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On isomorphisms between certain subalgebras of $B(X)$

by

H. I. BROWN, F. P. CASS* and I. J. W. ROBINSON (London, Ontario)

Abstract. Let X be a non-reflexive Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators on X with the norm given by $\|T\| = \sup[\|Tx\|: \|x\| \leqslant 1]$, for $T \in B(X)$.

Wilansky [11] introduced two classes of subalgebras of $B(X)$, $[\Omega_w]$ and $[\Gamma_w]$, defined as follows:

$$\Omega_w = \{T \in B(X): T^{**}w \in w \oplus \hat{X}\}, \quad w \in X^{**} \setminus \hat{X}$$

and

$$\Gamma_w = \{T \in B(X): T^{**}w \in \langle w \rangle\}, \quad w \in X^{**}.$$

Brown and Cho [1] have studied the subalgebras Ω_w and Γ_w in the special case where $X = c$, the Banach space of convergent sequences. In this paper their results are examined in the case of general X. Where the results of [1] extend, the proofs are simplified and in one case the result is improved in the case $X = c$. It is shown that Ω_w and Γ_w are the commutants of an operator in $B(X)$ only in trivial instances. The form of an algebraic isomorphism between Γ_w and Γ_z for $w, z \in X^{**} \setminus \hat{X}$ is determined, and from this it is shown that the subalgebras Γ_w are not all isomorphic when $X = l_1$ in contrast to the case where $X = c$ as shown in [1].

1. Introduction. Let X be a non-reflexive Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators on X with the norm given by $\|T\| = \sup\{\|Tx\|: \|x\| \leqslant 1\}$, for $T \in B(X)$.

Wilansky [11] introduced two classes of subalgebras of $B(X)$, $\{\Omega_w\}$ and $\{\Gamma_w\}$, defined as follows:

$$(1) \qquad \Omega_w = \{T \in B(X): T^{**}w \in w \oplus \hat{X}\}, \quad w \in X^{**} \setminus \hat{X}$$

and

$$(2) \qquad \Gamma_w = \{T \in B(X): T^{**}w \in \langle w \rangle\}, \quad w \in X^{**}.$$

Here X^{**} is the second dual of X, $\hat{X}$ is the image of X under the natural embedding of X into X^{**}, T^{**} is the adjoint of the adjoint T^* of T, $w \oplus \hat{X} = \{\lambda w + \hat{x}: \lambda \text{ is a scalar and } \hat{x} \in \hat{X}\}$, and $\langle w \rangle = \{\lambda w: \lambda \text{ is a scalar}\}$. It is convenient for some purposes to extend the definition of Ω_w to elements $w \in \hat{X}$. We take $\Omega_w = B(X)$ when $w \in \hat{X}$. This is natural, because for every $T \in B(X)$ we have $T^{**}(\hat{X}) \subseteq \hat{X}$.

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Our notation differs from that of Wilansky, in that we write Ω_w instead of Γ_w^a, but our notation agrees with that of Brown and Cho [1], who had previously introduced these subalgebras in the case where $X = c$ (the Banach space of convergent sequences with $\|x\| = \sup_{n\geq 1}|x_n|$, for $x \in c$).

When $w \in X^{**}\setminus\hat{X}$, define $\varrho_w\colon \Omega_w \to C$, where C is the field of complex numbers, by the equation

$$T^{**}w = \varrho_w(T)w + \hat{x}, \quad T \in \Omega_w. \tag{3}$$

In [11] it is shown that for $w \in X^{**}\setminus\hat{X}$, Ω_w is a closed subalgebra of $B(X)$ and ϱ_w is a non-zero continuous scalar homomorphism. Of course if $w \in \hat{X}$, (3) makes no sense; in fact, it is shown in [2] that $B(c)$ does not support a non-zero continuous scalar homomorphism. It is obvious that Γ_w is a closed subalgebra of $B(X)$ for every $w \in X^{**}$.

