STUDIA MATHEMATICA, T. LXIII. (1978)

Restricted and unrestricted convergence of approximate identities in product spaces

by

CHONG RAE KIM (Ossining, N. Y.)

Abstract. For multiple Fourier series, if $f \in L^p$, $p > 1$, then the $(C, 1)$ sum of the Fourier series of f converges almost everywhere to f, (with the aid of unrestricted sums), and for $f \in L^1$ we have restricted almost everywhere convergence.

Our purpose is to extend these facts to general approximate identities. A measure of generality is whether the result includes the Kantorovich polynomials. In the unrestricted case we obtain a result which does include the Kantorovich polynomials and in the restricted case we obtain a result which does not include them.

1. Terminology and notation. 1.1. A *kernel* is a sequence $\{\Phi_n(x, t)\}$ of functions defined in the square $\{(x, t)\colon a < x < b,\ a \leqslant t \leqslant b\}$ and such that, for all x,

$$\lim_{n\to\infty} \int_a^b \Phi_n(x, t)\, dt = 1.$$

1.2. A kernel $\{\Phi_n(x, t)\}$ is called an *approximate identity* if

$$\lim_{n\to\infty} \left\{ \int_a^\alpha |\Phi_n(x, t)|\, dt + \int_\beta^b |\Phi_n(x, t)|\, dt \right\} = 0$$

for $a \leqslant \alpha < x < \beta \leqslant b$.

1.3. For a set of approximate identities, if

$$\int f(t_1, \ldots, t_k) \prod_i \Phi_{n_i}(x_i, t_i)\, dt_1 \ldots dt_k$$

converges to $f(x_1, \ldots, x_k)$ as $n_1, \ldots, n_k$ tend to $+\infty$ independently (or dependently) of one another, we say that it *converges unrestrictedly* (or *restrictedly*).

1.4. A function $\Psi(x, t)$ is called a *majorant* (*having the monotonicity property*) of the function $\Phi(x, t)$ if

$$|\Phi(x, t)| \leqslant \Psi(x, t)$$

and if, for fixed x, $\Psi(x, t)$ increases on the closed interval $[a, x]$ and decreases on the closed interval $[x, b]$.

2. Unrestricted case.

THEOREM. *Let* $[\{\Phi_n^i(x,t)\}_{n=1}^{\infty}: i = 1, 2, \ldots, k]$ *be a finite set of approximate identities each of which is defined in the square* $(a_i \leqslant t \leqslant b_i,\ a_i < x < b_i)$. *Suppose that for each i there exists a sequence* $\{\Psi_n^i(x,t)\}$ *such that*

(i) *for each* n, $\Psi_n^i(x,t)$ *is a majorant having the monotonicity property of* $\Phi_n^i(x,t)$, *and*

(ii) $\int_{a_i}^{b_i} \Psi_n^i(x,t)\,dt \leqslant C_i < \infty$, *where* C_i *is independent of* n *and* x. *Let* $f(t_1, \ldots, t_k)$ *be a function of* $k \geqslant 2$ *variables defined on the closed* k-*cell* $Q = \{(t_1, \ldots, t_k): a_i \leqslant t_i \leqslant b_i,\ i = 1, \ldots, k\}$. *If* $|f|(\log^+|f|)^{k-1}$ *is integrable on* Q, *then*

$$\int_Q f(t_1, \ldots, t_k) \prod_{i=1}^{k} \Phi_{n_i}^i(x_i, t_i)\,dt_1 \ldots dt_k$$

converges almost everywhere to $f(x_1, \ldots, x_k)$ *as* $n_1, \ldots, n_k$ *tend to* $+\infty$ *independently of one another.*

Examples are the product kernels of Fejér, Kantorovich, Landau, and de la Vallée Poussin, as well as the product Poisson kernel.

The proof of the theorem needs the following lemmas.

