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A topological version of some ergodic theorems

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Abstract. The theorem given in Section 2 is a consolidation of known results in that it shows that some known weakly ergodic theorems are corollaries of a single theorem concerning topological semigroups. In Section 4 the characteristic differences between these results and results not covered by the theorem of Section 2 are discussed.

1. Introduction. Since the work of Perron [8] and Markov [6], the present century has witnessed a considerable amount of investigation of ergodic properties of certain types of matrices. The bulk of this work is neatly summarized in the recent book by Seneta [9]. Various types of ergodicity have been established for infinite products of matrices. Under some conditions the products converge to a fixed matrix (e.g. Bernstein [2]); under other conditions to the set of rank 1 matrices (e.g. Lopez [5]); under yet other conditions, to the set of rank d matrices (e.g. Taylor [11]).

The aims of the present paper are:

(i) to show that a number of these ergodic theorems are merely special cases of a simpler general theorem on topological semigroups rather than sets of matrices;

(ii) to expose the relation between the theorems covered in (i) and a number of other known results which do not follow as corollaries of our general theorem;

(iii) to extend these results to non-matrix applications.

2. A weakly ergodic theorem on topological semigroups.

Notation. 1. Let S be a topological semigroup. If $\{s_i\}$ is a sequence of elements of S, let $s_{i,j}$ denote the product $s_i s_{i+1} \ldots s_j$.

2. For a topological semigroup S and for $T \subset S$, let $A(T)$ denote the set $\{t \in T\colon\ tu = t \text{ for some } u \in T\}$.

THEOREM (weakly ergodic). *Let S be a topological semigroup and suppose that there exists a sequentially compact subset T of S and an integer N such that $S^p \subset T$ whenever $p \geqslant N$. Then $A(T) \neq \emptyset$, and, if $\{s_i\}$ is an arbitrary sequence in S, then, for each fixed i, the sequence $\{s_{i,j}\}$ converges to $A(T)$.*

Proof. Fix i arbitrarily. Now for j sufficiently large, $\{s_{i,j}\} \in T$ and so, by sequential compactness of T, $\{s_{i,j}\}$ has a convergent subsequence $\{s_{i,j_k}\}$ with limit $s \in T$. Define $t_{i,k} = s_{j_k+1,j_{k+1}}$, so that $s_{i,j_{k+1}} = s_{i,j_k} t_{i,k}$. Without loss of generality we may assume $j_{k+1} - j_k \geqslant N$. Therefore, again by sequential compactness, $\{t_{i,k}\}$ has a convergent subsequence $\{t_{i,k_m}\}$ with limit $t \in T$. We now have $s_{i,j_{k_m+1}} = s_{i,j_{k_m}} t_{i,k_m}$, where both of the s terms converge to s and the t term converges to t as $m \to \infty$. By continuity, therefore, $st = s$, i.e. $s \in A(T)$, and so s_{i,j_k} converges to $A(T)$ as $k \to \infty$. It has now been shown that the sequence $\{s_{i,j}\}$ has a subsequence convergent to $A(T)$. Starting with an arbitrary subsequence of $\{s_{i,j}\}$ rather than $\{s_{i,j}\}$ itself, one can apply the same reasoning as above to deduce that any subsequence of $\{s_{i,j}\}$ has a subsequence convergent to $A(T)$. The theorem then follows.

3. Ergodic theorems as corollaries. We now see that some key ergodic theorems follow as simple corollaries of the above theorem. We shall require the following lemma in which the (i,j)-element of a matrix M is denoted by $m(i,j)$.

LEMMA 1. *Let $\{M_i\}$ be a sequence of $n \times n$ matrices over the field of real numbers and suppose that, for each i, there exists N such that $M_{i,p} = M_i M_{i+1} \dots M_p > 0$ whenever $p \geqslant N$. If $0 < \alpha \leqslant \min^+_{j,k} m_i(j,k)$ and $\max_{j,k} m_i(j,k) \leqslant \beta < \infty$ for each i, where $\min^+_{j,k}$ denotes the minimum among all strictly positive elements, then there exists a number γ such that*

$$0 < \gamma \leqslant \min_{j,k} m_{i,p}(j,k) / \max_{j,k} m_{i,p}(j,k) \quad \textit{whenever} \quad p \geqslant N.$$

This lemma is well known. It is essentially proved by Seneta ([9], p. 71) and in a restricted form by Lopez [5].