We now state some results established by Brown and Cho [1], which are valid for the special case $X = c$. Let e be the sequence $\{x_n\}$ where $x_n = 1$ for all n, and e^k be the sequence $\{x_n\}$ where $x_k = 1$ and $x_n = 0$ for $n \neq k$, $k = 1, 2, 3, \ldots$ We make the usual identification of c^{**} with m, the Banach space of bounded sequences with $\|x\| = \sup|x_n|$, for $x \in m$. (See [10], p. 102.) Under this identification $\hat{c}$ is identified with those convergent sequences which converge to their first term. Now $e^1 \in c^{**}\setminus\hat{c}$, $\Gamma_{e^1} = \Gamma$ the algebra of conservative matrices, and $\Omega_{e^1} = \Omega$ the algebra of almost matrices; see [1] and [2]. The following results are established in [1]. The identity operator is denoted by I.

I. If $w \notin \hat{c}$, then $\Omega_w = \Omega_z$ if and only if $z \in (w \oplus \hat{c})\setminus\hat{c}$. ([1], Theorem 5)

II. $w \in \hat{c}$ if and only if $\Omega_w = B(c)$. ([1], Lemma 3)

III. $\Gamma_z = \Gamma$ if and only if $z = \mu e^1$ with $\mu \neq 0$. ([1], Theorem 13)

IV. $\bigcap\{\Omega_w\colon w \in c^{**}\} = \langle I\rangle \oplus K$, where K denotes the two sided ideal of compact operators. ([1], Corollary 11)

V. $\bigcap\{\Gamma_w\colon w \in c^{**}\} = \langle I\rangle$. ([1], Theorem 12)

VI. If $w, z \in \hat{c}$, then Ω_w is isomorphic to Ω_z and Γ_w is isomorphic to Γ_z. ([1], Theorem 10)

Our principal purpose in this paper is to examine these results in $B(X)$. Where they extend, our proofs are simpler than those of Brown and Cho. Summarizing, we find that for general X: I and II are false (Theorem 1); III is true (Corollary 4) and even improved in $B(c)$; IV is true if the compact operators are replaced by the weakly compact operators (Theorem 5); V is true (Theorem 6); VI is false if we take $X = l$, the Banach space of absolutely convergent series with $\|x\| = \sum_{k=1}^{\infty}|x_k|$, for $x \in l$ (Theorem 10).

2. The James space. In [7] James gives an example of a non-reflexive Banach space X for which $\hat{X}$ has codimension 1 in X^{**}. Thus, if $w \in X^{**}\setminus\hat{X}$, we have $X^{**} = w \oplus \hat{X}$ so that $\Omega_w = B(X)$. This establishes the following theorem.

THEOREM 1. *There is a non-reflexive Banach space X for which $\Omega_w = B(X)$ for every $w \in X^{**}\setminus\hat{X}$.*

Remark 1. If a Banach space X has $\Omega_w = B(X)$ for some $w \in X^{**}\setminus\hat{X}$, then $B(X)$ supports the non-zero continuous scalar homomorphism ϱ_w defined by (3). This is in contrast to the case of $B(c)$. See [2].

Remark 2. It is a consequence of Theorem 5, that for the James space X, every $T \in B(X)$ can be uniquely expressed as the sum of a weakly compact operator and a scalar multiple of the identity operator.

3. Simple properties of the subalgebras. The identity operator on X is denoted by I and the kernel of ϱ_w by $\varrho_w^{\perp}$.

THEOREM 2. *Let X be a non-reflexive Banach space. Then*

(a) *$I \in \Gamma_w$ and $\Gamma_w \subseteq \Omega_w$ for $w \in X^{**}$;*

(b) *$\Gamma_0 = B(X)$;*

(c) *if $z \in w \oplus \hat{X}$, then $\Omega_w \subseteq \Omega_z$;*

(d) *if $z \in (w \oplus \hat{X})\setminus\hat{X}$, then $\Omega_w = \Omega_z$;*

(e) *$\Omega_w = \langle I\rangle \oplus \varrho_w^{\perp}$ for $w \in X^{**}\setminus\hat{X}$.*

4. One-dimensional operators in $B(X)$. The one-dimensional operators in $B(X)$ reveal a lot of information about the structure of the subalgebras Ω_w and Γ_w. For $z \in X$ and $f \in X^*$ we define $z \otimes f \in B(X)$ by

$$(z \otimes f)x = f(x)z. \tag{4}$$

The range of $z \otimes f$ is at most one-dimensional, so it is a compact operator. It is clear that if $T \in B(X)$ and its range is at most one-dimensional, then $T = z \otimes f$ for some $z \in X$ and $f \in X^*$.

