measure) if

(i) for any sequence of mutually orthogonal projectors $P_1, P_2, \ldots$ from S_H the series

$$\sum_i \xi(P_i) \tag{0.1}$$

is weakly convergent to $\xi(\sum_i P_i)$;

(ii) $$\sup_{P\in S_H} \|\xi(P)\| = K < \infty.$$

By the well-known theorem of Orlicz [4], the accepted definition immediately implies unconditional and strong convergence of (0.1).

Gleason's theorem [2], giving the general form of a probability measure on the lattice of projective operators, is the basic tool in the study of VG-measures. This theorem states that if $\xi\colon S_H\to \mathbf{R}^+$ is a non-negative VG-measure, then there exists an s-operator M such that

(0.2) $$\xi(P) = \operatorname{tr} MP, \quad P\in S_H.$$

In 1971 Sherstniev [6] gave the following generalization of Gleason theorem for bounded real-valued measures.

THEOREM 0.1 Let $X=\mathbf{R}$. *For each* VG-*measure* $\xi\colon S_H\to\mathbf{R}$ *there exists a self-adjoint* s-*operator* M *such that* (0.2) *holds.*

Theorem 1 in §1 can be obtained as an easy consequence of Sherstniev's theorem. The proof given by us differs from that of Sherstniev and makes use only of the Gleason theorem.

§1

1.1. Now we shall give some description of a general VG-measure.

THEOREM 1. *If* $\xi\colon S_H\to X$ *is a* VG-*measure which takes values in a Banach space* X, *then for every* $x^*\in X^*$ *there exist self-adjoint trace-class operators* $M^1_{x^*}$, $M^2_{x^*}$ *such that*

(1.0) $$\langle \xi(P), x^*\rangle = \operatorname{tr} M^1_{x^*}P + i\operatorname{tr} M^2_{x^*}P$$

for any operator $P\in S_H$.

If H is a complex Hilbert space, then of course, we can write

$$\langle \xi(P), x^*\rangle = \operatorname{tr} M_{x^*}P,$$

where M_{x^*} is an s-operator of the form $M^1_{x^*}+iM^2_{x^*}$.

First we shall show the following

Remark 1. *For a finitely-dimensional subspace* $X\subset H$ *there exist uniquely defined linear self-adjoint operators* M^1_X, M^2_X *acting in* X *such that*

(1.1) $$(\xi(P), x^*) = \operatorname{tr} M^1_X P + i\operatorname{tr} M^2_X P \quad (P\in S_X),$$

and for two finitely-dimensional subspaces $X\subset X'\subset H$

(1.2) $$M^i_X x = P_X M^i_{X'} x \quad (i=1,2;\ x\in X)$$

(P_X an orthogonal projection on X).

Proof of Remark 1. Let $X\subset Z\subset H$ and $3\leqslant \dim Z<\infty$. The functions

$$\nu_1\colon P\to c\operatorname{tr} P + \operatorname{Re}\big(\xi(P), x^*\big),$$
$$\nu_2\colon P\to c\operatorname{tr} P + \operatorname{Im}\big(\xi(P), x^*\big),$$

where $c=\sup_{x\in H}\big|\big(\xi(\hat{x}), x^*\big)\big|$ ($\hat{x}$ means one-dimensional projection on the line spanned by $x\in H$), are positive VG-measures on S_Z. Hence, by Gleason's theorem,

$$\nu_i(P) = \operatorname{tr} M^i P \quad (i=1,2;\ P\in S_Z)$$

for some self-adjoint operators M^1, M^2 in Z and for $M^i_Z = M^i - c1_Z$, $i=1,2$ (1_Z the identity operator on Z) we obtain

$$\big(\xi(P), x^*\big) = \operatorname{tr} M^1_Z P + i\operatorname{tr} M^2_Z P \quad (P\in S_Z).$$

To obtain (1.1) it suffices to put

$$M^i_X x = P_X M^i_Z x \quad (i=1,2)$$

for $x\in X$, where P_X is the orthogonal projection on X.

Condition (1.2) is also satisfied, as the operator M^1_X is uniquely defined by the function $x\to(M^1_X x, x) = \|x\|^2\operatorname{tr} M^1_X\hat{x}$ on X, i.e. by the function $x\to\|x\|^2\operatorname{Re}\big(\xi(\hat{x}), x^*\big)$, and M^2_X is defined by the function $x\to \|x\|^2\operatorname{Im}\big(\xi(\hat{x}), x^*\big)$.

Proof of Theorem 1. For $x, y\in H$, put

(1.3) $$a^i(x,y) = (M^i_X x, y) \quad (i=1,2)$$

where X is the space spanned by vectors x, y.

$a^i(x,y)$ is then uniquely defined and homogeneous. Let now Z be the space spanned by vectors $x, x', y\in H$. Then, by (1.2),

$$\begin{aligned} a^i(x+x', y) &= \big(M^i_Z(x+x'), y\big) = (M^i_Z x, y) + (M^i_Z x', y)\\ &= a^i(x,y) + a^i(x', y) \quad (i=1,2). \end{aligned}$$

Similarly we obtain

$$a^i(x, y+y') = a^i(x,y) + a^i(x,y') \quad (x, y, y'\in H,\ i=1,2).$$

We also have

$$\sup_{\substack{x\in H\\ \|x\|=1}} |(a^i x, x)| \leqslant \sup_{\|x\|=1}\big|\big(\xi(x), x^*\big)\big| \leqslant K\|x^*\|$$

and thus there exist bounded linear operators $M^1_{x^*}$, $M^2_{x^*}$ such that

$$(a^i x, y) = (M^i_{x^*}x, y) \quad (x, y\in H,\ i=1,2).$$

Clearly, $M^1_{x^*}$, $M^2_{x^*}$ are s-operators since for any orthonormal sequence $z_1, z_2, \ldots$ in H we have

$$\sum_i (M^1_{x^*}z_i, z_i) = \sum_i \operatorname{Re}\big(\xi(\hat{z}_i), x^*\big) = \operatorname{Re}\Big(\xi\Big(\sum_i \hat{z}_i\Big), x^*\Big),$$
$$\sum_i (M^2_{x^*}z_i, z_i) = \operatorname{Im}\Big(\xi\Big(\sum_i \hat{z}_i\Big), x^*\Big),$$

and (1.0) is satisfied.

1.2. COROLLARY 1. *Each VG-measure taking values in a Banach space X can be extended to a continuous linear operator $\tilde{\xi}: L(H) \to X$ identical with ξ on S_H* ($L(H)$ and X are endowed with uniform and strong topologies, resp.). (1)

Proof. Let us call $A \in L(H)$ a *simple operator* if

$$A = \sum_{i=1}^{n} \lambda_i P_i, \tag{1.4}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ (or $\mathbf{C}$) and $P_1, \ldots, P_n \in S_H$ ($P_1, \ldots, P_n$ need not be mutually orthogonal), and let

$$\tilde{\xi}(A) = \sum_{i=1}^{n} \lambda_i \xi(P_i)$$

if A is simple. By Theorem 1, $\tilde{\xi}$ is uniquely defined and

(1) *$\tilde{\xi}$ is a linear operator on the space of simple operators.*

Moreover, by condition (ii) of Definition 0,

(2) *for any simple operator A*

$$\|\tilde{\xi}(A)\| \leqslant 4K\|A\|, \quad \textit{where} \quad K = \sup_{P \in S_H} \|\xi(P)\|.$$

Indeed, we have

$$\|\tilde{\xi}(A)\| = \big(\tilde{\xi}(A), x_0^*\big),$$

for some $x_0^* \in X^*$, $\|x_0^*\| = 1$ and, by Theorem 1,

$$\|\tilde{\xi}(A)\| = \operatorname{tr} M^1 A + i \operatorname{tr} M^2 A.$$

Let P^+, P^-, Q^+, Q^- be such projections that

$$|M^1| = P^+ M^1 - P^- M^1, \quad |M^2| = Q^+ M^2 - Q^- M^2,$$

where $|M^i| = \sqrt{(M^i)^2}$ $(i = 1, 2)$. Then, by (ii),

$$\begin{aligned} \operatorname{tr} M^1 A + i \operatorname{tr} M^2 A &\leqslant \|A\|(\operatorname{tr}|M^1| + \operatorname{tr}|M^2|) \\ &\leqslant \|A\|(\|\xi(P^+)\| + \|\xi(P^-)\| + \|\xi(Q^+)\| + \|\xi(Q^-)\|) \\ &\leqslant 4K\|A\|. \end{aligned}$$

For any operator $A \in L(H)$ there is a sequence of simple operators $A_1, A_2, \ldots$, which tends uniformly to A if H is complex and to $\frac{1}{2}(A + A^+)$, where A^+ is the operator adjoint to A, if H is real. Then we can put

$$\tilde{\xi}(A) = \lim_{n \to \infty} \xi(A_n). \ (^2)$$

(1) The space X must be complex when the space H is a complex one.

