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Finite dimensional subspaces of L_p

by

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Abstract. If $1 < p < \infty$ and $E \subset L_p(\mu)$ is an n-dimensional subspace, then $d(E, l_2^n) \leqslant n^{|1/2-1/p|}$; further, there is a projection u of $L_p(\mu)$ onto E with $\|u\| \leqslant n^{|1/2-1/p|}$.

For convenience only real normed spaces are considered. The notation and terminology is standard; we mention only that the p-absolutely summing, p-integral and L_p-factorization norms of operators are denoted by π_p, i_p and γ_p, respectively (cf. [13], [12], [7]). The main result of this paper is the following.

THEOREM 1. *Let E be an n-dimensional subspace of $L_p(\mu)$, $1 \leqslant p < \infty$.*

(1) *There is a basis $(f_i)_{i\leqslant n}$ of E so that for all $x \in l_2^n$,*

$$n^{-1}\|x\|_2^2 = \int \Big|\sum_{i\leqslant n} x_i f_i\Big|^2 |f|^{p-2}\, d\mu, \quad \textit{where} \quad f = \Big(\sum_{i\leqslant n} |f_i|^2\Big)^{1/2}.$$

(2) *If $(h_i)_{i\leqslant n}$ is another basis for E satisfying* (1), *there is an $n\times n$ orthogonal matrix (a_{ik}) such that*

$$h_k = \sum_{i\leqslant n} a_{ik} f_i, \qquad i \leqslant k \leqslant n.$$

The proof requires an easy lemma.

LEMMA 2. *For $1 \leqslant p < \infty$ and $u\colon l_2^n \to L_p(\mu)$ any operator,*

$$\pi_p(u) = \Big\|\sup_{\|x\|=1} |u(x)|\Big\|.$$

Proof of Lemma. The Hilbert space l_2^n is a quotient of an L_q-space, $1/p + 1/q = 1$, so $\pi_p(u') \leqslant \pi_p(u)$ by [7]. Since the domain of u' is $L_q(\mu)$, the Kwapień–Schwartz theorem [6] shows that $u = u''$ maps the unit ball of l_2^n into an order bounded set of the lattice $L_p(\mu)$, and that $\|f\| \leqslant \pi_p(u')$ for $f = \sup_{\|x\|=1} |u(x)|$. The other inequality is obvious.

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Proof of Theorem 1. By [8] there is an isomorphism $u\colon l_2^n\to E$ with $\pi_p(u)=1$ and $i_q(u^{-1})=n$, $1/p+1/q=1$. Write $(e_i)_{i\leqslant n}$ for the unit vector basis of l_2^n, $f_i=u(e_i)$ and $f=\big(\sum_{i\leqslant n}|f_i|^2\big)^{1/2}$. There is no harm in supposing $f>0$ μ-a.e. since the f_i are a basis for E. The operator u^{-1} has an extension $w\colon L_p(\mu)\to l_2^n$ satisfying $i_q(w)=i_q(u^{-1})=n$. Let $g_i=w'(e_i^*)$, where $e_i^*\in(l_2^n)'$ is inner product with e_i, and $g=\big(\sum_{i\leqslant n}|g_i|^2\big)^{1/2}$.

It is clear that $f=\sup_{\|x\|=1}|u(x)|$ and $g=\sup_{\|x\|=1}|w'(x)|$. By the lemma, $\|f\|_p=1$ and $\|g\|_q\leqslant\pi_q(w)\leqslant n$, again by the Kwapień–Schwartz theorem. Since $\langle f_i,g_k\rangle=\delta_{ik}$,

$$n=\int\sum_{i\leqslant n}f_ig_i\,d\mu\leqslant\int fg\,d\mu\leqslant\|f\|_p\|g\|_q\leqslant n.$$

Thus $\langle f,g\rangle=n$ and $\|g\|_q=n$. For $1<p<\infty$ this clearly implies $g=n|f|^{p-1}$ because $\|f\|_p=1$. In case $p=1$, we have $g=n$ μ-a.e. since $f>0$ μ-a.e. Also $\sum_{i\leqslant n}f_ig_i=fg$ so that $f_if^{-1}=g_ig^{-1}$ for $i=1,\ldots,n$. Combining equalities, $g_k=n|f|^{p-2}f_k$ for each k and hence

$$\delta_{ik}=\langle f_i,g_k\rangle=n\int f_if_k|f|^{p-2}\,d\mu,$$

which is enough to establish (1).