We now state a lemma and give three simple consequences.

LEMMA 1. *Let $w \in X^{**}$, $f \in X^*$ and $z \in X$. Then*

$$(z \otimes f)^{**}w = w(f)\hat{z}, \tag{5}$$

where $\hat{z}$ is the image of z under the natural embedding of X into X^{**}.

COROLLARY 1. *Let $w \in X^{**}\setminus\hat{X}$ and $z \neq 0$. Then $z \otimes f \in \Gamma_w$ if and only if $w(f) = 0$.*

COROLLARY 2. *If $w \in \hat{X}$ and $w = \hat{z}$, then $z \otimes f \in \Gamma_w$ for every $f \in X^*$.*

COROLLARY 3. *Suppose $w \in \hat{X}$, $w = \hat{y} \neq 0$ and $x \notin \langle y\rangle$. Then $x \otimes f \notin \Gamma_w$ whenever $f(y) \neq 0$.*

Remark 3. In Corollary 3, we are assuming that $w \neq 0$, so that $y \neq 0$ and $w^\perp = \{f \in X^*: w(f) = 0\} \neq X^*$. Thus, there are functionals $f \in X^*$ for which $w(f) = \hat{y}(f) = f(y) \neq 0$.

5. Further properties of the subalgebras.

THEOREM 3. *Let* $w_1, w_2 \in X^{**}$.

(a) *If* $w_1 = \mu w_2$ *where* $\mu \neq 0$, *then* $\Gamma_{w_1} = \Gamma_{w_2}$.

(b) *If* $w_i \neq 0$, $i = 1, 2$, *and* $w_1 \notin \langle w_2 \rangle$, *then* $\Gamma_{w_1} \setminus \Gamma_{w_2} \neq \emptyset$ *and* $\Gamma_{w_2} \setminus \Gamma_{w_1} \neq \emptyset$.

(c) *If* $w_1 \neq 0$, *then* $\Gamma_{w_1} \neq B(X)$.

The proof of Theorem 3 is a routine application of Corollaries 1, 2, and 3.

COROLLARY 4. $\Gamma_{w_1} = \Gamma_{w_2}$ *if and only if there is a number* $\mu \neq 0$ *such that* $w_1 = \mu w_2$.

We can sharpen Theorem 3 (c) to the following

THEOREM 4. *If* $w \in X^{**}$ *and* $w \neq 0$, *then* $\Gamma_w \neq \Omega_w$. (*Of course, we always have* $\Gamma_w \subseteq \Omega_w$ *by Theorem* 2 (a).)

To facilitate the proof of Theorems 5 and 6, we observe the following lemma and corollary.

LEMMA 2. *Let* $T \in \bigcap \{\Omega_w: w \in X^{**} \setminus \hat{X}\}$. *Then* $\varrho_w(T)$ *has the same value for every* $w \in X^{**} \setminus \hat{X}$.

COROLLARY 5. $\bigcap \{\Omega_w: w \in X^{**} \setminus \hat{X}\} = \langle I \rangle \oplus \bigcap \{\varrho_w^\perp: w \in X^{**} \setminus \hat{X}\}$.

Let W denote the two-sided ideal of weakly compact operators in $B(X)$.

THEOREM 5. (a) $W = \bigcap \{\varrho_w^\perp: w \in X^{**} \setminus \hat{X}\}$;

(b) $\langle I \rangle \oplus W = \bigcap \{\Omega_w: w \in X^{**} \setminus \hat{X}\}$.

Proof. (a) In [3], p. 482, Theorem 2, it is shown that $T \in W$ if and only if $T^{**}(X^{**}) \subseteq \hat{X}$, from which (a) follows simply. Conclusion (b) follows from (a) and Corollary 5. See [11] (Theorem 3) in connection with Theorem 5 (a).

Remark 4. Since weak and strong sequential convergence are equivalent in c^* $(= l)$, it follows that in $B(c)$ the weakly compact operators and the compact operators are the same. Thus Theorem 5 includes as a special case [1], Corollary 11,

THEOREM 6. $\bigcap \{\Gamma_w: w \in X^{**}\} = \bigcap \{\Gamma_w: w \in X^{**} \setminus \hat{X}\} = \langle I \rangle$.