LEMMA 1. *Let* $[\{\Phi_n^i(x,t)\}_{n=1}^{\infty}: i = 1, 2, \ldots, k]$ *be a set of approximate identities of the Theorem. If* $f \in C(Q)$, *then*

$$\int_Q f(t_1, \ldots, t_k) \prod_{i=1}^{k} \Phi_{n_i}^i(x_i, t_i)\,dt_1 \ldots dt_k$$

converges to $f(x_1, \ldots, x_k)$ *at all points* $(x_1, \ldots, x_k)$ *inside of* Q *as* $n_1, \ldots, n_k$ *tend to* $+\infty$ *independently of one another.*

LEMMA 2. *Let* $\{\Psi_n(x,t)\}$ *be a sequence of positive functions such that for fixed* n *and* x, *the function* $\Psi_n(x,t)$, *as a function of* t *only, increases on the closed interval* $[a, x]$ *and decreases on the closed interval* $[x, b]$ *and*

$$\int_a^b \Psi_n(x,t)\,dt \leqslant C, \quad \text{where } C \text{ is a constant.}$$

Let $f(t)\log^+|f(t)|$ *be integrable over* $[a, b]$ *and*

$$\Psi^*(x) = \sup_n \int_a^b |f(t)|\Psi_n(x,t)\,dt, \tag{1}$$

$$f^*(x) = \sup_{h \neq 0} \left\{ \frac{1}{h} \int_x^{x+h} |f(u)|\,du \right\}. \tag{2}$$

Then

$$\Psi^*(x) \leqslant 3Cf^*(x).$$

LEMMA 3. *If* $f(x)$, $a \leqslant x \leqslant b$, *is integrable and*

$$f^*(x) = \sup_{h \neq 0} \left\{ \frac{1}{h} \int_x^{x+h} |f|\,dt \right\},$$

then

$$\|f^*\|_p \leqslant A_p \|f\|_1 \quad (0 < p < 1) \tag{3}$$

and

$$\int_a^b f^*(\log^+ f^*)^{a-1}\,dt \leqslant A_a \int_a^b |f|(\log^+|f|)^a\,dt + A_a \quad (a \geqslant 1), \tag{4}$$

where A_p *and* A_a *are constants depending only on* p *and* a, *respectively.*

We omit the proofs of Lemma 1 and Lemma 3 since Lemma 1 can be obtained easily, and Lemma 3 is a result of Zygmund [3].

Proof of Lemma 2. Put $\varphi(t) = \int_a^t |f(u)|\,du$. For fixed x, $a < x < b$,

$$\begin{aligned}
\int_a^b |f(t)|\Psi_n(x,t)\,dt &= \int_a^b \Psi_n(x,t)\,d\varphi(t) \\
&= [\Psi_n(x,t)\varphi(t)]_a^b - \int_a^b \varphi(t)\,d(\Psi_n(x,t)) \\
&= [\Psi_n(x,t)\varphi(t)]_a^b - \varphi(x)\int_a^b d(\Psi_n(x,t)) + \\
&\quad + \int [\varphi(x) - \varphi(t)]\,d(\Psi_n(x,t)).
\end{aligned}$$

Let

$$H_1 = [\Psi_n(x,t)\varphi(t)]_a^b - \varphi(x)\int_a^b d(\Psi_n(x,t))$$

and

$$H_2 = \int_a^b [\varphi(x) - \varphi(t)]\,d(\Psi_n(x,t)).$$

We have

$$H_1 = [\varphi(b) - \varphi(x)]\Psi_n(x,b) + \varphi(x)\Psi_n(x,a).$$

Since $\Psi_n(x,t)$ is increasing on $[a, x]$ and decreasing on $[x, b]$,

$$c \geqslant \int_x^b \Psi_n(x,t)\,dt \geqslant (b-x)\Psi_n(x,b), \quad \text{or} \quad \Psi_n(x,b) \leqslant \frac{c}{b-x}$$

and

$$c \geqslant \int_a^x \Psi_n(x,t)\,dt \geqslant (x-a)\Psi_n(x,a), \quad \text{or} \quad \Psi_n(x,a) \leqslant \frac{c}{x-a}.$$