To prove (2), let $(f_i)_{i\leqslant n}$ be any basis for which (1) is true, and define $u\colon l_2^n\to E$, $w\colon L_p(\mu)\to l_2^n$ by $u(x)=\sum_{i\leqslant n}x_if_i$ and $w(h)=n(\langle h,f_i|f|^{p-2}\rangle)_{i\leqslant n}$. Clearly, wu is the identity, $\pi_p(u)=1$ by the lemma and $n=\operatorname{tr}(uu^{-1})\leqslant\pi_p(u)i_q(w)\leqslant i_q(w)$. Notice that $|w'(x)|\leqslant n|f|^{p-1}$ μ-a.e. whenever $\|x\|_2\leqslant 1$. For $1<p<\infty$ this implies $i_q(w')\leqslant n$ and hence $i_q(w)\leqslant i_q(w')\leqslant n$ since l_2^n is a quotient of an L_p-space [7]. In the case $p=1$, $w\colon L_1(\mu)\to l_2^n$ has norm $\leqslant n$; since l_2^n is a quotient of a $C(K)$ and w has the lifting property, $i_\infty(w)=\gamma_\infty(w)\leqslant n$. In any event the isomorphism $u\colon l_2^n\to E$ which takes the ith unit vector to f_i satisfies $\pi_p(u)=1$ and $i_q(u^{-1})=n$ if the basis (f_i) satisfies (1). Now if $(h_i)_{i\leqslant n}$ also satisfies (1), then $\pi_p(v)=1$ and $i_q(v^{-1})=n$, where $v\colon l_2^n\to E$ maps e_i to h_i. By Theorem 1.1 of [8], $u^{-1}v$ is an isometry of l_2^n and representing $u^{-1}v$ as a matrix with respect to the unit vector basis proves (2).

Theorem 3. *Let $1<p<\infty$, $E\subset L_p(\mu)$ and F be an n-dimensional space. Each operator $u\colon E\to F$ has an extension $w\colon L_p(\mu)\to F$ with $\|w\|\leqslant n^{|1/2-1/p|}\|u\|$; in case $2\leqslant p<\infty$ the extension may be chosen to satisfy $\gamma_2(w)\leqslant n^{1/2-1/p}\|u\|$.*

Proof. First suppose $2<p<\infty$ and consider the special case in which $E=F$ and u is the identity. Let $f_1,f_2,\ldots,f_n\in E$ and let f be as in Theorem 1; set $d\gamma=|f|^p d\mu$ and let $w=w_3w_2w_1$, where $w_1\colon L_p(\mu)\to L_2(\gamma)$ is multiplication by f^{-1}, $w_2\colon L_2(\gamma)\to w_1(E)$ is the orthogonal projection and $w_3\colon w_1(E)\to E$ is multiplication by f. Clearly, w is a projection onto E. By Hölder's inequality,

$$\|w_1(h)\|=\Big[\int|h|^2|f|^{p-2}d\mu\Big]^{1/2}\leqslant\|h\|_p\|f\|_p^{(p-2)/2p}$$

for all $h\in L_p(\mu)$, and hence $\|w_1\|\leqslant 1$. Also for $x\in\mathbf{R}^n$

$$\Big\|w_3\Big[\sum_{i\leqslant n}x_i(f_if^{-1})\Big]\Big\|^p=\int\Big|\sum_{i\leqslant n}x_if_i\Big|^2\Big|\sum_{i\leqslant n}x_if_i\Big|^{p-2}d\mu$$
$$\leqslant\int\Big|\sum_{i\leqslant n}x_if_i\Big|^2\|x\|_2^{p-2}|f|^{p-2}d\mu=n^{(p/2)-1}\Big[\int\Big|\sum_{i\leqslant n}x_i(f_if^{-1})\Big|^2|f|^p d\mu\Big]^{p/2}$$

by Theorem 1, so $\|w_3\|\leqslant n^{1/2-1/p}$. This establishes the special case. More generally, for $2<p<\infty$, let φ be the norm one bilinear form

$$\varphi\colon L\big(L_p(\mu),l_2\big)\times L(l_2,F)\to L(E,F)$$

defined by $\varphi(a,b)=ba|E$, and also denote by φ the induced linear operator on the projective tensor product of $L\big(L_p(\mu),l_2\big)$ and $L(l_2,F)$. After making the natural identification $L(E,F)'=i_1(F,E)$ and $L(l_2,F)'=i_1(F,l_2)$ (possible since F is finite dimensional) the adjoint

$$\varphi'\colon i_1(F,E)\to L\Big(L\big(L_p(\mu),l_2\big),i_1(F,l_2)\Big)$$

is given by $\varphi'(u)(v)=vju$, where $j\colon E\to L_p(\mu)$ is the natural embedding. For $u\in i_1(F,E)$ the special case proven above shows that there are operators $\alpha\colon L_p(\mu)\to l_2$, $\beta\colon l_2\to ju(F)$ with $\|\alpha\|\leqslant 1$, $\|\beta\|\leqslant n^{1/2-1/p}$ and $\beta\alpha|ju(F)=$ identity. For clarity we temporarily write $i_1(\gamma\colon A\to B)$ to denote the i_1-norm of an operator γ considered as a map from A to B. With this notation,

$$\begin{aligned}i_1(u\colon F\to E)&\leqslant i_1\big(u\colon F\to u(F)\big)\\&=i_1\big(ju\colon F\to ju(F)\big)\\&=i_1\big(\beta\alpha ju\colon F\to ju(F)\big)\\&\leqslant\|\beta\|i_1(\alpha ju\colon F\to l_2)\\&\leqslant n^{1/2-1/p}\|\varphi'(u)\|.\end{aligned}$$