6. Some results concerning commutants. For $T \in B(X)$, define the *commutant* of T, $\mathrm{Com}\, T$ by

$$\mathrm{Com}\, T = \{S \in B(X): TS = ST\}.$$

It is clear that $\mathrm{Com}\, T$ is a closed subalgebra of $B(X)$.

THEOREM 7. *Let* $w \in X^{**}$ *and* $T \in B(X)$. *If* T *commutes with every (one-dimensional) compact operator in* Γ_w, *then* $T \in \langle I \rangle$, *and* $\mathrm{Com}\, T = B(X)$.

Proof. Suppose that $T \notin \langle I \rangle$. Choose an $x \neq 0$ such that $Tx \notin \langle x \rangle$. Let $w \in X^{**} \setminus \hat{X}$. Now $w^\perp$ is total over X (see [10], p. 104, exercise 2) so that there is a functional $f \in w^\perp$ with $f(x) \neq 0$. Then $x \otimes f \in \Gamma_w$ but $x \otimes f \notin \mathrm{Com}\, T$.

For $w \in \hat{X}$ use Corollary 2 and the Hahn–Banach theorem.

COROLLARY 6. (a) $\mathrm{Com}\, T = B(X)$ *if and only if* $T \in \langle I \rangle$.

(b) Ω_w *is the commutant of an operator if and only if* $\Omega_w = B(X)$.

(c) Γ_w *is the commutant of an operator if and only if* $w = 0$. (Use Theorem 3 (c).)

(d) *If* $w \in c^{**} \setminus \hat{c}$, *then* Ω_w *is not the commutant of any operator in* $B(c)$. (Since $\Omega_w \neq B(c)$, see [1].)

7. The nature of isomorphisms between certain subalgebras of $B(X)$. Let A_1 and A_2 be subalgebras of $B(X)$. We say that A_1 and A_2 are *isomorphic as algebras*, and we write $A_1 \cong A_2$, if a bijective linear transformation $\varphi: A_1 \to A_2$ exists which satisfies $\varphi(ST) = \varphi(S)\varphi(T)$ for all S, $T \in A_1$. Such a transformation φ is called an *algebra isomorphism*.

We wish to determine the form of an arbitrary algebra isomorphism $\varphi: \Gamma_w \to \Gamma_z$, where $w, z \in X^{**} \setminus \hat{X}$. We first show that φ is necessarily continuous, and hence, by the open mapping theorem, a homeomorphism. We give a definition and three lemmas.

Let X be a Banach space and $\mathscr{S}$ a total linear subspace of X^*. For $x \in X$, define $\|x\|_{\mathscr{S}} = \|\hat{x} \mid \mathscr{S}\| = \sup\{|f(x)|: \|f\| \leqslant 1 \text{ and } f \in \mathscr{S}\}$. If $\| \ \|_{\mathscr{S}}$ is equivalent to the original norm in X, we say that $\mathscr{S}$ is *norming*. See [10], p. 105, exercise 24.

LEMMA 3. *Let* X *be a non-reflexive Banach space and let* $w \in X^{**} \setminus \hat{X}$. *Then* $w^\perp$ *is norming*.

Proof. Now $w^\perp$ is total, because for $x \in X$ with $x \neq 0$, we have $w^\perp \setminus x^\perp \neq \emptyset$. Also $\hat{X} \oplus w^{\perp\perp}$ is closed in X^{**}, because $\hat{X}$ is closed and $w^{\perp\perp}$ is one-dimensional. Here $w^{\perp\perp} = \{q \in X^{**}: q(f) = 0 \text{ for all } f \in w^\perp\}$. The result now follows from [10], p. 201, exercise 20.

LEMMA 4. *Suppose that* $\{y_n\}$ *is a sequence in a non-reflexive Banach space* X, $\|y_n\| \to \infty$ *as* $n \to \infty$, *and* $w \in X^{**} \setminus \hat{X}$. *Then there is a functional* $f \in w^\perp$ *such that* $\limsup_{n\to\infty} |f(y_n)| = \infty$.