Thus

$$H_1 \leqslant c\,\frac{\varphi(b)-\varphi(x)}{b-x} + c\,\frac{\varphi(x)}{x-a} \leqslant 2cf^*(x),$$

$$H_2 = \int_a^b [\varphi(x)-\varphi(t)]\,d\big(\psi_n(x,t)\big) = \int_a^b \frac{\varphi(x)-\varphi(t)}{x-t}(x-t)\,d\big(\Psi_n(x,t)\big)$$

$$\leqslant \sup_{t\neq x}\left|\frac{\varphi(x)-\varphi(t)}{x-t}\right| \int_a^b \big|(x-t)\,d\big(\Psi_n(x,t)\big)\big|$$

$$\leqslant f^*(x)\int_a^b \big|(x-t)\,d\big(\Psi_n(x,t)\big)\big|.$$

We know that

$$\int_a^b \big|(x-t)\,d\big(\Psi_n(x,t)\big)\big| = \int_a^x (x-t)\,d\big(\Psi_n(x,t)\big) + \int_x^b (t-x)\,d\big(-\Psi_n(x,t)\big)$$

$$= [(x-t)\Psi_n(x,t)]_a^x - \int_a^x \Psi_n(x,t)\,d(x-t) +$$

$$+\big[(t-x)\big(-\Psi_n(x,t)\big)\big]_x^b - \int_x^b \big(-\Psi_n(x,t)\big)\,d(t-x)$$

$$= -(x-a)\Psi_n(x,a) - (b-x)\Psi_n(x,b) +$$

$$+\int_a^b \Psi_n(x,t)\,dt \leqslant c$$

since $(x-a)\Psi_n(x,a)\geqslant 0$, $(b-x)\Psi_n(x,t)\geqslant 0$ and $\int_a^b \Psi_n(x,t)\,dt \leqslant c$. Therefore, $H_2 \leqslant cf^*(x)$. This shows that $\Psi^*(x) = H_1+H_2 \leqslant 3cf^*(x)$. This completes the proof of Lemma 2.

Now we are ready to prove the theorem by a method similar to the method of Zygmund.

Proof of Theorem. Let

$$\Phi^*(x_1,\ldots,x_k;f) = \sup_{(n_1,\ldots,n_k)} \int_Q f(t_1,\ldots,t_k)\prod_{i=1}^k \Phi^i_{n_i}(x_i,t_i)\,dt_1\ldots dt_k.$$

We wish to find a measurable function f_k such that

$$\Phi^*(x_1,\ldots,x_k;f) \leqslant c_0 f_k(x_1,\ldots,x_k)$$

and for every p, $0<p<1$,

$$\|f_k\|_p \leqslant c\,\|\,|f|(\log^+|f|)^{k-1}\|_1 + c,$$

i.e.,

$$\Big[\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k} f_k^p(x_1,\ldots,x_k)\,dx_1\ldots dx_k\Big]^{1/p}$$

$$< c\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k} |f|(\log^+|f|)^{k-1}dx_1\ldots dx_k + c,$$

where c_0 and c are constants.