Thus $\|\varphi'(u)\|\leqslant i_1(u)\leqslant n^{1/2-1/p}\|\varphi'(u)\|$ for all $u\in i_1(F,E)$ so that φ' is an $n^{1/2-1/p}$ — into isomorphism and hence φ, on the tensor product, is $(1+\varepsilon)n^{1/2-1/p}$-quotient for every $\varepsilon>0$.

For $u\colon E\to F$ and $\varepsilon>0$, let t_ε in the projective tensor product satisfy $\varphi(t_\varepsilon)=u$ and $|t_\varepsilon|\leqslant(1+\varepsilon)n^{1/2-1/p}\|u\|$. Expand t_ε as an absolutely convergent series $t_\varepsilon=\sum_{k\geqslant 1}\lambda_ka_k\otimes b_k$, with $(a_k)_{k\geqslant 1}$ and $(b_k)_{k\geqslant 1}$ sequences in the closed unit balls of $L\big(L_p(\mu),l_2\big)$ and $L(l_2,F)$, respectively, and $\lambda=(\lambda_k)\in l_1$

a positive sequence with $\|\lambda\|_1 \leqslant (1+\varepsilon)|t_\varepsilon|$. Define operators $a\colon L_p(\mu) \to (\bigotimes_{k\geqslant 1} l_2)_2$ and $b\colon (\bigotimes_{k\geqslant 1} l_2)_2 \to F$ by $a(h) = (\lambda_k^{1/2} a_k(h))_{k\geqslant 1}$ and $b((x_k)_{k\geqslant 1}) = (\lambda_k^{1/2} b_k(x_k))_{k\geqslant 1}$. Set $w_\varepsilon = ba$. Since $\|a\|$ and $\|b\|$ are both at most $\|\lambda\|_1^{1/2}$, $\gamma_2(w) \leqslant \|\lambda\|_1$. Since $\varphi(t_\varepsilon) = u$, $w_\varepsilon | E = u$ and combining inequalities shows $\gamma_2(w_\varepsilon) \leqslant (1+\varepsilon)^2 n^{1/2-1/p}\|u\|$. For each $\varepsilon > 0$ choose such an extension w_ε. Then $\gamma_2(L_p(\mu), F) = \gamma_2^*(F, L_p(\mu))'$ since F is finite dimensional, so as ε tends to zero the net $(w_\varepsilon)_{\varepsilon>0}$ closters wk^* to some $w \in \gamma_2(L_p(\mu), F)$. Clearly, w is the desired extension of u.

The case $1 < p < 2$ follows by duality. By a theorem of Maurey ([10], or [11], Proposition 9.2) the conclusion of Theorem 8 holds for $1 < p < 2$ if the inequality $i_q(v) \leqslant n^{1/2-1/q}\pi_q(v)$ is true for every operator v defined on an n-dimensional space F, where $1/p + 1/q = 1$. To see this let $v\colon F \to G$ be any map, let C be an injective space containing G isometrically and factor v as

$$F \xrightarrow{a} L_\infty(\mu) \xrightarrow{\beta} L_q(\mu) \xrightarrow{\gamma} C,$$

for μ a probability measure, β inclusion and a and γ operators satisfying $\|a\|\|\gamma\| = \pi_q(v)$. Since $q > 2$, the previously proven part of the theorem gives a projection w of $L_q(\mu)$ onto $\beta a(F)$ with $\|w\| \leqslant n^{1/2-1/q}$. Since γ maps $\beta a(F)$ into G, $i_q(v) = i_q(\gamma w \beta a) \leqslant \|w\|\|a\|\|\gamma\|$, which completes the proof.

COROLLARY 4. *If $E \subset L_p(\mu)$ is n-dimensional and $1 < p < \infty$, there is a projection $w\colon L_p(\mu) \to E$ with $\|w\| \leqslant n^{|1/2-1/p|}$.*

Proof. In Theorem 4 take $E = F$ and u the identity.

The *Banach–Mazur distance* between isomorphic spaces E and F is defined as $d(E, F) = \inf \|u\|\|u^{-1}\|$, with the infimum taken over all isomorphisms $u\colon E \to F$. The next corollary answers a question raised in [1].