(2) In the real case we have $\tilde{\xi}(A) = \tilde{\xi}((A + A^+)/2)$.

By (1) and (2), $\tilde{\xi}$ is a well-defined and continuous operator on $L(H)$, what completes the proof.

Obviously, it is also possible to extend any VG-measure ξ to an operator on some space of unbounded (integrable) operators on H but such "theory of integration" will be the aim of a subsequent paper.

1.3. We shall now single out an important class of VG-measures,

DEFINITION 1. VG-measure $\xi: S_H \to \mathscr{H}$ taking values in a Hilbert space $\mathscr{H}$ is called an *orthogonal Gleason measure* (OG-*measure*) if

(j) *for any mutually orthogonal operators $P, Q \in S_H$, the vectors $\xi(P)$ and $\xi(Q)$ are mutually orthogonal.*

It can easily be verified that for VG-measure, taking values in the Hilbert space, conditions (i) and (j) imply (ii). Indeed,

$$\|\xi(P)\|^2 = \|\xi(I_H)\|^2 - \|\xi(I_H - P)\|^2 \leqslant \|\xi(I_H)\|^2$$

for any operator $P \in S_H$. Let us notice the following trivial

PROPOSITION 1. *For each OG-measure $\xi: S_H \to \mathscr{H}$ there is a uniquely defined self-adjoint* s-*operator M such that*

$$\big(\xi(P), \xi(Q)\big) = \operatorname{tr} MPQ$$

for any commuting operators $P, Q \in S$.

Proof. By Gleason's theorem there exists an s-operator M such that

$$\|\xi(P)\|^2 = \operatorname{tr} MP$$

for any $P \in S_H$. For commuting operators $P, Q \in S$ we obtain, by (j),

$$\big(\xi(P), \xi(Q)\big) = \|\xi(PQ)\|^2 = \operatorname{tr} MPQ.$$

The following theorem gives the general form of OG-measure.

THEOREM 2. *Let $\xi: S_H \to \mathscr{H}$ be an arbitrary orthogonal Gleason measure. Then*

I. *If both the spaces H and $\mathscr{H}$ are complex, then there exist* s-*operators M' and M'' acting in H such that*

$$\big(\xi(P), \xi(Q)\big) = \operatorname{tr} M'PQ + \operatorname{tr} M''QP \tag{1.5}$$

for any projections $P, Q \in S_H$. When $\dim H = \infty$, the operators M' and M'' are uniquely determined by the measure ξ; if $\dim H = n < \infty$, then the correlation function of ξ can be uniquely recorded as

$$\big(\xi(P), \xi(Q)\big) = \operatorname{tr} M'PQ + \operatorname{tr} M''QP + \omega \operatorname{tr} PQ, \tag{1.5'}$$

where $\omega \geqslant 0$ is a certain constant, and we additionally require the operators M' and M'' to vanish on some subspace of H.

II. *If both the spaces H and $\mathscr{H}$ are real, then there exists strictly one s-operator M acting in H such that*

$$(1.6)\qquad \big(\xi(P),\,\xi(Q)\big) = \operatorname{tr} MPQ = \operatorname{tr} MQP, \quad P, Q \in S_H.$$

The authors wish to thank Professor C. Ryll-Nardzewski for having noticed that the original version of Theorem 2 was incorrect, which enabled them to improve their paper. They would also like to thank Dr E. Hensz for her help in giving shape to the corrected proof.

Let us notice that Theorem 2.II follows immediately from Corollary 1. Indeed, let $\tilde{\xi}\colon L(H)\to\mathscr{H}$ be the "integral" of the measure ξ. Then there is an s-operator M, uniquely determined, such that

$$\big(\xi(P),\,\xi(Q)\big) = \|\xi(PQ)\|^2 = \operatorname{tr} MPQ$$

for any commuting operators $P, Q \in S_H$, and M is now self-adjoint. Therefore, when the operator $A \in L(H)$ is a finite linear combination of mutually orthogonal projections, then

$$\|\tilde{\xi}(A)\|^2 = \operatorname{tr} MA^2.$$

For any operators $P, Q \in S_H$ we have

$$P+Q = \lim_{n\to\infty} A_n,$$

where each operator A_n is a finite linear combination of mutually orthogonal projections and convergency is uniform. Thus

$$\|\tilde{\xi}(P+Q)\|^2 = \lim_{n\to\infty} \|\tilde{\xi}(A_n)\|^2 = \operatorname{tr} M(P+Q)^2$$

and, as M is self-adjoint,

$$\|\tilde{\xi}(P+Q)\|^2 = 2\operatorname{tr} MPQ + \operatorname{tr} MP + \operatorname{tr} MQ.$$

Therefore

$$2\big(\xi(P),\,\xi(Q)\big) = \|\xi(P)+\xi(Q)\|^2 - \|\xi(P)\|^2 - \|\xi(Q)\|^2 = 2\operatorname{tr} MPQ.$$

The proof of Theorem 2.I is more complicated. First we shall introduce some notations and prove some auxiliary lemmas.

For any set $M \subset H$, let $[M]$ denote the subspace of H spanned by vectors from M. $[M]$ is the real Hilbert space if H is real, and $[M]$ is complex if H is complex.

The projective operator which projects on $[M]$ will be denoted by the same symbol. The field of real (resp. complex) numbers will be denoted, as usual, by $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) and $\bar{\alpha}$, for $\alpha \in \boldsymbol{C}$, will denote the number conjugate to α.

We shall prove the following

LEMMA 1. *If H_n is the n-dimensional complex Hilbert space with an orthonormal basis $e_1, \ldots, e_n$, and the matrix $(T_{pqrs})_{pqrs=1,\ldots,n}$ satisfies the conditions*

$$(1.7)\qquad T_{pqrs} = \overline{T_{rspq}}, \quad p, q, r, s = 1, \ldots, n,$$

$$(1.8)\qquad \sum_{p,q,r,s=1}^{n} T_{pqrs}(Pe_p, e_q)\,\overline{(Qe_r, e_s)} = 0$$

for any commuting operators $P, Q \in S_{H_n}$, then

$$(1.9)\qquad T_{abcd} = 0 \quad \text{if} \quad a \neq c \text{ and } b \neq d,$$

$$(1.10)\qquad T_{atbt} = \overline{T_{btat}} = -T_{tbta} = -\overline{T_{tatb}} = \mu_{ab} \quad \text{if} \quad a \neq b \text{ and } t = 1, \ldots, n$$

(i.e. T_{atbt} does not depend on t) and

$$(1.11)\qquad T_{abab} = -T_{baba} \quad (\text{in particular, } T_{aaaa} = 0),$$

$$(1.12)\qquad T_{abab} + T_{bcbc} + T_{caca} = 0$$

for any $a, b, c, d = 1, \ldots, n$.

