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A version of the Harris–Spitzer "random constant velocity" model for infinite systems of particles

by

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Abstract. In this paper a one-dimensional system of infinitely many elastic particles is considered. If the initial positions and velocities are independent random variables, then the actual motion of the 0th particle converges to the Gaussian process, which is in general non-Markovian.

0. Introduction. We shall consider a system of particles with equal masses (point masses) on the real line. This system will be one with a "random constant velocity", i.e. the position $x_k(t)$ of the kth trajectory at time $t \geqslant 0$ (if the particles can penetrate each other) is described by the formula

$$x_k(t) = x_k + v_k \cdot t \quad \text{for} \quad k = 0, \pm 1, \ldots, \; t \geqslant 0,$$

where $\{x_k - k\}_{-\infty}^{+\infty}$, $\{v_k\}_{-\infty}^{+\infty}$ are independent systems of independent random variables identically distributed in each of the systems.

We shall consider the billiard-ball case, i.e. whenever two particles meet we assume that they collide *elastically*, that is, the collision conserves the energy and momentum. This implies that they simply exchange trajectories.

If $E[x_k - k] = 0$ and $E[v_k] = 0$, we define, by the deterministic theorem of Harris [6], the actual motion of the kth elastic particle $y_k(t)$.

We restrict our attention to the trajectory $y(t) = y_0(t)$. In this model, which we call *model* D, we shall prove the convergence of the finite-dimensional distributions of the processes $Y_A(t) := y(At)/A^{1/2}$, $t \geqslant 0$, to the joint distributions of the Gaussian process $X(t)$, as $A \to \infty$, with

$$E[X(t)] = 0,$$

$$E[X(t) \cdot X(s)] = \min(t, s) E|v| - E[\min(tu^-, sv^-) + \min(tu^+, sv^+)];$$

here u and v are independent random variables, with the same distribution law as v_k's, and

$$a^- := -\min(0, a), \quad a^+ := \max(0, a).$$

The process $X(t)$, $t \in [0, 1]$, can be realized on $C_{[0,1]}$.

In the case of symmetric distributions of the variables $x_k - k$ and v_k we shall prove, moreover, that $Y_\Delta(\cdot) \Rightarrow X(\cdot)$ in the interval $[0, 1]$, where $\Rightarrow$ means weak convergence in the space of measures on the space of continuous functions on $[0, 1]$. In this symmetric case the correlation function of the process $X(\cdot)$ is simply

$$\min(t, s)E[|v|] - \tfrac{1}{2}E\big[\min[t|u|, s|v|]\big].$$

The process $X(\cdot)$ then has the following structure:

$$(E|v|)^{1/2}W(t) = X(t) + 2^{-1/2}Z(t),$$

where $Z(t, \omega) = \int Y(t|v|, \omega)P(dv)$, and the processes $Y(\cdot)$, $W(\cdot)$ are standard Brownian motions, the processes $X(\cdot)$ and $Z(\cdot)$ being independent. Physically, weak convergence gives the convergence of the observables.

In some cases covariance can be marked out explicitly. We shall do this when the distribution of velocities U is normal.

We shall also discuss the non-Markovian character of the process $X(\cdot)$ and we shall prove the non-differentiability of its trajectories.

Finally we compare this model with the model of Harris–Spitzer [6], in which $\{x_k\}_{-\infty}^{+\infty}$ is a Poisson system.

This paper is an extension of the author's lecture given at the 4th Probability Winter School in Karpacz [11] and was inspired by the note of Frank Spitzer in [9].

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1. Definition and existence of collision processes. Let us assume that

(D1) $x_0, v_0, x_1, v_1, x_{-1}, v_{-1}, \ldots$ *are independent random variables.*

Let the distributions $\mathcal{N}_k$ of the random variables $x_k - k$ and the distributions $\mathcal{U}_k$ of the random variables v_k, $k = 0, \pm 1, \pm 2, \ldots$, satisfy the following assumptions:

(D2) $\mathcal{N}_k = \mathcal{N}$ *for* $k = \pm 1, \pm 2, \ldots$, *where* $E[x_k - k] = 0$ *and* $\mathcal{N}_0$ *is concentrated at the origin.*

(D3) $\mathcal{U}_k = \mathcal{U}$ *for* $k = 0, \pm 1, \pm 2, \ldots$, *where* $E(v_k) = 0$.