Let us fix $x_2,\ldots,x_k$, and let $f_1(x_1,\ldots,x_k)$ be obtained from $f(x_1,\ldots,x_k)$ in the same way as (2), that is,

$$f_1(x_1,\ldots,x_k) = \sup_{h\neq 0}\Big\{\frac{1}{h}\int_{x_1}^{x_1+h} |f(u,x_2,\ldots,x_k)|\,du\Big\}.$$

Similarly we obtain $f_2(x_1,\ldots,x_k)$ from $f_1(x_1,\ldots,x_k)$, this time fixing $x_1,x_3,\ldots,x_k$, and so on. We get from (4) of Lemma 3

$$\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k} f_{k-1}(x_1,\ldots,x_k)\,dx_1\ldots dx_k$$

$$\leqslant A\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k} f_{k-2}\log^+ f_{k-2}\,dx_1\ldots dx_k + A_1 \leqslant \ldots$$

$$\ldots \leqslant A_{k-1}\int_{a_k}^{b_k}\ldots\int_{a_k}^{b_k} |f|(\log^+|f|)^{k-1}dx_1\ldots dx_k + A_{k-1}.$$

Let us now observe that, using Lemma 2, we have

$$\Big|\int_{a_k}^{b_k}\ldots\int_{a_1}^{b_1} f(t_1,\ldots,t_k)\prod_{i=1}^k \Phi^i_{n_i}(x_i,t_i)\,dt_1\ldots dt_k\Big|$$

$$\leqslant \int_{a_k}^{b_k}\ldots\int_{a_1}^{b_1} |f(t_1,\ldots,t_k)|\prod_{i=1}^k \Psi^i_{n_i}(x_i,t_i)\,dt_1\ldots dt_k$$

$$\leqslant \beta_1\int_{a_k}^{b_k}\ldots\int_{a_2}^{b_2} f_1(x_1,t_2,\ldots,t_k)\prod_{i=2}^k \Psi^i_{n_i}(x_i,t_i)\,dt_2\ldots dt_k$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leqslant \beta_k\int_{a_k}^{b_k} f_{k-1}(x_1,\ldots,x_{k-1},t_k)\Psi^k_{n_k}(x_k,t_k)\,dt_k$$

$$\leqslant \beta_k f_k(x_1,\ldots,x_k).$$

The above inequality shows that

$$\Phi^*(x_1,\ldots,x_k;f) \leqslant c_0 f_k(x_1,\ldots,x_k), \quad\text{where}\quad c_0 = \beta_k. \tag{6}$$

Now we want to show that $\|f_k\|_p \leqslant c\int |f|(\log^+|f|)^{k-1}+c$. Using Hölder's inequality, we get

$$\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k}[f_k(x_1,\ldots,x_k)]^p\,dx_1\ldots dx_k$$
$$=\int_{a_1}^{b_1}\ldots\int_{a_{k-1}}^{b_{k-1}}dx_1\ldots dx_{k-1}\Big[\int_{a_k}^{b_k}(f_k)^p\,dx_k\Big]$$
$$\leqslant\Big\{\int_{a_1}^{b_1}\ldots\int_{a_{k-1}}^{b_{k-1}}dx_1\ldots dx_{k-1}\Big[\int_{a_k}^{b_k}(f_k)^p\,dx_k\Big]^{1/p}\Big\}^p\Big(\int_{a_1}^{b_1}\ldots\int_{a_{k-1}}^{b_{k-1}}dx_1\ldots dx_{k-1}\Big)^{1-p}$$
$$=c\Big\{\int_{a_1}^{b_1}\ldots\int_{a_{k-1}}^{b_{k-1}}dx_1\ldots dx_{k-1}\Big[\int_{a_k}^{b_k}(f_k)^p\,dx_k\Big]^{1/p}\Big\}^p.$$

By inequality (3) of Lemma 3,

$$\Big\{\int_{a_k}^{b_k}(f_k)^p\,dx_k\Big\}^{1/p}\leqslant A_p\int_{a_k}^{b_k}f_{k-1}\,dx_k.$$

Thus

$$\Big\{\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k}(f_k)^p\,dx_1\ldots dx_k\Big\}^{1/p}\leqslant A_p\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k}f_{k-1}\,dx_1\ldots dx_k.$$

Combining this and inequality (5), we get

$$\|f_k\|_p\leqslant c\,\||f|(\log^+|f|)^{k-1}\|_1+c.$$

Having obtained our desired function f_k, we are now ready to prove that

$$\int_Q f(t_1,\ldots,t_k)\prod_{i=1}^{k}\Phi_{n_i}^i(x_i,t_i)\,dt_1\ldots dt_k\to f(x_1,\ldots,x_k)$$

almost everywhere.

