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A Holsztyński theorem for spaces of continuous vector-valued functions

by

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Abstract. W. Holsztyński has shown that if X and Y are compact Hausdorff spaces, and if A is any isometry of $C(X)$ into $C(Y)$, then there exists a closed subset $B(A) \subseteq Y$, a continuous function τ mapping $B(A)$ onto X, and an element $a \in C(Y)$ with $\|a\|_\infty = 1$, and $|a(y)| = 1$ for $y \in B(A)$, such that $(A(f))(y) = a(y)f(\tau(y))$ for all $y \in B(A)$ and $f \in C(X)$.

Here we obtain a formulation of this theorem for spaces of vector-valued functions. It is shown that if E, E_1 are normed linear spaces with E_1 strictly convex, and A is an isometry of $C(X, E)$ into $C(Y, E_1)$, then there exists a subset $B(A) \subseteq Y$, a continuous function $y \to \mathscr{A}_y$ from Y into the space of bounded operators on E into E_1, when this latter space is given its strong operator topology, with $\|\mathscr{A}_y\| \leqslant 1$ for all $y \in Y$ and $\|\mathscr{A}_y\| = 1$ for $y \in B(A)$, and there exists a continuous function τ from $B(A)$ onto X such that $(A(F))(y) = \mathscr{A}_y F(\tau(y))$ for $F \in C(X, E)$ and $y \in B(A)$. If E is finite dimensional, then $B(A)$ is a closed subset of Y.

Throughout this paper X and Y will denote compact Hausdorff spaces, E a normed linear space, and $C(X, E)$ the space of all continuous functions from X into E. We will denote by $C(X)$ the space of continuous functions from X into the scalar field associated with E.

The Banach–Stone theorem states that

(1) *if A is an isometry of $C(X)$ onto $C(Y)$, there exists a homeomorphism τ of Y onto X and a function $a(y) \in C(Y)$, with $|a(y)| = 1$ for all $y \in Y$, such that $\big(A(f)\big)(y) = a(y)f\big(\tau(y)\big)$ for $f \in C(X)$ $y \in Y$* ([1], p. 442).

This theorem has been generalized by W. Holsztyński, who considered isometries A of $C(X)$ into $C(Y)$ which are not necessarily surjective, [3]. What Holsztyński showed is, essentially, that the image of $C(X)$ under such an isometry behaves well at least on a subset of Y. More precisely, if we modify the statement of (1) by changing τ from a homeomorphism of Y onto X to a continuous function from a closed subset $B(A) \subseteq Y$ onto X, and require only that $\|a\|_\infty = 1$ and $|a(y)| = 1$ for $y \in B(A)$, then the modified statement of (1) is valid in the case of an arbitrary isometry A of $C(X)$ into $C(Y)$. (A concise proof of Holsztyński's theorem may be found in [5]. A similar result had previously been obtained by K. Gęba and Z. Semadeni for spaces of real-valued functions and isometries that are isotonic, [2].)

In [4] M. Jerison showed that if E is a strictly convex Banach space, then the exact vector analogue of (1) can be established. He found that

(2) *if A is an isometry of $C(X, E)$ onto $C(Y, E)$, there exists a homeomorphism τ of Y onto X and a continuous function $y \to \mathscr{A}_y$ from Y into the space of bounded operators on E, when this latter space is given its strong operator topology, such that for all $y \in Y$, $\mathscr{A}_y$ is an isometry of E onto itself, and such that $(A(F))(y) = \mathscr{A}_y F(\tau(y))$ for $F \in C(X, E)$, $y \in Y$* ([4] p. 317).

The object of this article is to show that Holsztyński's theorem has a natural formulation in the vector case. We prove the following

THEOREM. *Let E, E_1 be normed linear spaces with E_1 strictly convex, and let A be an isometry of $C(X, E)$ into $C(Y, E_1)$. Then there exists a subset $B(A) \subseteq Y$, a continuous function $y \to \mathscr{A}_y$ from Y into the space of bounded operators on E into E_1, when this latter space is given its strong operator topology, with $\|\mathscr{A}_y\| \leqslant 1$ for all $y \in Y$ and $\|\mathscr{A}_y\| = 1$ for $y \in B(A)$, and there exists a continuous function τ from $B(A)$ onto X such that $(A(F))(y) = \mathscr{A}_y F(\tau(y))$ for $F \in C(X, E)$, $y \in B(A)$. If E is finite dimensional, then $B(A)$ is a closed subset of Y.*

The proof of the theorem will be established by a sequence of lemmas and corollaries. The author is indebted to the referee, Dr T. Figiel, for his contribution in simplifying the proof of the theorem as it appeared in the original version of the article.

The following notational conventions will be used throughout the article. We denote elements of $C(X, E)$, for the most part, by F, and those of $C(X)$ by f or g. For $e \in E$, we let $\underset{\sim}{e}$ stand for that element of $C(X, E)$ which is constantly equal to e. The norms in E, E_1 and $\mathscr{B}(E, E_1)$, the space of bounded linear operators on E into E_1, will be denoted by $\|\cdot\|$, while norms in $C(X)$, $C(X, E)$ and $C(Y, E_1)$ are denoted by $\|\cdot\|_\infty$. The letter S will stand for the surface of the unit ball in E.