Proof. Putting in (1.8) $P = [e_a]$, $Q = [e_b]$ (where $a = 1, \ldots, n$, $b = 1, \ldots, n$ are taken independently), we obtain

$$(1.13)\qquad T_{aabb} = 0, \quad a, b = 1, \ldots, n.$$

Now if we put $P = Q = [\alpha e_a + \beta e_b]$ (where $\alpha, \beta \in \boldsymbol{C}$, $|\alpha|^2 + |\beta|^2 = 1$; $a \neq b$, $a, b = 1, \ldots, n$), then the only non-vanishing matrix elements (Pe_p, e_q), (Qe_r, e_s) are

$$(Pe_a, e_a) = (Qe_a, e_a) = |\alpha|^2,$$
$$(Pe_a, e_b) = (Qe_a, e_b) = \bar{\alpha}\beta,$$
$$(Pe_b, e_a) = (Qe_b, e_a) = \alpha\bar{\beta},$$
$$(Pe_b, e_b) = (Qe_b, e_b) = |\beta|^2,$$

and condition (1.8) by (1.13) gives

$$T_{aaab}|\alpha|^2\alpha\bar{\beta} + T_{aaba}|\alpha|^2\bar{\alpha}\beta + T_{abaa}|\alpha|^2\bar{\alpha}\beta + T_{abab}|\alpha|^2|\beta|^2 + T_{abba}\bar{\alpha}^2\beta^2 + T_{abbb}\bar{\alpha}\beta|\beta|^2 + T_{baaa}|\alpha|^2\alpha\bar{\beta} + T_{baab}\alpha^2\bar{\beta}^2 + T_{baba}|\alpha|^2|\beta|^2 + T_{babb}\alpha\bar{\beta}|\beta|^2 + T_{bbab}\alpha\bar{\beta}|\beta|^2 + T_{bbba}\bar{\alpha}\beta|\beta|^2 = 0.$$

This polynomial with respect to α, β is homogeneous. Therefore the condition $|\alpha|^2 + |\beta|^2 = 1$ is not relevant and it is easy to check that the coefficients at all different products of variables $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ must vanish, so we have

$$(1.14)\qquad T_{abbb} = -T_{bbba}, \quad T_{abaa} = -T_{aaba}$$

and

(1.15) $$T_{abab} = -T_{baba}, \quad T_{abba} = T_{baab} = 0,$$

by the vanishing of the coefficients at $\bar{\alpha}\beta|\beta|^2$, $|\alpha|^2\bar{\alpha}\beta$, $|\alpha|^2|\beta|^2$, $\bar{\alpha}^2\beta^2$, $\alpha^2\bar{\beta}^2$.

Similarly, putting in (1.8)

$$P = [\alpha e_a + \beta e_b], \quad Q = [e_a, e_b] \quad (|\alpha|^2+|\beta|^2 = 1),$$

as a result of the vanishing of the coefficient at $\bar{\alpha}$, β we obtain

$$T_{abbb} = -T_{abaa}.$$

Thus by (1.14)

$$T_{abbb} = -T_{bbba} = -T_{abaa} = T_{aaba}$$

and by (1.7) we have already obtained (1.10) for $t \in \{a, b\}$. Note that (1.9), for $c, d \in \{a, b\}$, is reduced to

$$T_{aabb} = T_{bbaa} = T_{abba} = T_{baab} = 0.$$

Therefore by (1.13), (1.15) all formulas (1.9)–(1.11) are satisfied in the case of the indices a, b, c, d, t taking two different values at the most and μ_{ab} has already been defined for any $a, b = 1, \ldots, n$, $a \neq b$.

Formula (1.10) is now a consequence of

(1.16) $$T_{acbc} = -T_{cbca} = \mu_{ab}, \quad a \neq b \neq c \neq d,$$

and by (1.7), the condition

(1.17) $$T_{ccab} = T_{abbc} = 0$$

implies (1.9) when three of the indices a, b, c, d at the most may be mutually different.

If we put $P = [\alpha e_a + \beta e_b, e_c]$, $Q = [\delta(\alpha e_a + \beta e_b) + \gamma e_c]$ $(\alpha, \beta, \gamma, \delta \in C$, $|\alpha|^2+|\beta|^2 = |\gamma|^2+|\delta|^2 = 1)$, then $PQ = QP$ (as $Q \subset P$), and (1.8) gives a homogeneous polynomial with respect to the variables γ, δ. The restriction $|\gamma|^2+|\delta|^2 = 1$ is not relevant and as a coefficient at $\delta\bar{\gamma}$ we may write

$$T_{aaac}|\alpha|^2\alpha + T_{aabc}|\alpha|^2\beta + T_{abac}|\alpha|^2\beta + T_{abbc}\bar{\alpha}\beta^2 + \\ + T_{baac}\alpha^2\bar{\beta} + T_{babc}\alpha|\beta|^2 + T_{bbac}\alpha|\beta|^2 + \\ + T_{bbbc}\beta|\beta|^2 + T_{ccac}\alpha(|\alpha|^2+|\beta|^2) + T_{ccbc}\beta(|\alpha|^2+|\beta|^2) = 0.$$

In this form the coefficient is a homogeneous polynomial with respect to α, β and as coefficients at $\bar{\alpha}\beta^2$, $|\alpha|^2\beta$ and $\alpha|\beta|^2$ we get

(1.18) $$T_{abbc} = 0,$$

(1.19) $$T_{abac} + T_{aabc} + T_{ccbc} = 0, \\ T_{bbac} + T_{babc} + T_{ccac} = 0.$$

Now we put

$$P = [\varrho(\alpha e_a + \beta e_b + e_c)],$$
$$Q = [\varrho'(\alpha' e_a + \beta' e_b - (\bar{\alpha}\alpha' + \bar{\beta}\beta')e_c)],$$

where

$$\varrho = (|\alpha|^2+|\beta|^2+1)^{-1/2},$$
$$\varrho' = (|\alpha'|^2+|\beta'|^2+|\bar{\alpha}\alpha'+\bar{\beta}\beta'|^2)^{-1/2}$$

and $\alpha, \alpha', \beta, \beta' \in C$ are arbitrary, satisfying $|\alpha'|+|\beta'| > 0$. Then $P \perp Q$, otherwise (1.8) holds once again. The polynomial (with respect to $\alpha, \beta, \alpha', \beta'$) given by (1.8) is now extremely long, but we can immediately write the coefficients at $\alpha'\bar{\beta}'$ and $\bar{\alpha}\beta\alpha'\bar{\beta}'$

(1.20) $$T_{ccab}\varrho^2\varrho'^2 = 0,$$

(1.21) $$\varrho'^2\varrho'^2(T_{abab} - T_{cbcb} - T_{acac} + T_{cccc}) = 0.$$

In fact, by an analysis of all non-vanishing elements (Pe_p, e_q) and (Qe_r, e_s) it can be noticed that if we treat the products $(Pe_p, e_q)(Qe_r, e_s)$ as polynomials with respect to $\alpha, \alpha', \beta, \beta'$, then the monomial $\alpha'\bar{\beta}'$ appears only in

$$(Pe_c, e_c)(Qe_a, e_b) = \varrho^2\varrho'^2\alpha'\bar{\beta}',$$

and $\bar{\alpha}\beta\alpha'\bar{\beta}'$ occurs only in

$$(Pe_a, e_b)(Qe_a, e_b) = \varrho^2\varrho'^2\bar{\alpha}\beta\alpha'\bar{\beta}',$$
$$(Pe_c, e_b)(Qe_c, e_b) = \varrho^2\varrho'^2(-\bar{\alpha}\beta\alpha'\bar{\beta}' - |\beta|^2|\beta'|^2),$$
$$(Pe_a, e_c)(Qe_a, e_c) = \varrho^2\varrho'^2(-\bar{\alpha}\beta\alpha'\bar{\beta}' - |\alpha|^2|\alpha'|^2),$$
$$(Pe_c, e_c)(Qe_c, e_c) = \varrho^2\varrho'^2(|\alpha|^2|\alpha'|^2 + |\beta|^2|\beta'|^2 + \bar{\alpha}\alpha'\beta\bar{\beta}' + \alpha\bar{\alpha}'\bar{\beta}\beta').$$

Formulas (1.17) and (1.16) follow from (1.18), (1.20) and (1.19). Condition (1.12) is (by (1.13), (1.11)) a consequence of (1.21), thus Lemma 1 is proved when $\dim H_n = n = 3$ (and the indices a, b, c, d, t can take three different values at the most).