First of all, we fix p such that $0<p<\frac{1}{2}$ and apply the above inequality to the function Mf, where M is a positive constant so large that in the resulting inequality

(8)
$$\Big\{\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k}(f_k)^p\,dx_1\ldots dx_k\Big\}^{1/p}\leqslant c\int_{a_1}^{b_1}\ldots\int_{a_k}^{b_k}|f|(\log^+|Mf|)^{k-1}\,dx_1\ldots dx_k+c/M$$

the last term c/M on the right is $<(\varepsilon/2)^{1/p}$. Then we make a decomposition $f=g+h$, where g is a continuous function and

(9)
$$\int_Q |h|\,dx_1\ldots dx_k<\varepsilon,$$

(10)
$$c\int_Q |h|(\log^+|Mh|)^{k-1}<(\varepsilon/2)^{1/p}.$$

Applying inequality (8) to the function h, we attain

$$\Big\{\int_Q (h_k)^p\,dx_1\ldots dx_k\Big\}^{1/p}<(\varepsilon/2)^{1/p}+(\varepsilon/2)^{1/p}<\varepsilon^{1/p}.$$

This together with (9) shows that the set $E(\varepsilon)$ of points $(x_1,\ldots,x_k)$, where either $|h(x_1,\ldots,x_k)|>\sqrt{\varepsilon}$ or $\{h_k(x_1,\ldots,x_k)\}^p>\sqrt{\varepsilon}$, is of measure $<2\sqrt{\varepsilon}$. Since

$$\int_Q f(t_1,\ldots,t_k)\prod_{i=1}^{k}\Phi_{n_i}^i(x_i,t_i)\,dt_1\ldots dt_k-f(x_1,\ldots,x_k)$$
$$=\int_Q g(t_1,\ldots,t_k)\prod_{i=1}^{k}\Phi_{n_i}^i(x_i,t_i)\,dt_1\ldots dt_k-g(x_1,\ldots,x_k)+$$
$$+\int_Q h(t_1,\ldots,t_k)\prod_{i=1}^{k}\Phi_{n_i}^i(x_i,t_i)\,dt_1\ldots dt_k-h(x_1\ldots,x_k)$$

and, by Lemma 1,

$$\int_Q g(t_1,\ldots,t_k)\prod_{i=1}^{k}\Phi_{n_i}^i(x_i,t_i)\,dt_1\ldots dt_k-g(x_1,\ldots,x_k)\to 0$$

as $n_1,\ldots,n_k\to\infty$, we see that if $(x_1,\ldots,x_k)\notin E(\varepsilon)$ and $\varepsilon<1$, then

$$\limsup_{n_1,\ldots,n_p}\Big|\int_Q f(t_1,\ldots,t_k)\prod_{i=1}^{k}\Phi_{n_i}^i(x_i,t_i)\,dt_1\ldots dt_k-f(x_1,\ldots,x_k)\Big|$$
$$\leqslant\Phi^*(x_1,\ldots,x_k;h)+|h(x_1,\ldots,x_k)|$$
$$\leqslant c_1h_k(x_1,\ldots,x_k)+|h(x_1,\ldots,x_k)|\leqslant(1+c_0)\sqrt{\varepsilon}.$$

Since the number ε may be as small as we please, and the measure of $E(\varepsilon)$ tends to 0 with ε, the theorem follows.