(D4) *The equality* $P(x_i + v_i \cdot t = x_j + v_j \cdot t$ *for some interval* $[0, T]) = 0$ *holds for all pairs* $i \neq j$, $i, j = 0, \pm 1, \ldots$

Our model will be realized on the probability space of the initial conditions

$$(\Omega, \mathfrak{F}, P) = \prod_{k=-\infty}^{+\infty} (\boldsymbol{R}^2, \mathfrak{B}, \gamma_k),$$

where $\mathfrak{B}$ is the Borel field in the plane $\boldsymbol{R}^2$ and $\gamma_k = \mathcal{N}_k^{(1)} \otimes \mathcal{U}$, where $\mathcal{N}_k^{(1)}$ is the distribution of x_k.

We restrict our attention to $\Omega_0 \subset \Omega$ such that for every $\omega \in \Omega_0$ and $l, n = 0, 1, 2, \ldots$ we have

$$\sum_{k=-\infty}^{-1} \chi(x_k + v_k \cdot l > -n) < +\infty \quad \text{and} \quad \sum_{k=1}^{\infty} \chi(x_k + v_k \cdot l < n) < +\infty.$$

But it follows from the finitness of the expectations $E[x_k] = E[x] + k \equiv k$ and $E[v_k] = E[v] \equiv 0$ that

$$\sum_{k=1}^{\infty} P(x_k + v_k \cdot l < n) = \sum_{k=1}^{\infty} P(x + v \cdot l - n < -k) < +\infty,$$

where x and v are independent random variables with the distribution laws $\mathcal{N}$ and $\mathcal{U}$, respectively. Similarly,

$$\sum_{k=-\infty}^{-1} P(x_k + v_k \cdot l > -n) < +\infty.$$

Hence, from the Borel–Cantelli lemma, we immediately infer that $P(\Omega_0) = 1$. It is easy to see that the set $\bigcup_{k=-\infty}^{+\infty} \{x_k(\omega)\}$ has no point of concentration for any $\omega \in \Omega_0$. Let us order by $\prec$ the points of $\{x_k\}$:

$$x_k \prec x_j \quad \text{iff} \quad [x_k < x_j \text{ or } (x_k = x_j \text{ and } k < j)].$$

Since the set $\{x_k(\omega)\}_{k=-\infty}^{\infty}$ has no point of concentration for any $\omega \in \Omega_0$, the order $\prec$ allows such a numbering $k \to n(k)$ that

$$\ldots \leqslant \tilde{x}_{-1} \leqslant \tilde{x}_0 = 0 \leqslant \tilde{x}_1 \leqslant \ldots,$$

where

$$\tilde{x}_k := x_{n(k)} \quad \text{for} \quad k = 0, \pm 1, \ldots$$

Let us write $\tilde{v}_k = v_{n(k)}$ for $k = 0, \pm 1, \ldots$, and define

$$\tilde{x}_k(t) = \tilde{x}_k + \tilde{v}_k \cdot t \quad \text{for} \quad t \geqslant 0.$$

For every $\omega \in \Omega_0$ the family of continuous functions $\tilde{x}_k(t)$, $t \geqslant 0$, satisfies the following assumptions of the deterministic theorem of Harris [6], i.e.

(H1) $\tilde{x}_i(0) \leqslant \tilde{x}_{i+1}(0)$ *for* $i = 0, \pm 1, \ldots$ *and* $x_0(0) = 0$.

(H2) *For each* i *and* $T \geqslant 0$, $\inf_{0 \leqslant t \leqslant T} \tilde{x}_i(t)$ *and* $\sup_{0 \leqslant t \leqslant T} \tilde{x}_i(t)$ *are finite and satisfy*

$$\lim_{i\to\infty} \inf_{0\leqslant t\leqslant T} \tilde{x}_i(t) = +\infty, \quad \lim_{i\to-\infty} \sup_{0\leqslant t\leqslant T} \tilde{x}_i(t) = -\infty.$$

(H3) *If* $i \neq j$, *then* $\{t\colon \tilde{x}_i(t) = \tilde{x}_j(t)\}$ *does not contain a (non-degenerate) interval.*