COROLLARY 1 OF THE THEOREM. *Let $\{M_i\}$ be a sequence of $n \times n$ nonnegative matrices for which there exists an integer N such that any product of N of the M_i is strictly positive. Suppose also that there exist α, β such that $0 < \alpha \leqslant \min^+_{j,k} m_i(j,k)$ and $\max_{j,k} m_i(j,k) \leqslant \beta < \infty$ for each i. Then, for given i, $M_{i,p} / \min^+_{j,k} m_{i,p}(j,k)$ converges to the set of strictly positive matrices of rank 1 as $p \to \infty$.*

Proof. Let Q be the set of matrices such that all sequences $\{M_i\}$ in Q satisfy the hypotheses of the corollary. Now, for each $M \in Q$, define $\tilde{M} = M / \min^+_{j,k} m(j,k)$, and let $\tilde{Q}$ be the set obtained from Q on replacing each M by $\tilde{M}$. Then Q can be made a semigroup by the multiplication rule $\tilde{M}_1 \tilde{M}_2 = \widetilde{M_1 M_2}$, and this is metrizable with metric $d(\tilde{M}_1, \tilde{M}_2) = \max_{j,k} |\tilde{m}_1(j,k) - \tilde{m}_2(j,k)|$. Now define $S = \tilde{Q}$ and (sequentially) compact $T = \{\tilde{M} : \tilde{m}(j,k) \leqslant \gamma^{-1}\}$, where γ is the number appearing in the above lemma. The lemma shows that $S^p \subset T$ for $p \geqslant N$.

It is clear that all matrices in T are strictly positive. Now if $M \in A(T)$, then there exists $L \in T$ such that $ML = M$, i.e. each row of M is either zero or a nonnegative left eigenvector of L. The first of these alternatives is ruled out by the fact that $L > 0$ for all $L \in T$, and by Perron's theorem there is only one eigenvector (apart from scalar multiples) of L. Such an eigenvector is strictly positive. Thus $M > 0$ and has rank 1. This shows that $A(T)$ is contained in the set of $n \times n$ strictly positive matrices of rank 1.

As a second corollary we can obtain the more general result obtained by matrix methods by Taylor [11]. We require an extension of Lemma 1. This is

LEMMA 2. *Let $\{M_i\}$ be a sequence of $n \times n$ nonnegative matrices over the field of real numbers and suppose that, for each fixed i, there exists an integer N and an irreducible nonnegative matrix D with period d such that $M_{i,p}$ has the same graph as D^{p+1-i} whenever $p+1-i \geqslant N$. If $0 < \alpha \leqslant \min^+_{j,k} m_i(j,k)$ and $\max_{j,k} m_i(j,k) \leqslant \beta < \infty$ for each i, then there exists a number γ such that $0 < \gamma \leqslant \min^+_{j,k} m_{i,p}(j,k) / \max_{j,k} m_{i,p}(j,k)$ whenever $p \geqslant N$.*

Proof. The result follows easily from Lemma 1 if the index set of the M_i (and of D) is permuted so that $M_{i,i+d-1}$ assumes its canonical form:

$$\begin{bmatrix} Z^{(1)} & & & \\ & Z^{(2)} & & 0 \\ & & \ddots & \\ 0 & & & Z^{(d)} \end{bmatrix}$$

where each square block $Z^{(h)}$ is a primitive matrix.