Proof. Suppose $\sup_{n\geqslant 1} |f(y_n)| = \sup_{n\geqslant 1} |\hat{y}_n(f)| < \infty$ for all $f \in w^\perp$. Now $\hat{y}_n$ restricted to the Banach space $w^\perp$ is a member of $(w^\perp)^*$. The Banach–Steinhaus theorem shows that $\sup_{n\geqslant 1} \|y_n\|_{w^\perp} < \infty$. But since $w^\perp$ is norming, we must have $\|y_n\|_{w^\perp}\to\infty$ because $\|y_n\|\to\infty$. This contradiction yields the result.

The next lemma is a modification of Lemma 1 of [4].

LEMMA 5. *A set $H\subseteq\Gamma_w$ is bounded if and only if the following condition is satisfied. For every $S\in\Gamma_w$, there is a number $a>0$ such that, if λ is a real number satisfying $|\lambda|<a$ and $U\in H$, then $I-\lambda SU$ has an inverse in Γ_w.*

Proof. *Necessity*: Suppose $\|U\|\leqslant M$ for all $U\in H$ and let $S\in\Gamma_w$. We may suppose $S\neq 0$ since I is invertible. Put $a=1/M\|S\|$. Then $\|\lambda SU\|<a\|S\|\,M=1$ for $|\lambda|<a$ and $U\in H$, and hence $I-\lambda SU$ has an inverse. See [10], p. 259, fact (ii).

Sufficiency: Suppose H is not bounded and let $\{U_n\}$ be a sequence of elements of H with $\|U_n\|\to\infty$. The Banach–Steinhaus theorem implies the existence of $x_0\in X$ such that $\lim_{n\to\infty}\|U_n(x_0)\|=\infty$. Lemma 4, with $y_n=U_n(x_0)$ now gives a function $f\in x^\perp$ with $\limsup_{n\to\infty}|f(U_n(x_0))|=\infty$.

Choose $a>0$ arbitrarily, let $\lambda_n=1/f(U_n(x_0))$, choose an integer N such that $|\lambda_N|<a$, and let $S=x_0\otimes f$. Then $S\in\Gamma_w$ by Corollary 1. Since $(I-\lambda_N SU_N)x_0=0$, and $x_0\neq 0$, it follows that $I-\lambda_N SU_N$ has no inverse.

THEOREM 8. *If $\varphi\colon\Gamma_w\to\Gamma_z$ is an algebra isomorphism, then φ is continuous.*

Proof. Let H be a bounded subset of Γ_w. Then, by Lemma 5, for every $S\in\Gamma_w$ there is a number $a>0$ such that if λ is real and $|\lambda|<a$, we have $I-\lambda SU$ invertible for every $U\in H$. Hence $\varphi(I)-\lambda\varphi(S)\varphi(U)=I-\lambda\varphi(S)\varphi(U)$ is invertible in Γ_z. It now follows from Lemma 5 that $\varphi(H)$ is bounded in Γ_z. Thus φ is continuous.

We now give the following modification of Lemma 2 of [4].

LEMMA 6. *Let X be a non-reflexive Banach space and $w\in X^{**}\setminus\hat{X}$. An operator $U_0\in\Gamma_w$ is at most one dimensional if and only if for every $U\in\Gamma_w$ there is a scalar λ such that*

$$(UU_0)^2=\lambda UU_0. \tag{6}$$

Proof. *Necessity*: It is clear that for any zero-dimensional or one-dimensional operator V, there is a scalar λ such that $V^2=\lambda V$. But U_0 being an at most one-dimensional operator, UU_0 is always at most one-dimensional so (6) follows.

Sufficiency: Suppose that the range of U_0 contains two linearly independent elements y_1 and y_2 with

$$U_0x_1=y_1,\quad U_0x_2=y_2,\quad x_1,x_2,y_1,y_2\in X.$$

Now w, $\hat{y}_1$, and $\hat{y}_2$ are linearly independent vectors in X^{**}. It follows from [10], Theorem 3, p. 39, that $\hat{y}_2^\perp\not\supset w^\perp\cap\hat{y}_1$. Thus we can choose $f_1\in X^*$ such that $f_1\in(w\cap\hat{y}_1)\setminus\hat{y}_2$. Thus, after replacing f_1 by a suitable constant multiple of f_1, we have