3. Examples. 3.1. *Kantorovich kernel.* Let

$$K_n(x,t)=(n+1)\binom{n}{\nu}x^\nu(1-x)^{n-\nu},\qquad \frac{\nu}{n+1}<t\leqslant\frac{\nu+1}{n+1},$$
$$\nu=0,1,\ldots,n,\ 0<x<1.$$

$K_n(x,t)$ has the monotonicity property and

$$\int_0^1 K_n(x,t)\,dt=(n+1)\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}dt=\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}=1$$

for all n and x. Hence the Kantorovich kernel satisfies the conditions of the Theorem.

3.2. *Landau kernel*. Let

$$L_n(x,t) = \sqrt{\frac{n}{\pi}}\,[1-(t-x)^2]^n, \qquad 0 \leqslant t \leqslant 1,\ 0 < x < 1.$$

Then $L_n(x,t)$ has the monotonicity property,

$$\lim_{n\to\infty} \sqrt{\frac{n}{\pi}} \int_0^1 [1-(t-x)^2]^n\,dt = 1 \quad \text{for all } x,$$

and it converges uniformly. Hence, the Landau kernel satisfies the condition of the Theorem.

3.3. *Fejér kernel*. Let

$$F_n(x,t) = \frac{1}{2n\pi}\left[\frac{\sin n\dfrac{(t-x)}{2}}{\sin\dfrac{(t-x)}{2}}\right]^2, \qquad -\pi \leqslant t \leqslant \pi,\quad -\pi < x < \pi.$$

Then

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi}\left[\frac{\sin n\dfrac{t-x}{2}}{\sin\dfrac{t-x}{2}}\right]^2 dt = 1 \quad \text{for all } n \text{ and } x,$$

and the sequence $\{n\pi/(n^2(t-x)^2+4)\}$ is a majorant of the Fejér kernel having the monotonicity property. But

$$\int_{-\pi}^{\pi} \frac{n\pi\,dt}{n^2(t-x)^2+4} < \frac{\pi^2}{2},$$

and hence the Fejér kernel satisfies the conditions of the Theorem.

3.4. *De la Vallée Poussin kernel*. Let

$$V_n(x,t) = \frac{\sqrt{n}}{2\sqrt{\pi}}\cos^{2n}\left(\frac{t-x}{2}\right), \qquad -\pi < x < \pi, \quad -\pi \leqslant t \leqslant \pi.$$

Then

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\frac{\sqrt{n}}{2\sqrt{\pi}}\cos^{2n}\left(\frac{t-x}{2}\right)dt = 1 \quad \text{for all } x,$$

and it converges uniformly. Since $V_n(x,t)$ has the monotonicity property, the de la Vallée Poussin kernel satisfies the conditions of the Theorem.

4. Restricted convergence. Without detail we only state a result in the restricted case. This result includes many kernels such as Landau, Gauss–Weierstrass, Cauchy–Poisson, and de la Vallée Poussin, but does not include Kantorovich polynomials.

4.1. A sequence of functions $\{\varkappa_n(x)\}$ will be called a *kernel* (on the real line) if $\varkappa_n \in L^1$ for each n and

$$\int_{-\infty}^{\infty} \varkappa_n(u)\,du = 1.$$

4.2. A kernel $\{\varkappa_n(x)\}$ is called an *approximate identity* (on the real line) if there is some constant $M > 0$ with

$$\int_{-\infty}^{\infty} |\varkappa_n(u)|\,du \leqslant M, \qquad n = 1, 2, \ldots,$$

$$\lim_{n\to\infty}\int_{\delta\leqslant|u|} |\varkappa_n(u)|\,du = 0 \qquad (\delta > 0).$$