To prove Lemma 1 for $n > 3$ it is now enough to exhibit (1.9) for mutually different numbers (a, b, c, d). For the purpose we put $P = [\alpha e_a + \beta e_b]$, $Q = [\gamma e_c + \delta e_d]$ $(|\alpha|^2+|\beta|^2 = |\gamma|^2+|\delta|^2 = 1)$ in (1.8) and thus we obtain $T_{abcd} = 0$ as the coefficient at $\bar{\alpha}\beta\gamma\bar{\delta}$.

Let J denote the set of all positive integers if $\dim H = \infty$, and let $J = \{1, \ldots, \dim H\}$ if $\dim H < \infty$. We denote by $(e_j)_{j\in J}$ an orthonormal basis in H such that

(1.22) $$Me_j = \lambda_j e_j, \quad j \in J,$$

where M is given in Proposition 1, and put

$$H_n = [e_1, \ldots, e_n], \quad n \in J.$$

Lemma 2. *There exists a matrix* $(U_{pqrs})_{pqrs\in J}$ *such that for any projectors* $P, Q \in S_{H_n}$ *(for some* $n \in J$*)*

$$(\xi(P), \xi(Q)) = \sum_{pqrs=1}^{n} U_{pqrs}(Pe_p, e_q)\overline{(Qe_r, e_s)}, \tag{1.23}$$

and

$$U_{pqrs} = \overline{U_{rspq}} \tag{1.24}$$

for any $p, q, r, s = 1, \ldots, n$.

Proof. Let $\tilde{\xi}$ be the linear extension of the measure ξ onto the whole space $L(H)$. The correlation function $(A, B) \to (\tilde{\xi}(A), \tilde{\xi}(B))$, when $A, B \in L(H_n)$, is linear with respect to A and anti-linear with respect to B, and by the well-known properties of bilinear transformations for a fixed $n \in J$ there exists a matrix $(U^n_{pqrs})_{p,q,r,s=1,\ldots,n}$ such that

$$(\tilde{\xi}(A), \tilde{\xi}(B)) = \sum_{pqrs=1}^{n} U^n_{pqrs}(Ae_p, e_q)\overline{(Be_r, e_s)}, \quad \text{for} \quad A, B \in L(H_n).$$

Let us define the operator

$$E_{ab}e_p = \begin{cases} e_b & \text{if} \quad p = a, \\ 0 & \text{if} \quad p \neq a \end{cases}$$

for arbitrary numbers $a, b = 1, \ldots, n$. Now we have

$$U^n_{abcd} = \sum_{pqrs=1}^{n} U^n_{pqrs}(E_{ab}e_p, e_q)\overline{(E_{cd}e_r, e_s)}$$
$$= (\tilde{\xi}(E_{ab}), \tilde{\xi}(E_{cd})) = \overline{(\tilde{\xi}(E_{cd}), \tilde{\xi}(E_{ab}))} = U^{n'}_{abcd} = \overline{U^{n'}_{cdab}}$$

for any $a, b, c, d = 1, \ldots, n$ and $n \leqslant n'$.

Thus, to obtain (1.23), (1.24), it is enough to put

$$U_{pqrs} = U^n_{pqrs}, \quad p, q, r, s \in J,$$

where $n = \max(p, q, r, s)$.

Lemma 3. *Theorem 2.I is valid if we in* (1.5) *additionally require that the operators* $P, Q \in S_{H_n}$ *with the fixed* $n \in J$.

Proof. Let the operator M be given by Proposition 1. For any commuting operators $P, Q \in S_{H_n}$ we have by (1.22)

$$(\xi(P), \xi(Q)) = \operatorname{tr} MPQ = \sum_{p=1}^{n} \lambda_p(Pe_p, Qe_p)$$
$$= \sum_{pqrs=1}^{n} \lambda_p \delta_{pr}\delta_{qs}(Pe_p, e_q)\overline{(Qe_r, e_s)}$$

(where $\delta_{pr} = 1$ if $p = r$ and $\delta_{pr} = 0$ if $p \neq r$).

If we put

$$T_{pqrs} = U_{pqrs} - \lambda_p \delta_{pr}\delta_{qs}, \quad p, q, r, s = 1, \ldots, n,$$

then, by Lemma 2, (1.7) and (1.8) are valid for any commuting operators $P, Q \in S_{H_n}$ and consequently, by Lemma 1, (1.9)–(1.12) hold for any $p, q, r, s = 1, \ldots, n$ and thus for any $p, q, r, s \in J$.

First we shall examine consequences of relations (1.11), (1.12). If we put

$$\tilde{\alpha}_1 = 0, \quad \tilde{\alpha}_p = T_{1p1p}, \quad p \in J, \; p \geqslant 2,$$

then

$$T_{pqpq} = T_{1q1q} - T_{1p1p} = \tilde{\alpha}_q - \tilde{\alpha}_p \quad \text{for} \quad p, q \in J.$$

Thus, for $\tilde{\beta}_p = \lambda_p - \tilde{\alpha}_p \quad (p \in J)$,

$$U_{pqpq} = T_{pqpq} + \lambda_p = \tilde{\beta}_p + \tilde{\alpha}_q.$$

Moreover, (if $a, b \leqslant n \in J$)

$$\tilde{\beta}_a + \tilde{\alpha}_b = U_{abab} = \sum_{pqrs=1}^{n} U_{pqrs}(E_{ab}e_p, e_q)\overline{(E_{ab}e_r, e_s)} = \|\tilde{\xi}(E_{ab})\|^2 \geqslant 0.$$

Thus for

$$\alpha_a = \tilde{\alpha}_a - \inf_{p\in J} \tilde{\alpha}_p, \quad \beta_a = \lambda_a - \alpha_a$$

we have

$$\alpha_a + \beta_a = \lambda_a, \quad \beta_a + \alpha_b = U_{abab}, \quad \alpha_a \geqslant 0 \tag{1.25}$$

and moreover

$$\beta_a \geqslant 0 \tag{1.26}$$

for any $a, b \in J$, $a \neq b$. Indeed, for an arbitrary $\varepsilon > 0$ there exist $\alpha_a < \varepsilon$ $(a \in J)$ and $\alpha_a + \beta_b = \tilde{\alpha}_a + \tilde{\beta}_b \geqslant 0$ for any $b \in J$. Thus $\beta_b \geqslant -\alpha_a > -\varepsilon$ and (1.26) is satisfied.

Taking advantage of relations (1.9), (1.10), (1.25) and putting $\mu_{aa} = 0$ $(a \in J)$ we can verify that

$$U_{pqrs} = \mu_{pr}\delta_{qs} - \overline{\mu_{qs}}\delta_{pr} + (\beta_p + \alpha_q)\delta_{pr}\delta_{qs}$$

with $\mu_{pr} = \overline{\mu_{rp}}$ $(p, q, r, s \in J)$. Now we shall take into consideration the case of the infinitely-dimensional Hilbert space H apart from the finitely-dimensional one.

If $\dim H = \infty$, then we define the matrices

$$\begin{aligned} m''_{ab} &= \mu_{ab} + \delta_{ab}\beta_a, \\ m'_{ab} &= -\mu_{ab} + \delta_{ab}\alpha_a, \quad a, b \in J. \end{aligned} \tag{1.27}$$

For any operators $A, B \in L(H_n)$ we now have

$$(1.28)\qquad (\tilde{\xi}(A), \tilde{\xi}(B)) = \sum_{pqrs=1}^{n} U_{pqrs}(Ae_p, e_q)\overline{(Be_r, e_s)}$$
$$= \sum_{pqrs=1}^{n} \big(m''_{pr}\delta_{qs}(Ae_p, e_q)\,\overline{(Be_r, e_s)} + \overline{m'_{qs}}\,\delta_{pr}(Ae_p, e_q)\,\overline{(Be_r, e_s)}\big)$$
$$= \sum_{pr=1}^{n} m''_{pr}(B^+Ae_p, e_r) + \sum_{qs=1}^{n} m'_{qs}(AB^+e_q, e_s),$$

where by B^+ we denote the operator adjoint to B.