It remains to prove only (H2). In our model, (H2) is equivalent to the conditions:

$$\lim_{k\to\infty} \min(\tilde{x}_k(T), \tilde{x}_k(0)) = +\infty,$$

$$\lim_{k\to-\infty} \max(\tilde{x}_k(t), \tilde{x}_k(0)) = -\infty.$$

But this follows immediately from the definition of Ω_0. Following Harris [6], if we define $y_i(t)$ by the limit $y_i(t) = \lim_{n\to\infty} \operatorname{med}(x_{i-n}(t), \ldots, x_{i+n}(t))$ where med is the median of the $2n+1$ trajectories $x_{i-n}(t), \ldots, x_{i+n}(t)$, then $y_i(\cdot)$, $i = 0, \pm 1, \ldots$, is the unique set of functions with the following properties:

(Y1) $y_i(0) = x_i(0)$ *for* $i = 0, \pm 1, \ldots$

(Y2) $y_i(t) \leqslant y_{i+1}(t)$ *for* $t \geqslant 0$, $i = 0, \pm 1, \ldots$

(Y3) *For each* $t \geqslant 0$

$$\lim_{i\to\infty} \inf_{0\leqslant\tau\leqslant y} y_i(\tau) = +\infty,$$

$$\lim_{i\to-\infty} \sup_{0\leqslant\tau\leqslant t} y_i(\tau) = -\infty.$$

(Y4) *The union of the graphs of the* $y_i(\cdot)$ *is identical to the union of the graphs of the* $x_i(\cdot)$.

(Y5) *For* $i \neq j$, *the set* $\{t\colon y_i(t) = y_j(t)\}$ *does not contain any interval.*

(Y6) *The* $y_i(\cdot)$ *are continuous at* t.

The functions $y_i(t)$ may be viewed as an actual motion of the ith (after ordering) elastic particle in a one-dimensional chaotic bath formed by the remaining particles.

2. Convergence of the joint distributions. Let $y(t) = y_0(t)$ be the "particle" under observation. Assuming that conditions (D1)–(D4) are satisfied, we shall prove the following

THEOREM 1. *The finite-dimensional distributions of the processes* $Y_A(t) = y(At)/A^{1/2}$, $t \geqslant 0$, *where* $A \to \infty$, *converge to the joint distributions of the Gaussian process* $X(\cdot)$ *with*

$$E[X(t)] = 0 \quad \text{for} \quad t \geqslant 0,$$

$$E[X(t)\cdot X(s)] = \min(t, s) E|v| - E[\min(tu^-, sv^-) + \min(tu^+, sv^+)], \; t, s \geqslant 0,$$

where u *and* v *are independent random variables, with the same distribution law* U *(cf.* (D2)).

Proof. Following Spitzer [8], it suffices to show that for every $m \in N = \{1, 2, \ldots\}$ the distribution of the random vector with m components,

$$A^{-1/2}\Big\{\sum_{k=1}^{\infty} \chi(\tilde{x}_k + \tilde{v}_k A t_i < A^{1/2}\alpha_i) - \sum_{k=-\infty}^{-1} \chi(\tilde{x}_k + \tilde{v}_k A t_i > A^{1/2}\alpha_i)\Big\} - \alpha_i, \quad 1 \leqslant i \leqslant m,$$

approaches the m-dimensional Gaussian distribution with the mean and covariance matrix as above corresponding to $(t_1, \ldots, t_m)$. But an asymptotical behavior of this random vector on Ω_0 is the same as that of the random vector with m components

$$A^{-1/2}\Big\{\sum_{k=1}^{\infty} \chi(x_k + v_k A t_i < A^{1/2}\alpha_i) - \sum_{k=-\infty}^{-1} \chi(x_k + v_k A t_i \geqslant A^{1/2}\alpha_i)\Big\} - \alpha_i.$$

Thus we shall work on sums of independent random vectors. We use the following multidimensional central limit theorem:

For every $k = 1, 2, \ldots, i_n$ *let* $(\vec{X}^{(n,k)}) = (X_1^{(n,k)}, \ldots, X_m^{(n,k)})$ *be independent random vectors. If*

(C1) $$\exists_{\vec{b}=(\beta_1,\ldots,\beta_m)} \; \forall_{1\leqslant j\leqslant m} \; \sum_{k=1}^{i_n} E(X_j^{(n,k)}) - \beta_j \xrightarrow[n\to\infty]{} 0,$$