Given $e \in S$ and $x \in X$, we let $\mathscr{F}_{e,x} = \{F \in C(X, E): F(x) = \|F\|_\infty \cdot e\}$ and then set $B(e, x) = \{y \in Y: \|(A(F))(y)\| = \|F\|_\infty \text{ for all } F \in \mathscr{F}_{e,x}\}$. We let $B(x) = \bigcup_{e \in S} B(e, x)$ and $B(A) = \bigcup_{x \in X} B(x)$. We denote by B that subset of $S \times X \times Y$ defined by $B = \{(e, x, y): e \in S, x \in X \text{ and } y \in B(e, x)\}$. Finally, for $y \in Y$, we define a map $\mathscr{A}_y: E \to E_1$ by $\mathscr{A}_y(e) = (A(\underset{\sim}{e}))(y)$, $e \in E$. $\mathscr{A}_y$ is a linear operator from E into E_1 with $\|\mathscr{A}_y\| \leqslant 1$. If $y \in B(A)$, then clearly $\|\mathscr{A}_y\| = 1$.

LEMMA 1. *For each pair $(e, x) \in S \times X$, $B(e, x)$ is nonempty.*

Proof. Since $B(e, x)$ is the intersection of the family of closed sets $M_F = \{y \in Y: \|(A(F))(y)\| = \|F\|_\infty\}$, for $F \in \mathscr{F}_{e,x}$, it suffices to show that this family has the finite-intersection property. Thus suppose that $\{F_1, \ldots, F_n\}$ is a finite subset of $\mathscr{F}_{e,x}$, and define $F_0 \in C(X, E)$ by $F_0 = \sum_{i=1}^{n} F_i$. Choose a $y \in Y$ with $\|(A(F_0))(y)\| = \|F_0\|_\infty$. We have $(A(F_0))(y) = \sum_{i=1}^{n} (A(F_i))(y)$, and hence

$$(3) \qquad \sum_{i=1}^{n} \|(A(F_i))(y)\| \geqslant \|(A(F_0))(y)\| = \|F_0\|_\infty \geqslant \|F_0(x)\| = \Big\| \sum_{i=1}^{n} \|F_i\|_\infty \cdot e \Big\| = \sum_{i=1}^{n} \|F_i\|_\infty .$$

Since for each i we have $\|(A(F_i))(y)\| \leqslant \|F_i\|_\infty$, equality must hold throughout in (3), and thus $\|(A(F_i))(y)\| = \|F_i\|_\infty$, $1 \leqslant i \leqslant n$. That is, $y \in M_{F_i}$ for $1 \leqslant i \leqslant n$ and hence $\bigcap_{i=1}^{n} M_{F_i} \neq \emptyset$.

LEMMA 2. *If $y \in B(x)$, then for each $F \in C(X, E)$ we have $(A(F))(y) = \mathscr{A}_y(F(x))$.*

Proof. If $y \in B(x)$, then $y \in B(x, e)$ for some $e \in S$. We first assume that F vanishes on some neighborhood U of x and prove that $(A(F))(y) = 0$. Choose a function $f \in C(X)$ with $f(x) = \|f\|_\infty > \|F\|_\infty$, and such that the support of f is contained in U. Define $F_1 \in C(X, E)$ by $F_1(z) = f(z) \cdot e$, $z \in X$. Then let $F_2 = F + F_1$ and $F_3 = \frac{1}{2}(F_1 + F_2)$. Then for $1 \leqslant i \leqslant 3$ $\|F_i\|_\infty = f(x)$, and $F_i(x) = f(x) \cdot e$, so that $F_i \in \mathscr{F}_{e,x}$. Since $y \in B(e, x)$, we have $\|(A(F_i))(y)\| = f(x)$, $1 \leqslant i \leqslant 3$. Since E_1 is strictly convex and $(A(F_3))(y) = \frac{1}{2}[(A(F_1))(y) + (A(F_2))(y)]$, we have $(A(F))(y) = (A(F_2))(y) - (A(F_1))(y) = 0$ as claimed.

Now let F be an arbitrary element of $C(X, E)$. Given $\varepsilon > 0$, pick an element $g \in C(X)$ such that $\|g\|_\infty = 1$, $g(z) = 1$ for z belonging to a neighborhood of x, and such that the support of g is contained in the set $\{z \in X: \|F(z) - F(x)\| < \varepsilon\}$. Let F_1 be that element of $C(X, E)$ which is constantly equal to $F(x)$, and define $F_2 \in C(X, E)$ by $F_2(z) = g(z)[F(z) - F(x)]$, $z \in X$. Then let F_3 be defined by $F = F_1 + F_2 + F_3$. Now F_3 vanishes in a neighborhood of x, so that, by what we have proved in the previous paragraph, $(A(F_3))(y) = 0$. Also $\|F_2\|_\infty < \varepsilon$, so that $\|(A(F_2))(y)\| < \varepsilon$. Since $(A(F_1))(y) = \mathscr{A}_y(F(x))$ and $(A(F))(y) = \sum_{i=1}^{3} (A(F_i))(y)$, we have $\|(A(F))(y) - \mathscr{A}_y(F(x))\| = \|(A(F_2))(y)\| < \varepsilon$. And as ε is an arbitrary positive number, the proof of the lemma is complete.