$$w(f_1)=0,\quad f_1(y_1)=0\quad\text{and}\quad f_1(y_2)=1.$$

Similarly we can choose $f_2\in(w\cap\hat{y}_2^\perp)\setminus\hat{y}_1^\perp$ so that

$$w(f_2)=0,\quad f_2(y_2)=0\quad\text{and}\quad f_2(y_1)=1.$$

But

$$U=x_1\otimes f_1+x_2\otimes f_2.$$

Corollary 1 shows that $U\in\Gamma_w$. Now we have $UU_0x_1=x_2$ and $UU_0x_2=x_1$; hence, on setting $V=(UU_0)^2$, we have $Vx_2=x_2$. But (6) implies $Vx_2=\lambda U[U_0x_2]=\lambda x_1$. Thus, $x_2=\lambda x_1$, consequently $y_2=\lambda y_1$ contrary to the assumption that y_1 and y_2 are linearly independent.

We are now able to prove the main theorem of this section.

THEOREM 9. *Let X be a non-reflexive Banach space and let $w,z\in X^{**}\setminus\hat{X}$. If $\varphi\colon\Gamma_w\to\Gamma_z$ is an algebra isomorphism, then there is a linear homeomorphism $T\in B(X)$ such that*

$$\varphi(U)=TUT^{-1}\quad\text{for}\quad U\in\Gamma_w. \tag{7}$$

Proof. We determine T just as in [4], Theorem 2. Choose $f_0\in w^\perp$ and $x_0\in X$ such that $f_0(x_0)=1$. Put

$$U_0x=f_0(x)x_0=(x_0\otimes f_0)x\quad\text{for}\quad x\in X.$$

Thus $U_0\in\Gamma_w$. Consider $V_0=\varphi(U_0)$. Since U_0 is one-dimensional, we have by Lemma 6 and the properties of φ that V_0 is a one-dimensional operator in Γ_z. Say

$$V_0y=g_0(y)y_0=(y_0\otimes g_0)y\quad\text{for}\quad y\in X, \tag{8}$$

where $y_0\in X$ and $g_0\in X^*$. Since $U_0\neq 0$, we have $V_0\neq 0$, hence $y_0\neq 0$ and $g_0\neq 0$. By Corollary 1 we also have $z(g_0)=0$. We define the operator T as follows. Let $x\in X$; we choose an operator $U\in\Gamma_w$ (for example, $x\otimes f_0$) such that $Ux_0=x$ and set $Tx=Vy_0$ where $V=\varphi(U)$ and y_0 is determined by (8). Exactly as in [4] we have that T is well defined, bijective, linear, and continuous. T is thus a linear homeomorphism in $B(X)$. Moreover, just as in [4] we have (7) holds.

COROLLARY 7. *Let X be a non-reflexive Banach space and let $w,z\in X^{**}\setminus\hat{X}$. An isomorphism $\varphi\colon\Gamma_w\to\Gamma_z$ induces an inner automorphism $\tilde{\varphi}\colon B(X)\to B(X)$ such that $\tilde{\varphi}(\Omega_w)=\Omega_z$.*

Proof. If T is the linear homeomorphism constructed in Theorem 9, we show that $T^{**}w=\mu z$ for some $\mu\neq 0$. For if $T^{**}w\notin\langle z\rangle$, then $T^{**-1}(z)\notin\langle w\rangle$. Let $w_1=T^{**-1}(z)$ where $w_1\notin\langle w\rangle$. Choose $f\in w^\perp\setminus w_1^\perp$ and $x\neq 0$. Then $x\otimes f\in\Gamma_w$ but $T^{**}(x\otimes f)^{**}T^{**-1}(z)=T^{**}w_1(f)\hat{x}\notin\langle z\rangle$.

So $T(x\otimes f)T^{-1} \notin \Gamma_z$, i.e., $\varphi(x\otimes f) \notin \Gamma_z$ which is a contradiction. Thus $T^{**}w = \mu z$ and $\mu \neq 0$, because $w \neq 0$ and T^{**} is an isomorphism. Now let $S \in \Omega_w$ and $S^{**}w = \varrho_w(S)w + \hat{x}$. Then $(\tilde{\varphi}(S))^{**} = T^{**}S^{**}T^{**-1}(z) = \varrho_w(S)z + \widehat{Tx}$. So $\tilde{\varphi}(S) \in \Omega_z$. Similarly, $\tilde{\varphi}^{-1}(\Omega_z) \subseteq \Omega_w$. Thus, $\tilde{\varphi}(\Omega_w) = \Omega_z$.