(C2) $$\exists_{[\sigma_{jl}]} \; \forall_{1\leqslant j,l\leqslant m} \; \sum_{k=1}^{i_n} E\big((X_j^{(n,k)} - E(X_j^{(n,k)}))(X_l^{(n,k)} - E(X_l^{(n,k)}))\big) \xrightarrow[n\to\infty]{} \sigma_{jl},$$

(C3) $$\forall_{\varepsilon>0} \; \forall_{1\leqslant j\leqslant m} \; \sum_{k=1}^{i_n} E\{(X_j^{(n,k)} - E(X_j^{(n,k)}))^2 \chi(|X_j^{(n,k)} - E(X_j^{(n,k)})| > \varepsilon)\} \xrightarrow[n\to\infty]{} 0,$$

then

$$\sum_{k=1}^{i_n} (\vec{X}^{(n,k)} - \vec{b}) \underset{n\to\infty}{\Rightarrow} N(\vec{0}, [\sigma_{jl}]).$$

Let

$$X_j^{(n,2k-1)} = \chi(x_k + v_k A_n t_j < A_n^{1/2}\alpha_j)\cdot A_n^{-1/2},$$

$$X_j^{(n,2k)} = -\chi(x_{-k} + v_{-k} A_n t_j \geqslant A_n^{1/2}\alpha_j)\cdot A_n^{-1/2},$$

$k = 1, 2, \ldots$, and let A_n tend to infinity. First we shall check the corresponding conditions $(C1)^\infty$, $(C2)^\infty$, $(C3)^\infty$ for the infinite systems $(X_j^{(n,k)}, k = 1, 2, \ldots)$, $n \geqslant 1$.

$(C3)^\infty$ is obvious.

$(C1)^\infty$ means: For every real number α, $0 \leqslant t \leqslant 1$,

$$A^{-1/2}\Big\{E\Big[\sum_{k=1}^{\infty}\chi(x_k+v_kAt < A^{1/2}\alpha)+\sum_{k=-\infty}^{-1}\big(\chi(x_k+v_kAt < A^{1/2}\alpha)-1\big)\Big]\Big\}\to\alpha$$

as $A\to\infty$.

We shall write $[\cdot]$ as an integer part and use the notation

$$g_A \Leftrightarrow f_A \quad \text{iff} \quad \lim_{A\to\infty} A^{-1/2}\cdot g_A = \lim_{A\to\infty} A^{-1/2}\cdot f_A.$$

We have

$$E\Big\{\sum_{k=1}^{\infty}\chi(x_k+v_kAt < A^{1/2}\alpha)+\sum_{k=-\infty}^{-1}\big(\chi(x_k+v_kAt < A^{1/2}\alpha)-1\big)\Big\}$$

$$\Leftrightarrow -\sum_{k=-\infty}^{[A^{1/2}\alpha]+1} P(k+x+vAt \geqslant A^{1/2}\alpha)+\sum_{k=[A^{1/2}\alpha]+1}^{\infty} P(k+x+vAt < A^{1/2}\alpha)+$$

$$+\sum_{k=1}^{[A^{1/2}\alpha]} P(k+x+vAt \geqslant A^{1/2}\alpha)+\sum_{k=1}^{[A^{1/2}\alpha]} P(k+x+vAt < A^{1/2}\alpha)$$

$$\Leftrightarrow A^{1/2}\alpha+\sum_{k=1}^{\infty}P(x+vAt < -k)-\sum_{k=1}^{\infty}P(x+vAt \geqslant k)$$

$$\Leftrightarrow A^{1/2}\alpha+E\{(x+vAt)^-\}-E\{(x+vAt)^+\} \Leftrightarrow A^{1/2}\alpha.$$

Now we shall prove $(C2)^\infty$. Here we shall use the notation

$$g_A \Leftrightarrow f_A \quad \text{iff} \quad \lim_{A\to\infty} A^{-1}(g_A) = \lim_{A\to\infty} A^{-1}(f_A).$$