COROLLARY 1. *The set $\Gamma = \{(y, x): x \in X \text{ and } y \in B(x)\}$ is the graph of a continuous function τ mapping $B(A)$ onto X.*

Proof. Let $(y, x) \in \Gamma$, and let U be a neighborhood of x in X. Since $y \in B(A)$, $\|\mathscr{A}_y\| = 1$, and we can thus choose an $F \in C(X, E)$ with support contained in U such that $\mathscr{A}_y(F(x)) \neq 0$. Let $V = \{w \in Y\colon (A(F))(w) \neq 0\}$. Since $(A(F))(y) = \mathscr{A}_y(F(x))$, V is a neighborhood of y in Y. If $y' \in V \cap B(A)$ and if $(y', x') \in \Gamma$, then $x' \in U$ since $\mathscr{A}_{y'}(F(x')) = (A(F))(y') \neq 0$. And as y can be an arbitrary element of $B(A)$, this completes the proof of the corollary.

Lemma 3. *B is a closed subset of $S \times X \times Y$.*

Proof. Suppose that (e_a, x_a, y_a) is a net in B which converges to $(e, x, y) \in S \times X \times Y$. We will show that $y \in B(e, x)$. Fix a nonzero $F \in \mathscr{F}_{e,x}$, and, for each a, define $F'_a, F_a \in C(X, E)$ by

$$F'_a(z) = F(z) + \|F\|_\infty \cdot e_a - F(x_a),$$
$$F_a(z) = F'_a(z)/\max\{1, \|F'_a(z)\|/\|F\|_\infty\}$$

for $z \in X$. We have $F_a(x_a) = \|F\|_\infty \cdot e_a$ and $\|F_a\|_\infty \leqslant \|F\|_\infty$. Hence $F_a \in \mathscr{F}_{e_a, x_a}$ and $\|(A(F_a))(y_a)\| = \|F\|_\infty$. Thus

$$\|(A(F))(y)\| = \lim_a \|(A(F))(y_a)\|$$
$$\geqslant \liminf_a [\|(A(F_a))(y_a)\| - \|(A(F-F_a))(y_a)\|] \geqslant \|F\|_\infty - \limsup_a \|F - F_a\|_\infty.$$

Noting that since $\|F\|_\infty \cdot e_a - F(x_a) \to 0$ we have $\lim_a \|F - F_a\|_\infty = 0$, it follows that $\|(A(F))(y)\| = \|F\|_\infty$ and we are done.

Corollary 2. *If E is finite dimensional, then $B(A)$ is closed in Y.*

Proof. If E is finite dimensional, then $S \times X \times Y$ is compact. By Lemma 3, B is a closed subset of $S \times X \times Y$ and is thus compact. The result now follows since $B(A)$ is the image of B under the continuous projection of $S \times X \times Y$ onto Y.

The proof of the theorem is now completed by the following easily established lemma.

Lemma 4. *The map $y \to \mathscr{A}_y$ is a continuous function from Y into $\mathscr{B}(E, E_1)$ when the latter space is given its strong operator topology.*

Remarks. Easy examples show that $\mathscr{A}_y$ need not be isometric at any point of $B(A)$. We conclude with an example to show that if E is infinite dimensional, then not only is it true that $B(A)$ need not be closed, but also it may be impossible to extend τ to a continuous map defined on the closure of $B(A)$ to X.

Let E be a separable, infinite-dimensional Hilbert space, with orthonormal basis $\{e_1, e_2, \ldots\}$. Let X be the discrete space consisting of two points, $X = \{1, 2\}$, and let Y be the one-point compactification of the

positive integers, with the point at infinity denoted by ∞. Define $A\colon C(X, E) \to C(Y, E)$ by

$$(A(F))(1) = F(1),$$
$$(A(F))(2) = F(2),$$
$$(A(F))(n) = \begin{cases} \langle F(1), e_n\rangle e_n, & n \text{ odd} \geqslant 3, \\ \langle F(2), e_n\rangle e_n, & n \text{ even} > 3, \end{cases}$$
$$(A(F))(\infty) = 0.$$

Then for each positive integer n, $2n \in B(e_{2n}, 2)$ so that $\tau(2n) = 2$, while $2n+1 \in B(e_{2n+1}, 1)$ so that $\tau(2n+1) = 1$. Here $B(A)$ is the set of positive integers, and it is obviously impossible to define τ at ∞ in such a way as to preserve continuity.

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