8. The subalgebras Ω_w and Γ_w for $B(l)$. We make the usual identification of l^* with m (see, for example, [10], p. 91); then l^{**} is identified with m^*. Now if $t = \{t_k\} \in l$ and $x = \{x_k\} \in m$, we have $\hat{t}(x) = x(t) = \sum_{k=1}^{\infty} t_k x_k$. Thus if $t \neq 0$ so that $t_k \neq 0$ for some k we have $\hat{t}(e^k) = t_k \neq 0$. Hence we have proved the following lemma.

LEMMA 7. *If $f \in l^{**}$ and $f(x) = \lim x$ for $x \in c$, then $f \in l^{**}\setminus\hat{l}$.*

We now give some examples of functionals in $l^{**}\setminus\hat{l}$. Let βN denote the Stone–Čech compactification of the natural numbers N. (For expositions on βN, the reader is referred to [5], [8] or [12].) Let $t \in \beta N\setminus N$ and $x \in m$. Now x can be regarded as a bounded continuous complex function defined on N. Thus x has a unique extension $\tilde{x}$ to βN. Define

$$\operatorname{Lim}_t x = \tilde{x}(t). \tag{9}$$

It is elementary to show that $\operatorname{Lim}_t \in m^*$ and that $\operatorname{Lim}_t x = \lim x$ for $x \in c$ (see [10], p. 270). Thus $\operatorname{Lim}_t \in l^{**}\setminus\hat{l}$ by Lemma 7. It is also clear, since βN is a compact Hausdorff space, that for $t_1, t_2 \in \beta N\setminus N$ with $t_1 \neq t_2$ we must have $\operatorname{Lim}_{t_1} x \neq \operatorname{Lim}_{t_2} x$ for some $x \in m$. However, $\operatorname{Lim}_{t_1} x = \operatorname{Lim}_{t_2} x = \lim x$ for $x \in c$ so that Lim_{t_1} and Lim_{t_2} are linearly independent members of $l^{**}\setminus\hat{l}$. The cardinality of $\beta N\setminus N$ is $2^{\mathfrak{c}}$ where $\mathfrak{c}$ denotes the cardinality of the continuum (see [5], p. 139, 90). Thus the cardinality of $l^{**}\setminus\hat{l}$ is at least $2^{\mathfrak{c}}$.

The Knopp–Lorentz theorem [6] shows that every $T \in B(l)$ is given by an infinite matrix $\{a_{nk}\}$ in the following manner.

$$Tx = \Big\{\sum_{k=1}^{\infty} a_{nk}x_k\Big\}, \quad x \in l,$$

where $\|T\| = \sup_{k\geqslant 1}\sum_{n=1}^{\infty}|a_{nk}| < \infty$. Thus the cardinality of $B(l)$ is $\mathfrak{c}$.

The above considerations together with Theorem 9 and Corollary 7 yield the following result.

THEOREM 10. *There are points $w, z \in l^{**}\setminus\hat{l}$ such that Γ_w is not isomorphic to Γ_z.*

We can also give a 'positive' result. Let $f\colon \beta N\to\beta N$ be a homeomorphism. Then f induces a permutation π of N. Define $\varphi\colon B(l)\to B(l)$ as follows. If $T \in B(l)$ and T is given by the matrix $\{a_{nk}\}$, set $\varphi(T) = S$

where S is given by the matrix $\{b_{nk}\}$ with $b_{nk} = a_{\sigma(n)\sigma(k)}$ where $\sigma = \pi^{-1}$. It is easy to check that φ is an algebra isomorphism which is also an isometry. It can also be checked that if $t_1 \in \beta N\setminus N$ and $t_2 = f(t_1)$, then $\varphi(\Omega_{\operatorname{Lim}_{t_1}}) = \Omega_{\operatorname{Lim}_{t_2}}$ and $\varphi(\Gamma_{\operatorname{Lim}_{t_1}}) = \Gamma_{\operatorname{Lim}_{t_2}}$ where Lim_{t_1} and Lim_{t_2} are defined as in (9).

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