COROLLARY 5. *If F is n-dimensional and isometric to a quotient of a subspace of $L_p(\mu)$, $1 < p < \infty$, then $d(F, l_2^n) \leqslant n^{|1/2-1/p|}$.*

Proof. A subspace of a quotient of $L_p(\mu)$ is isometric to a quotient of a subspace of $L_p(\mu)$, so it suffices to consider the case $p > 2$. If $E \subset L_p(\mu)$ and $u\colon E \to F$ is a quotient map, then choosing w as in Theorem 3 shows $\gamma_2(u') = \gamma_2(u) \leqslant \gamma_2(w) \leqslant n^{1/2-1/p}$, so the isometric embedding $u'\colon E' \to F'$ factors nicely through a Hilbert space.

COROLLARY 6. *Let $v\colon E \to F$ be any linear operator and suppose that one of E, F is n-dimensional.*

(1) *For $1 \leqslant p \leqslant \infty$, $i_p(v) \leqslant n^{|1/2-1/p|}\pi_p(v)$.*

(2) *For $2 \leqslant p \leqslant \infty$, $\pi_2(v) \leqslant (\pi/2)^{1/2} n^{1/2-1/p}\pi_p(v)$.*

(3) *For $1 \leqslant q \leqslant 2$, $i_q(v) \leqslant (\pi/2)^{1/2} n^{1/q-1/2}\pi_2(v)$.*

Proof. The extreme cases $p = 1$ and $p = \infty$ follow from John's Theorem [5], phrased as in [2] to say that $i_\infty(w) \leqslant \pi_2(w) \leqslant n^{1/2}\|w\|$ for each operator w on or into an n-dimensional space; further, in (2) and (3) the constant $(\pi/2)^{1/2}$ can clearly be replaced by 1. To prove (1) and (2) assume $\dim F = n$, let C be an injective space containing F, and write $v = \gamma\beta a$, where μ is some probability measure, $a\colon E \to L_\infty(\mu)$, $\beta\colon L_\infty(\mu) \to L_p(\mu)$ is inclusion and $\gamma\colon L_p(\mu) \to C$ satisfy $\|a\|\|\gamma\| = \pi_p(v)$. Choose an extension $w\colon L_p(\mu) \to F$ of $\gamma|\gamma^{-1}(F)$ as in Theorem 3, so that $i_p(v) = i_p(w\beta a) \leqslant n_1^{|1/2-1/p|}\pi_p(v)$. For $p > 2$, Grothendieck's Theorem (cf. [9], or [2] for the constant $(\pi/2)^{1/2}$) shows that $\pi_2(v) = \pi_2(w\beta a) \leqslant (\pi/2)^{1/2}\gamma_2(w)\gamma_\infty(\beta a) \leqslant (\pi/2)^{1/2} n^{1/2-1/p}\|a\|\|\gamma\|$, which proves (2). The case $\dim E = n$ may be handled as in the proof of Theorem 3, and (2) and (3) are equivalent by a standard duality argument.

Remarks. (1) The distance estimate of Corollary 5 is best possible since $d(l_p^n, l_2^n) = n^{|1/2-1/p|}$ [4]. The corollary implies that given $p, q \in (1, +\infty)$, there is an $a < 1$ such that $d(E, F) \leqslant n^a$ whenever $E \subset L_p$ and $F \subset L_q$ are n-dimensional. It would be of interest to determine the smallest such a. In particular, is it true that $d(E, F) \leqslant \max d(l_r^n, l_s^n)$, $r, s \in \{p, q, 2\}$?

(2) Given $p, q \in (1, +\infty)$, there is a $\beta > 0$ such that if $E \subset L_p$ and $F \subset L_q$ are any n-dimensional subspaces, there is an operator $u\colon E \to F$ with $\|u\| \leqslant 1$ and $i_1(u) \geqslant n^\beta$. This follows by composing the isomorphisms given by Corollary 5, since the identity on l_2^n has 1-integral norm n.

(3) The norm estimates of Theorem 3, Corollary 4 and Corollary 6 (1) are asymptotically best possible. Sobczyk [14] has shown that for n a power of 2 and any p, there is an n-dimensional subspace S of l_p such that every projection onto S has norm at least $2^{-1}[(2n)^{|1/2-1/p|}-1]$. By Maurey's Theorem [10], if Corollary 6 (1) is true with $n^{|1/2-1/p|}$ replaced by constant c, then Theorem 3 is also true with constant c. The estimates of Corollary 6 (2) and (3) are also asymptotically best possible. Garling and Gordon [2] show that $i_p(u) = \pi_p(u) = n^{1/p}$, $1 \leqslant p \leqslant \infty$, for u the identity on l_∞^n.

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