COROLLARY 2 OF THE THEOREM. *Let $\{M_i\}$ be a sequence of $n \times n$ nonnegative matrices for which there exists an integer N and an irreducible nonnegative matrix D, of rank d such that, if $q \geqslant N$, then any product of q of the M_i has the same graph as D^q. Suppose also that there exist α, β such that $0 < \alpha \leqslant \min^+_{j,k} m_i(j,k)$ and $\max_{j,k} m_i(j,k) \leqslant \beta < \infty$ for each i. Then, for given i, $M_{i,p} / \min^+_{j,k} m_{i,p}(j,k)$ converges to the set of $n \times n$ nonnegative matrices of rank $\leqslant d$.*

Proof. The wording is exactly as for Corollary 1 down to the definition of S. Now define (sequentially) compact $T = \{\tilde{M} : \text{graph}(\tilde{M}) = \text{graph}(D^q)$ for some $q \geqslant N$, $\max_{j,k} \tilde{m}(j,k) \leqslant \gamma^{-1}\}$, where N, γ are the numbers appearing in Lemma 2. The proof now continues just as for Corollary 1 except that:

1. $L \in T$ implies that graph $(L) \doteq$ graph (D^q) for some integer q, $0 \leqslant q < d$ instead of $L > 0$; and

2. $L \in T$ implies that L has no more than d distinct nonnegative eigenvectors (they are not positive if $d > 1$) instead of no more than 1.

By this means we deduce that $A(T)$ is contained in the set of $n \times n$ nonnegative matrices of rank $\leqslant d$.

In the remaining corollaries we move away from just matrix applications. The next corollary is a natural extension of Corollary 1 to a case of composition of *nonlinear* functions.

DEFINITION. Consider a function $f\colon A \to \mathbf{R}^m$, where $A \subset \mathbf{R}^m$. If, for any real constant $K > 0$ and any $x \in A$, we have $f(Kx) = K^p f(x)$, where p is some real constant, then we say that f is *homogeneous of degree* p.

COROLLARY 3 OF THE THEOREM. *Let $\{f_i\}$, $i = 1, 2, \ldots$, be a sequence of homogeneous self-maps on A, the interior of the positive orthant of $\mathbf{R}^m$. Suppose that the Jacobian matrices of these functions are such that, for any sequence $\{x^{(i)}\}$, $\{Df_i(x^{(i)})\}$, $i = 1, 2, \ldots$, satisfies the same conditions as $\{M_i\}$ in Corollary 1. Define $f_{i,i} = f_i$ and $f_{i,j} = f_i(f_{i+1,j})$, for $j > i$. Then there is a sequence $\{r_j\}$ of rays emanating from the origin, such that for fixed i, and any sequence $\{x_j\}$ in A, the angle between $f_{i,j}(x_j)$ and $\{r_j\}$ converges to zero as $j \to \infty$.*

Proof. Let Q be the set of all functions f_i such that all sequences f_i in Q satisfy the hypotheses of the corollary. Now identify functions in Q which are merely scalar multiples of each other by defining $\tilde{f}_i = f_i / \inf_x \min_{j,k}{}^{+} Df(j, k)(x)$, and, in the same manner as in Corollary 1, it can be shown that, for $p \geqslant N$, $S^p \subset T$, a compact metric space consisting of functions f_i with Df_i always strictly positive matrices. As in Corollary 1, it then follows that $f_{i,j}$ converges, with increasing j, to the set of homogeneous functions which have Jacobian matrices of rank 1 at each point. It is clear, from the fact that homogeneous functions preserve rays emanating from the origin and the rank of the Jacobian matrix, that $f_{i,j}$ converges to the set of functions which map all points of the domain to a single ray emanating from the origin.

Such results as this have been extended (Taylor [12]) to the case of functions f_i which are not homogeneous but retain the essential positivity properties in their Jacobian matrices. Proofs of these results could also be developed as corollaries of the present theorem.

As a final corollary, we choose a slightly more novel setting, by proving a central limit theorem of statistics.

COROLLARY 4 OF THE THEOREM. *Let $X_1, X_2, \ldots$ be independent nondegenerate random variables with $E[X_i] = \mu_i < \infty$ and $V[X_i] = \sigma_i^2 \leqslant K$, a fixed real number. Define*

$$Y_i = \Big(\sum_{k=1}^{i} X_k - \sum_{k=1}^{i} \mu_k\Big) \Big/ \Big(\sum_{k=1}^{i} \sigma_k^2\Big)^{1/2}, \qquad i = 1, 2, \ldots$$

Then the sequence Y_i converges in distribution to a random variable that is standard normal.