We shall demonstrate that the matrices $\{m'_{qs}\}$, $\{m''_{pr}\}$ are positive-defined. For an arbitrary finite sequence $\eta_1, \ldots, \eta_k$ of complex numbers and $\varepsilon > 0$ let us find $\alpha_{n_0} < \varepsilon / \sum_{i=1}^{k} |\eta_i|^2$ (by (1.25), (1.26) $\alpha_a \leqslant \lambda_a$ and $\alpha_a \to 0$ as $a \to \infty$). The operator

$$A = \sum_{i=1}^{k} \eta_i E_{in_0}$$

belongs to $L(H_n)$ for $n = \max(n_0, k)$, i.e.

$$0 \leqslant (\tilde{\xi}(A), \tilde{\xi}(A)) = \sum_{pr=1}^{k} m''_{pr}\eta_p\overline{\eta}_r + \sum_{q=1}^{k} \overline{m}'_{n_0n_0}\eta_q\overline{\eta}_q$$
$$= \sum_{pr=1}^{k} m''_{pr}\eta_p\overline{\eta}_r + \alpha_{n_0}\sum_{q=1}^{k} |\eta_q|^2,$$

that is,

$$\sum_{pr=1}^{k} m''_{pr}\eta_p\overline{\eta}_r \geqslant -\varepsilon$$

for every $\varepsilon > 0$. So, the matrix $\{m''_{pr}\}$ is non-negative-defined. Similarly, for the fixed numbers $\eta_1, \ldots, \eta_k \in C$, $\varepsilon > 0$, there exists an element

$$\beta_{n_0} < \varepsilon / \sum_{i=1}^{k} |\eta_i|^2$$

(since $\beta_n \leqslant \lambda_n \to 0$ as $n \to \infty$). Taking in (1.28)

$$A = B = \sum_{i=1}^{k} E_{n_0 i}$$

we find that

$$\sum_{qs=1}^{k} \overline{m}'_{qs}\eta_q\overline{\eta}_s \geqslant -\varepsilon,$$

which means that the matrix $\{m'_{qs}\}$ is also positive-defined. Hence there are s-operators M', M'' such that

$$(1.29)\qquad (e_a, M'e_b) = m'_{ab},$$
$$(1.30)\qquad (e_a, M''e_b) = m''_{ab}, \quad a, b = 1, 2, \ldots$$

In fact, since the matrix $\{m'_{ab}\}$ is positive-defined, by (1.27) we have

$$|m'_{ab}|^2 \leqslant m'_{aa}\cdot m'_{bb} = \alpha_a \cdot \alpha_b,$$

i.e., as $\sum_{a=1}^{\infty} \alpha_a \leqslant \sum_{a=1}^{\infty} \lambda_a = \operatorname{tr} M < \infty$, we have for any $x, y \in H$

$$\sum_{ab=1}^{\infty} |(x, e_a)(e_b, y)m'_{ab}| = \sum_{a=1}^{\infty} \alpha_a |(x, e_a)| \cdot \sum_{b=1}^{\infty} \alpha_b |(e_b, y)|$$
$$\leqslant \Big(\sum_{a=1}^{\infty} \alpha_a\Big)^{1/2} \|x\| \cdot \Big(\sum_{b=1}^{\infty} \alpha_b\Big)^{1/2} \|y\|;$$

hence the quadratic form

$$(x, y) \to \sum_{a,b=1}^{\infty} (x, e_a)(e_b, y)m'_{ab}, \quad x, y \in H,$$

is well defined and bounded. Consequently, there exists a linear bounded operator M' such that

$$(x, M'y) = \sum_{ab=1}^{\infty} (x, e_a)(e_b, y)m'_{ab}, \quad x, y \in H,$$

and (1.29) holds. Similarly, there exists a bounded operator M'' fulfilling (1.30), and M', M'' must be s-operators.

If $A, B \in L(H_n)$, then we have by (1.28)

$$(\tilde{\xi}(A), \tilde{\xi}(B)) = \operatorname{tr} M'AB^+ + \operatorname{tr} M''B^+A.$$

Besides, the measure ξ uniquely determines the extension $\tilde{\xi}$ (when the space H is complex) and consequently, the matrix U_{pqrs} and operators M', M'' are uniquely determined.

If $\dim H = n < \infty$, then for the matrices $\{m'_{ab}\}$, $\{m''_{ab}\}$ defined in (1.27) we put

$$\omega' = \min_{\{\eta_i\}} \sum_{q,s=1}^{n} m'_{qs}\eta_q\overline{\eta}_s,$$
$$\omega'' = \min_{\{\eta_i\}} \sum_{p,r=1}^{n} m''_{pr}\eta_p\overline{\eta}_r,$$

where $\{\eta_i\}$ is any complex sequence with n elements, satisfying $\sum_{i=1}^{n}\eta_i\bar{\eta}_i = 1$.

The matrices

$$\tilde{m}'_{qs} = m'_{qs} - \omega'\delta_{qs}, \quad q, s = 1, \ldots, n,$$
$$\tilde{m}''_{pr} = m''_{pr} - \omega''\delta_{pr}, \quad p, r = 1, \ldots, n,$$

are already positive defined and uniquely determined by the measure ξ. At the same time there exist non-vanishing sequences of complex numbers $\eta'_1, \ldots, \eta'_n$ and $\eta''_1, \ldots, \eta''_n$ such that

$$\sum_{q,s=1}^{n} \tilde{m}'_{qs}\eta'_q\bar{\eta}'_s = \sum_{p,r=1}^{n} \tilde{m}''_{pr}\eta''_p\bar{\eta}''_r = 0.$$

Taking now the operator

$$A = \sum_{q,p=1}^{n} \eta'_q\eta''_p E_{qp}$$

we will find by (1.28) for $\omega = \omega' + \omega''$

$$0 \leqslant \big(\tilde{\xi}(A), \tilde{\xi}(A)\big) = \sum_{p,q,r=1}^{n} |\eta'_q|^2 \tilde{m}''_{pr}\eta''_p\bar{\eta}''_r + \sum_{p,q,s=1}^{n} |\eta''_p|^2\tilde{m}'_{qs}\eta'_q\bar{\eta}'_s + \omega\Big(\sum_{q=1}^{n}|\eta'_q|^2\Big)\cdot\Big(\sum_{p=1}^{n}(|\eta''_p|^2\Big) = \omega\Big(\sum_{q=1}^{n}|\eta'_q|^2\Big)\cdot\Big(\sum_{p=1}^{n}|\eta''_p|^2\Big),$$

that is, $\omega \geqslant 0$. For the operators M', M'', fulfilling

$$(e_a, M'e_b) = \tilde{m}'_{ab}, \quad (e_a, M''e_b) = \tilde{m}''_{ab} \quad (a, b = 1, \ldots, n),$$

(1.5′) holds.

Proof of Theorem 2. I. By Lemma 3 it is enough to prove Theorem 2.I for $\dim H = \infty$. By Corollary 1 the measure ξ must be continuous, if S_H is endowed with the uniform topology. For any finitely-dimensional operator $P \in S_H$ there exist both a sequence of integers $n_1, n_2, \ldots$ and a sequence of projections $P_1, P_2, \ldots$ such that $P_a \in S_{H_{n_a}}$ $(a = 1, 2, \ldots)$ and $P_n \to P$ uniformly as $n \to \infty$. Thus, by the well-known properties of traces of operators, formula (1.5) is valid for any finitely-dimensional operators $P, Q \in S_H$.

For arbitrary operators $P, Q \in S_H$ we have

$$P = \sum_j [x_j], \quad Q = \sum_j [y_j]$$

(the series are finite or not), where $[x_j]$ (resp. $[y_j]$)$(j = 1, 2, \ldots)$ are mutually orthogonal one-dimensional projections. Let (³)

$$P_n = \sum_{j=1}^{n}[x_j], \quad Q_n = \sum_{j=1}^{n}[y_j].$$

Then

$$\xi(P) = \lim_{n\to\infty}\xi(P_n), \quad \xi(Q) = \lim_{n\to\infty}\xi(Q_n)$$

and, as it can easily be verified,

$$\operatorname{tr} M'PQ = \lim_{n\to\infty}\operatorname{tr} M'P_nQ_n.$$

Indeed, M' is an s-operator, and P_n (resp. Q_n) tends weakly to P (resp. Q) for $n \to \infty$. Similarly,

$$\operatorname{tr} M''QP = \lim_{n\to\infty}\operatorname{tr} M''Q_nP_n$$

and thus

$$\big(\xi(P), \xi(Q)\big) = \lim_{n\to\infty}(\operatorname{tr} M'P_nQ_n + \operatorname{tr} M''Q_nP_n) = \operatorname{tr} M'PQ + \operatorname{tr} M''QP.$$

This ends the proof of Theorem 2.