4.3. NL' is the set of those $f \in L'(R)$ which are normalized by $\int_{-\infty}^{\infty} f(u)\,du = 1$.

Result. *Let the kernels $\{\Phi_n(x)\}$ and $\{\Psi_n(y)\}$ be approximate identities. If there exist non-decreasing sequences $\{a_n\}$ and $\{b_n\}$ such that*

$$|\Phi_n(x)| \leqslant a_n, \qquad |\Phi_n(x)| \leqslant \frac{c}{(a_n)^a|x|^{1+a}} \qquad (a > c)$$

and

$$|\Psi_n(y)| \leqslant b_m, \qquad |\Psi_m(y)| \leqslant \frac{c}{(b_m)^\beta|y|^{1+\beta}} \qquad (\beta > 0),$$

then for every $f \in L^1(R^2)$ and for any pair $\{n_k\}$, $\{m_k\}$ of non-decreasing sequences,

$$\lim_{k\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+u, y+v)\,\Phi_{n_k}(u)\,\Psi_{m_k}(v)\,du\,dv = f(x,y)$$

almost everywhere on R^2.

We state a corollary which follows from the above Result.

Corollary. *Let $\Phi \in NL'$ and $\Psi \in NL'$ be bounded functions such that*

$$|\Phi(x)| \leqslant \frac{c_0}{|x|^{1+a}} \quad \text{and} \quad |\Psi(y)| \leqslant \frac{c_0}{|y|^{1+\beta}}$$

for positive numbers a and β. Then, for every $f \in L'(r^2)$ and for any pair $\{n_k\}$, $\{m_k\}$ of non-decreasing sequences,

$$\lim_{k\to\infty} n_k m_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+u, y+v)\,\Phi(n_k u)\,\Psi(m_k v)\,du\,dv = f(x, y)$$

almost everywhere on R^2.

References

[1] J. Marcinkiewicz and A. Zygmund, *On the summability of double Fourier series*, Fund. Math. 32 (1939), pp. 122–132.

[2] A. Zygmund, *On the summability of multiple Fourier series*, Amer. J. Math. 69 (1947), pp. 836–850.

[3] — *Trigonometric series, II*, Cambridge University Press, N. Y., 1959.

PURDUE UNIVERSITY

Received June 15, 1976
in revised form October 14, 1976 (1169)

STUDIA MATHEMATICA, T. LXIII. (1978)

Vector measures on the closed subspaces of a Hilbert space

by

R. JAJTE and A. PASZKIEWICZ (Łódź)

Abstract. The present paper is concerned with vector valued measures defined on the lattice of all orthogonal projectors in a separable Hilbert space H, with values in a Banach space X. Those measures can be extended to bounded linear operators on the space $L(H)$ of all linear operators in H. In particular, we consider the measures taking their values in a Hilbert space $\mathscr{H}$ and in $L(\mathscr{H})$. As a corollary we obtain a description of homomorphisms of a standard Hilbert logic into itself. This is the generalization of the well-known theorem of Wigner.

Introduction. Let H (or $\mathscr{H}$) denote a Hilbert space (real or complex). Throughout we always assume $\dim H \geq 3$. Let S_H (resp. $S_{\mathscr{H}}$) be the lattice of all orthogonal projectors in H (resp. $\mathscr{H}$) and let $L(H)$ be the space of all bounded linear operators acting in H.

An operator $M \in L(H)$, which is self-adjoint, nonnegative and trace-class will be called the s-*operator*.

For any subspace $H' \subset H$ we shall denote by $S_{H'}$ the lattice of all projective operators acting in H'.

$S_{H'}$ will also be treated as a set of operators from S_H which vanish on $H \ominus H'$.

Let X be a Banach space (real or complex).

Definition 0. The mapping $\xi\colon S_H \to X$ will be called the *vector Gleason measure* (VG-measure) if

(i) for any sequence of mutually orthogonal projectors $P_1, P_2, \ldots$ from S_H the series

$$\sum_i \xi(P_i) \tag{0.1}$$

is weakly convergent to $\xi(\sum_i P_i)$;

(ii) $$\sup_{P \in S_H} \|\xi(P)\| = K < \infty.$$

By the well-known theorem of Orlicz [4], the accepted definition immediately implies unconditional and strong convergence of (0.1).