Proof. Define B to be the set of nondegenerate distribution functions with variance $\leqslant K$. For $F, G \in B$, let $F*G$ denote convolution of F, G in the usual sense. Let $[B]$ denote the semigroup generated by B and $*$. This semigroup can be made a metric space by the introduction of the metric:

$$d(F, G) = |\sigma_F^2 - \sigma_G^2| + \sup_{A \in \mathscr{B}} \Big| \int_A d\tilde{F} - \int_A d\tilde{G} \Big|,$$

where $\mathscr{B}$ is the Borel σ-algebra on the real numbers, σ_F^2 is the variance associated with F, and $\tilde{F}$ is the d.f. of the standardized version of the random variable with d.f. F.

Let $S = \beta([B])$ be the Stone–Čech compactification of $[B]$. Now, if X denotes the space $\{\tilde{F}\colon F \in [B]\}$, then $[B] \subset X \times \mathbf{R}^2$, where $\mathbf{R}$ is the set of real numbers. Note that the family of measures associated with X is tight, a fact which follows easily from Chebyshev's inequality. Therefore, by Prokhorov's theorem, (Ash [1], p. 330), this family is relatively compact, and so compact in the above topology. Therefore, $S = \beta([B]) \subset X \times \beta(\mathbf{R}^2)$.

We now apply the theorem to the compact topological semigroup S with $*$ continuously extended from $[B]$, as assured by the Stone–Čech theorem. Let F_i be the d.f. of X_i, $i = 1, 2, \ldots$ Define $F_{i,j} = F_i * F_{i+1} * \ldots * F_j$. By the theorem, $F_{i,j}$ converges to $A(S)$ as $j \to \infty$. Now if $F \in A(S)$, then there is $G \in S$ such that $F * G^{n^*} = F$ for each n. Taking projection from S to X, we have $\widetilde{F * G^{n^*}} = \tilde{F}$, whence G is stable. But since $\tilde{G} \in X$, G has finite variance and so must be normal. From this and the fact that $\widetilde{F*G} = \tilde{F}$, it is a simple exercise in characteristic functions to show that $\tilde{F}$ is standard normal, thus completing the proof.

Corollary 4 above is a somewhat weakened version of the Lindeberg central limit theorem (Ash [1], p. 336). One pleasant feature of the above approach to the proof of this theorem is that it can be extended to vector-valued random variables with very little effort indeed.

4. Some theorems of a different type. It is interesting to notice that certain known ergodic theorems are not covered by the topological theorem of Section 2. Among these are theorems of Hajnal [3], [4], Paz and Reichaw [7], and, more generally, of Seneta [10] who demonstrate ergodicity of sequences of stochastic matrices in terms of a coefficient of ergodicity, which is essentially a measure of "amount of contraction" towards the set of matrices of rank 1 which is produced on multiplication by the matrix possessing the coefficient.

When the matter is put this way, the distinction between theorems of this type and the theorem of Section 2 becomes clear. Both types of theorem deal with the semigroup of nonnegative $n\times n$ matrices, but whereas the theorem of Section 2 imposes a topology on this semigroup, theorems involving a coefficient of ergodicity introduce a metric as well. The coefficient of ergodicity provides a measure of the distance between the matrix to which it belongs and the set of matrices of rank 1.

In view of this it is not surprising that the results obtainable from the coefficient of ergodicity type theorems are stronger than the topological theorem of this paper. Similar remarks apply to a comparison of Corollary 4, with its uniform condition on the basic random variables, and the usual form of the Lindeberg theorem.

One other point becomes clearer through the comparison of these two types of theorem. The uniform maxima and minima required of the elements of M_i in Corollaries 1 and 2 have appeared somewhat mysterious in the past. It becomes clear from the theorem of Section 2 that they comprise a compactness condition without which the topological theorem is false. The theorems based on coefficients of ergodicity show that once a metric is introduced into our topological space, the compactness requirement becomes unnecessary and we can choose ergodic sequences from noncompact spaces by reference to the coefficients of ergodicity.

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