Formulas (1.5) and (1.6) have the following interesting interpretation in the theory of random measures.

A stochastic process $\big(\xi(P): P \in S_H\big)$ is called a *Gaussian-Gleason measure* if

(1) $\xi(P)$ is a Gaussian random variable with the mean value $E\xi(P) = 0$;

(2) for any sequence of mutually orthogonal projectors $P_1, P_2, \ldots$ from S_H, the random variables $\xi(P_1), \xi(P_2), \ldots$ are independent and

$$\xi\Big(\sum_j P_j\Big) = \sum_j \xi(P_j) \quad \text{a.e.}$$

Formula (1.5) or (1.6) gives the general form of the covariance function for such a measure.

Using Theorem 2, we can also easily obtain the following corollary (see [3]):

Corollary 2. *Let H' and $\mathscr{H}$ be complex Hilbert spaces and* $\dim H' \geqslant 2$, *and let the mapping η: $H' \to \mathscr{H}$ satisfy*

(A) $\quad \operatorname{Im}(x, y) = 0$ *implies* $\big(\eta(x), \eta(y)\big) = (x, y)$

(³) If $\dim P < \infty$ (resp. $\dim Q < \infty$) it is enough to put $P_n = P$ (resp. $Q_n = Q$), $n = 1, 2, \ldots$

for $x, y \in H'$. Then there exists a real constant k, $|k| \leqslant 1$ such that

$$\text{(B)} \qquad \big(\eta(x), \eta(y)\big) = \operatorname{Re}(x, y) + ik \operatorname{Im}(x, y)$$

for any $x, y \in H'$.

Proof. Let the Hilbert space H be the orthogonal sum $H = H' \oplus [e]$ and $\|e\| = 1$. We can also assume the existence of a vector $f \in \mathscr{H}$, $\|f\| = 1$ such that $\eta(x) \perp f$ for any $x \in H'$. Let us extend η onto the space H by the formula

$$\text{(1.31)} \qquad \eta(ae + x) = af + \eta(x), \qquad a \in \boldsymbol{C},\ x \in H'$$

and let

$$\xi(P) = \eta(Pe), \qquad P \in S_H.$$

Then, for any mutually orthogonal operators $P_1, P_2, \ldots$ from S_H, η is an isometry on the set $\{(\sum_i P_i)e, P_1 e, P_2 e, \ldots\}$ (by (A)) and

$$\eta\Big(\Big(\sum_i P_i\Big)e\Big) = \sum_i \eta(P_i e).$$

Thus ξ is an OG-measure and, as the norm $\|\xi(P)\|$ is equal to $\|Pe\|$ ($P \in S_H$), formula (1.5) in Theorem 2 reduces to

$$\big(\xi(P), \xi(Q)\big) = \operatorname{Re}(Pe, Qe) + ik \operatorname{Im}(Pe, Qe) \qquad (P, Q \in S)$$

and $|k| \leqslant 1$. (4)

By (A) we have $\eta(ax) = a\eta(x)$ for any $a \in \boldsymbol{R}$, and

$$\begin{aligned}\eta(x+e) &= \eta\big(\|x+e\|^2(|x+e|e)\big)\\ &= \|x+e\|^2 \eta(|x+e|e) = \|x+e\|^2 \xi(|x+e|)\end{aligned}$$

for $x \in H'$. Thus, as $\xi(|e|) = \eta e = f$, for any $x \in H'$ we obtain, by (1.31),

$$\eta(x) = \eta(x+e) - f = \|x+e\|^2 \xi(|x+e|) - \xi(|e|).$$

Therefore

$$\begin{aligned}(\eta x, \eta y) &= \big(\|x+e\|^2 \xi([x+e]) - \xi([e]),\ \|y+e\|^2 \xi([y+e]) - \xi([e])\big)\\ &= \operatorname{Re}(\|x+e\|^2 [x+e]e - e,\ \|y+e\|^2 [y+e]e - e) +\\ &\qquad + ik \operatorname{Im}(\|x+e\|^2 [x+e]e - e,\ \|y+e\|^2 [y+e]e - e)\\ &= \operatorname{Re}(x, y) + ik \operatorname{Im}(x, y).\end{aligned}$$

Now we examine the case where the Hilbert space H is real and $\mathscr{H}$ is a complex one. The complex extension $\tilde{H}$ of the space H is constructed

(4) We have $(\xi(I-[e]), \xi(I-[e])) = \operatorname{tr} M'(I-[e]) + \operatorname{tr} M''(I-[e]) = 0$ and thus $M' = a[e]$, $M'' = \beta[e]$ for some $a, \beta \geqslant 0$. Moreover, $a + \beta = 1$, as $(\xi([e]), \xi([e])) = 1$, and it is enough to put $k = \beta - a$.

as a complement of the quotient space X/N, where X is a pre-Hilbert space of all formal finite linear combinations

$$\sum_{i=1}^{n} \eta_i x_i, \qquad \eta_i \in C,\ x_i \in H,\ i = 1, \ldots, n,$$

with the non-negative Hermitian form

$$\Big(\sum_{i=1}^{n} \eta_i x_i, \sum_{j=1}^{n} \zeta_j y_j\Big) = \sum_{i,j=1}^{n} \eta_i \bar{\zeta}_j (x_i, y_j),$$

and $N \subset X$ is a null subspace of X. The space H will be treated as a subset of $\tilde{H}$. Any orthonormal basis $\{e_i\}$ in the space H is also an orthonormal basis in the space $\tilde{H}$, and every bounded linear operator A acting in H can be extended to the operator $\tilde{A}$ acting in $\tilde{H}$ with the same matrix elements

$$(\tilde{A}e_i, e_j) = (Ae_i, e_j).$$

Let us notice that if the operator A is symmetric (trace-class, projective), then $\tilde{A}$ is self-adjoint (trace-class, projective).

COROLLARY 3. *If the spaces H and $\mathscr{H}$ are real and complex, respectively, then for any OG-measures $\xi\colon H \to \mathscr{H}$ there exists strictly one self-adjoint non-negative s-operator $\tilde{M}$ acting in $\tilde{H}$ such that*

$$\big(\xi(P), \xi(Q)\big) = \operatorname{tr} \tilde{M}\tilde{P}\tilde{Q} = \overline{\operatorname{tr} \tilde{M}\tilde{Q}\tilde{P}}, \qquad P, Q \in S_H.$$

Proof. Let $\tilde{\xi}\colon L(H) \to R$ be the "integral" of the measure ξ. For an arbitrary operator $\tilde{A} \in \tilde{L}(H)$ we may put

$$\tilde{\xi}(\tilde{A}) = \tilde{\xi}(A_r) + i\tilde{\xi}(A_i),$$

where $A_r, A_i \in L(H)$, and

$$\begin{aligned}(A_r e_k, e_1) &= \operatorname{Re}(\tilde{A}e_k, e_1),\\ (A_i e_k, e_1) &= \operatorname{Im}(\tilde{A}e_k, e_1)\end{aligned}$$

for a fixed basis $\{e_k\}$ in H. In this way $\tilde{\xi}$ can be treated as an "integral" of some OG-measures on $S_{\tilde{H}}$ and thus

$$\big(\xi(P), \xi(Q)\big) = \big(\tilde{\xi}(\tilde{P}), \tilde{\xi}(\tilde{Q})\big) = \operatorname{tr} \tilde{M}'\tilde{P}\tilde{Q} + \operatorname{tr} \tilde{M}''\tilde{Q}\tilde{P}.$$

Moreover, the operator $\tilde{M}$ with the matrix elements

$$(\tilde{M}e_k, e_l) = (\tilde{M}'e_k, e_l) + (\tilde{M}''e_1, e_l)$$

is self-adjoint, non-negative and trace-class and

$$\begin{aligned}\operatorname{tr} \tilde{M}\tilde{P}\tilde{Q} &= \sum_k (\tilde{M}PQe_k, e_k) = \sum_k (\tilde{M}'PQe_k, e_k) + \sum_k (\tilde{M}''QPe_k, e_k)\\ &= \operatorname{tr} \tilde{M}'\tilde{P}\tilde{Q} + \operatorname{tr} \tilde{M}''\tilde{Q}\tilde{P}\end{aligned}$$

for any $P, Q \in S_H$, which completes the proof.

§ 2

2.1. The well-known Wigner theorem [7] gives the general form of automorphism a of a standard quantum-mechanical Hilbert logic S_H. Namely every such automorphism is of the form

$$a(P) = UPU^{-1}, \quad P \in S_H, \tag{2.1}$$

where U is a unitary or antiunitary operator in H. Formula (2.1) describes some operator measure on S_H. This suggests the following definition.

DEFINITION. The mapping $\xi\colon S_H \to L(\mathscr{H})$ is called an *orthogonal operator measure* if for an arbitrary sequence of mutually orthogonal operators $P_1, P_2, \ldots$ from S_H, the operators $\xi(P_1), \xi(P_2), \ldots$ are mutually orthogonal, i.e.

$$\xi(P_i)^+ \xi(P_j) = 0 \quad \text{for} \quad i \neq j \ (i, j = 1, 2, \ldots)$$

and the series

$$\sum_i \xi(P_i) \tag{2.2}$$

converges in the weak operator topology to $\xi(\sum_i P_i)$. By Orlicz theorem the series (2.2) is then convergent in the strong operator topology.

The orthogonal operator measure taking values in $S_{\mathscr{H}}$ is called a *spectral measure.*

Theorem 1 enables the extension of an orthogonal measure to an "integral", i.e. for the orthogonal operator measure $\xi\colon S_H \to L(\mathscr{H})$ there exists a continuous linear operator $\bar{\xi}\colon L(H) \to L(\mathscr{H})$ equal to ξ on S_H.

Similarly to the well-known property of orthogonal operator measures on Boolean algebras, we have

PROPOSITION 1. *If $\xi\colon S_H \to L(\mathscr{H})$ is an orthogonal normed operator measure (i.e. $\xi(I_H) = I_{\mathscr{H}}$, where I_H and $I_{\mathscr{H}}$ are the unit operators in H and $\mathscr{H}$, respectively), then ξ is a spectral measure.*

Proof. For any operator $P \in S_H$ it is enough to consider the projections $\xi(P)$ and $\xi(I_H - P) = I_{\mathscr{H}} - \xi(P)$. Since

$$0 = \xi(I_H - P)^+ \xi(P) = \big(I_{\mathscr{H}} - \xi(P)\big)^+ \xi(P),$$

we have

$$\xi(P) = \xi(P)^+ \xi(P).$$

The operator on the right-hand side of this formula is self-adjoint, so $\xi(P) = \xi(P)^+$ and then $\xi(P) = \big(\xi(P)\big)^2$. Thus $\xi(P)$ is a projective operator in $\mathscr{H}$.

2.2. If a normed spectral measure ξ is a homomorphism of the lattice S_H into $S_{\mathscr{H}}$ (i.e. $\xi(\bigvee_{i=1}^{\infty} P_i) = \bigvee_{i=1}^{\infty} \xi(P_i)$ and $\xi(\bigwedge_{i=1}^{\infty} P_i) = \bigwedge_{i=1}^{\infty} \xi(P_i)$ for

any family $P_1, P_2, \ldots \in S_H$), then the dimension of the subspace $\xi([x])$, equal to $\operatorname{tr} \xi([x])$ (finite or not), is independent of x (see, for example, [3], Proposition 3). It will be proved that the same is fulfilled by any spectral measure $\xi\colon S_H \to S_{\mathscr{H}}$ for the real Hilbert space H. The spectral measure $\xi\colon S_H \to S_{\mathscr{H}}$ is, in some sense, characterized by its dimension defined as

$$\dim \xi = \dim \xi([x]), \quad x \in H,\ x \neq 0,$$

namely, we have

THEOREM 3. *For any normed spectral measure $\xi\colon S_H \to S_{\mathscr{H}}$, where H and $\mathscr{H}$ are real separable Hilbert spaces, and $\dim H \geqslant 3$, there exist unitary operators $U_i\colon H \to \mathscr{H}_i$, $i \in I$, where $I = \{1, \ldots, n\}$, (if $\dim \xi = n < \infty$) or $I = \{1, 2, \ldots\}$ (if $\dim \xi = \infty$), such that $\mathscr{H}$ is an orthogonal sum*

$$\mathscr{H} = \bigoplus_i \mathscr{H}_i \tag{2.3}$$

and

$$\xi(P) = \bigoplus_i U_i P U_i^{-1} \tag{2.4}$$

for any operator $P \in S$.

If the spectral measure $\xi\colon S_H \to S_{\mathscr{H}}$ is an isomorphism, then $\dim \xi = 1$ and we obtain the theorem of Wigner for a real Hilbert space H.

Proof of Theorem 3. For any vector $x \in \mathscr{H}$ the function

$$\eta_x(P) = \xi(P)x, \quad P \in S_H$$

is an OG-measure and, by Theorem 2,

$$\big(\eta_x(P), \eta_x(Q)\big) = \operatorname{tr} M_x PQ.$$

(as H is real), where M_x is some s-operator in H. When $x \in \xi([e])$ (5) for some vector $e \in H$, $\|e\| = 1$, then

$$\operatorname{tr} M_x[e_0] = \big(\xi([e_0])x, \xi([e_0])x\big) = \|x\|^2$$

and

$$\operatorname{tr} M_x = \big(\xi(I_H)x\big), \big(\xi(I_H)x\big) = \|x\|^2.$$

Thus (as M_x is self-adjoint and non-negative) $M_x = \|x\|^2\ [e]$ and for vectors $a, b \in H$, $\|a\| = \|b\| = 1$, we obtain

$$\begin{aligned} \big(\xi([a])x, \xi([b])x\big) &= \operatorname{tr} \|x\|^2 [e][a][b] \\ &= \|x\|^2 ([a]e, [b]e) \quad \text{if} \quad x \in \xi([e]). \end{aligned} \tag{2.5}$$

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(5) We shall often identify the projective operator P with the subspace on which it projects.

For each vector $x \in \xi([e])$, $e \in H$ ($\|e\| = 1$), we can define a function

$$U_e(x, a) = \frac{\|a\|^2}{(a, e)} \xi([a])x, \quad a \in H, \ (a, e) \neq 0. \tag{2.6}$$

Then we have

$$U_e(x+y, a) = U_e(x, a) + U_e(y, a), \tag{2.7}$$

$$x = U_e(x, e) \tag{2.8}$$

and by (2.5)

$$\big(U_e(x, a), U_e(x, b)\big) = \frac{\|a\|^2}{(a, e)} \frac{\|b\|^2}{(b, e)} \|x\|^2([a]e, [b]e) = \|x\|^2 (a, b) \tag{2.9}$$

for any vectors $x, y \in \xi([e])$, $a, b \in H$, $(a, e) \neq 0 \neq (b, e)$.

For a fixed x a function $U_e(x, \cdot)$ can be extended to the linear operator on the whole spaces H, and properties (2.7) and (2.9) will be still preserved. The operators $U_e(f, \cdot)$, where $|f| = 1$, $f \in \xi([e])$ for some $e \in H$, $\|e\| = 1$, have the following properties:

(a) $\big(U_e(f, a), U_e(f, b)\big) = (a, b)$, $a, b \in H$;

(b) $U_e(f, a) \in \xi([a])$, $a \in H$ $(a \neq 0)$;

(c) $f \perp f', f, f' \in \xi([e])$ *imply* $U_e(f, a) \perp U_e(f', b)$, $a, b \in H$;

(d) *if* $f' = U_e(f, e')$, $e' \in H$, $\|e'\| = 1$, *then* $U_{e'}(f', \cdot) = U_e(f, \cdot)$.

Property (a) follows immediately from (2.9).

Property (b), for $(a, e) \neq 0$, immediately follows from (2.6). If $(a, e) = 0$, $a \neq 0$, we put

$$a_n = a + \frac{1}{n} e \quad \text{and} \quad e_n = \frac{a_n}{\|a_n\|} \quad (n = 1, 2, \ldots).$$

Then

$$f_n = U_e(f, a_n) \in \xi([e_n]) \quad \text{and} \quad U_{e_n}\Big(f^n, \frac{a}{\|a\|}\Big) \in \xi([a]).$$

By (2.8), (2.9), we have

$$\Big\| U_e(f, a) - U_{e_n}\Big(f_n, \frac{a}{\|a\|}\Big)\Big\| \leqslant \|U_e(f, a) - f_n\| + \Big\| f_n - U_{e_n}\Big(f_n, \frac{a}{\|a\|}\Big)\Big\|$$

$$= \|U_e(f, (x - a_n))\| + \Big\| U_{e_n}\Big(f_n, \Big(e_n - \frac{a}{\|a\|}\Big)\Big)\Big\| \to 0 \quad (\text{as } n \to \infty)$$

and

$$U_e(f, a) = \lim_{n\to\infty} U_{e_n}(f_n, a) \in \xi([a]).$$

Now we shall prove (c). If $f \perp f'$, $f, f' \in \xi([e])$, $\|f\| = \|f'\| = 1$, then, by (2.7), (2.9),

$$\begin{aligned}\big(U_e(f, a), U_e(f', a)\big) &= \tfrac{1}{2}\Big[\big(U_e(f+f', a), U_e(f+f', a)\big) - \\ &\quad - \big(U_e(f, a), U_e(f, a)\big) - \big(U_e(f', a)\, U_e(f', a)\big)\Big] \\ &= \tfrac{1}{2}\|a\|^2(\|f+f'\|^2 - \|f\|^2 - \|f'\|^2) = 0\end{aligned}$$

for any $a \in H$. Let now $(e_i)_{i\in J}$ be an orthonormal basis in H. Then

$$U_e(f, e_i) \perp U_e(f', e_i)$$

and, by (b)

$$U_e(f, e_i) \perp U_e(f', e_j)$$

for $i \neq j$ $(i, j = 1, 2, \ldots)$. Therefore

$$U_e(f, a) \perp U_e(f', b)$$

for any vectors $a, b \in H$ (as the operators $U_e(f, \cdot)$ and $U_e(f^0, \cdot)$ are linear).

To prove (d), let us put (compare (2.8) and (b))

$$x = f = U_e(f, e),$$
$$y = f' = E_e(f, e') = U_{e'}(f', e')$$
$$P = \xi([a])$$

for some vectors $e, e', a \in H$, $f \in \xi([e])$, $\|e\| = \|e'\| = \|f\| = 1$ and $(a, e) \neq 0 \neq (a, e')$. Therefore, by (2.6),

$$Px = \frac{(a, e)}{\|a\|^2} U_e(f, a), \qquad Py = \frac{(a, e')}{\|a\|^2} U_{e'}(f', a)$$

and, by (2.9), the following can be verified (by (a), $\|f'\| = 1$)

$$\|Px\| = \frac{|(a, e)|}{\|a\|}, \qquad \|Py\| = \frac{|(a, e')|}{\|a\|},$$

$$(Px, y) = \frac{(a, e)(a, e')}{\|a\|^2}.$$

Thus $(Px, Py) = (Px, y) = \operatorname{sign}\big((a, e)(a, e')\big)\|Px\|\|Py\|$, which implies

$$Px = \frac{(a, e)}{(a, e')} Py$$

and $U_e(f, a) = U_{e'}(f', a)$. The last condition holds also if the vector a is orthogonal to e or to e' and (d) is proved.

To prove Theorem 3 it is enough to put $U_i(\cdot) = U_e(f_i, \cdot)$ $(i \in I)$ where $(f_i)_{i\in I}$ is an orthonormal basis in $\xi([e])$ for some vector $e \in H$, $\|e\| = 1$. By (a), U_i is a unitary operator from H into $\mathscr{H}$. Moreover, for any vector $e' \in H$, $\|e'\| = 1$, a sequence $(f_1' = U_i e')_{i\in I}$ is an orthonormal basis in $\xi([e'])$. Indeed, by (b), (c), $(f_i')_{i\in I}$ is an orthonormal sequence in $\xi([e'])$. If we assume existence of a vector $f' \in \xi([e'])$, $f' \perp f_i'$ $(i \in I)$, then, by (c) and (d), $U_{e'}(f', e) \perp U_{e'}(f_i', e) = U_e(f_i, e) = f_i$ for any $i \in I$. Thus $U_{e'}(f', e) = 0$ and $f' = 0$, so the whole space $\xi([e'])$ is spanned by $(f_i')_{i\in I}$. In conclusion, if $(e_j)_{j\in J}$ is an orthonormal basis in H, then

$$\mathscr{H} = \bigoplus_{j\in J} \xi([e_j]) = \bigoplus_{i\in I} \bigoplus_{j\in J} [U_i e_j] = \bigoplus_{i\in I} \mathscr{H}_i,$$

and (2.3) is satisfied. For any $e' \in H$, $\|e'\| = 1$, the operator $\xi([e']) = \sum_{i\in I} [U_i e']$ reduced to $\mathscr{H}_i = \bigoplus_{j\in J} [U_i e_j]$ is equal to $U_i[e'] U_i^{-1}$, so $\xi([e'])$ can be written as an orthogonal sum of operators:

$$\xi([e']) = \bigoplus_{i\in I} U_i [e'] U_i^{-1}.$$

For any operator $P \in S_H$ we have $P = [a_1] + [a_2] + \ldots$ So (2.4) holds and Theorem 3 is proved.

Remark 2. If Hilbert spaces H and $\mathscr{H}$ are real, then the extension of a spectral measure $\xi\colon S_H \to S_{\mathscr{H}}$ to the linear operator $\tilde{\xi}\colon L(H) \to L(\mathscr{H})$ has the form

$$\tilde{\xi}(A) = \bigoplus_{i\in I} U_i A U_i^{-1}, \quad A \in L(H),$$

where the operators $U_i\colon H \to \mathscr{H}_i$ are unitary and

$$\mathscr{H} = \bigoplus_{i\in I} \mathscr{H}_i.$$

Remark 3. Theorem 3 and Remark 2 are true and the proof need not any differences when the space H is real and $\mathscr{H}$ is complex. $U_i A U_i^{-1}$ is now the unique extension of the operator $U_i A U_i^{-1}\colon U_i(H) \to U_i(H)$ on the complex subspace $\mathscr{H}_i = [U_i(H)]$ (for any $A \in L(H)$).

The set $U_i(H) \subset \mathscr{H}$ is of course a real Hilbert space.

Let Λ_H (resp. $\Lambda_{\mathscr{H}}$) be the space of Hilbert–Schmidt operators in H (resp. $\mathscr{H}$) with the inner product $(A, B) = \operatorname{tr} AB^+$ for any operators A, B from Λ_H (resp. $\Lambda_{\mathscr{H}}$). If the space H is real, then:

Remark 4. *For a finitely-dimensional spectral measure $\xi\colon S_H \to S_{\mathscr{H}}$ the operator*

$$\eta = (\dim \xi)^{-1/2} \tilde{\xi}$$

is an isometry from Λ_H into $\Lambda_{\mathscr{H}}$.

The same is true when H and $\mathscr{H}$ are complex.

Proof. Observe that for every non-negative s-operator A we have

$$\operatorname{tr} \tilde{\xi}(A) = \dim \xi \operatorname{tr}(A), \qquad \tilde{\xi}(A^2) = (\tilde{\xi}(A))^2.$$

Using the above formulas to the operator $A = \hat{x} + \hat{y}$ we obtain after easy transformations

$$\operatorname{tr} \tilde{\xi}(\hat{x}) \tilde{\xi}(\hat{y}) = \dim \xi \operatorname{tr} \hat{x}\hat{y}. \tag{2.11}$$

Since one-dimensional projectors generate the space Λ_H, formula (2.11) establishes the required isometry.

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