We have to prove that, for every α, β, $0 \leqslant t, s \leqslant 1$,

$$\Big(\sum_{k=1}^{\infty}P(x_k+v_kAt < A^{1/2}\alpha,\ x_k+v_kAs < A^{1/2}\beta)+$$

$$+\sum_{k=-\infty}^{-1}P(x_k+v_kAt \geqslant A^{1/2}\alpha,\ x_k+v_kAs \geqslant A^{1/2}\beta)\Big)-$$

$$-\Big\{\sum_{k=1}^{\infty}P(x_k+v_kAt < A^{1/2}\alpha)P(x_k+v_kAs < A^{1/2}\beta)+$$

$$+\sum_{k=-\infty}^{-1}P(x_k+v_kAt \geqslant A^{1/2}\alpha)P(x_k+v_kAs \geqslant A^{1/2}\beta)\Big\}$$

$$\to\min(t, s)E|v|-E[\min(tu^-, sv^-)+\min(tu^+, sv^+)]$$

as A tends to infinity. Assuming x, y, u, v to be independent random variables, x, y have the distribution $\mathcal{N}$ and u, v have the distribution $\mathcal{U}$. Now the expresion in the parentheses ( ) is equal to:

$$\sum_{k=1}^{\infty}P(x+vAt-A^{1/2}\alpha < -k,\ x+vAs-A^{1/2}\beta < -k)+$$

$$+\sum_{k=1}^{\infty}P(x+vAt-A^{1/2}\alpha \geqslant k,\ x+vAs-A^{1/2}\beta \geqslant k)$$

$$=\sum_{k=1}^{\infty}P\big(\max^-(x+vAt-A^{1/2}\alpha,\ x+vAs-A^{1/2}\beta)\geqslant k\big)+$$

$$+\sum_{k=1}^{\infty}P\big(\min^+(x+vAt-A^{1/2}\alpha,\ x+vAs-A^{1/2}\beta)\geqslant k\big)$$

$$\Leftrightarrow E[\max^-(x+vAt-A^{1/2}\alpha,\ x+vAs-A^{1/2}\beta)]+$$

$$+E[\min^+(x+vAt-A^{1/2}\alpha,\ x+vAs-A^{1/2}\beta)]$$

$$\Leftrightarrow E[\max^-(vAt, vAs)+\min^+(vAt, vAs)] \Leftrightarrow A\min(t, s)\cdot E|v|.$$

The expresion in the braces $\{\cdot\}$ is equal to:

$$\sum_{k=1}^{\infty}P(x+vAt-A^{1/2}\alpha < -k)\cdot P(x+vAs-A^{1/2}\beta < -k)+$$

$$+\sum_{k=1}^{\infty}P(x+vAt-A^{1/2}\alpha \geqslant k)\cdot P(x+vAs-A^{1/2}\beta \geqslant k)$$

$$=\sum_{k=1}^{\infty}P\big(\max^-(x+uAt-A^{1/2}\alpha,\ y+vAs-A^{1/2}\beta) > k\big)+$$

$$+\sum_{k=1}^{\infty}P\big(\min^+(x+uAt-A^{1/2}\alpha,\ y+vAs-A^{1/2}\beta)\geqslant k\big)$$

$$\Leftrightarrow E[\max^-(x+uAt-A^{1/2}\alpha,\ y+vAs-A^{1/2}\beta)+$$

$$+\min^+(x+uAt-A^{1/2}\alpha,\ y+vAs-A^{1/2}\beta)]$$

$$\Leftrightarrow A\cdot E[\max^-(tu, sv)+\min^+(tu, sv)]$$

$$= A\cdot E[\min(tu^-, sv^-)+\min(tu^+, sv^+)].$$

From the definition of Ω_0 let i_n be such that, for every $1 \leqslant j \leqslant m$, $n \geqslant 1$,

$$\sum_{k=i_n+1}^{\infty} P(X_j^{(n,k)} \neq 0) < \frac{1}{n},$$

and for every $1 \leqslant j,\ l \leqslant m$, $n \geqslant 1$,

$$\sum_{k=i_n+1}^{\infty} \operatorname{cov}(X_j^{(n,k)}, X_l^{(n,k)}) < \frac{1}{n}.$$

Then for $\vec{X}^{(n,k)}$, $k=1,\ldots,i_n$, the assumptions (C1)–(C3) are satisfied with the same mean and covariance function. But

$$P\Big(\sum_{k=1}^{\infty}\vec{X}^{(n,k)}-\sum_{k=1}^{i_n}\vec{X}^{(n,k)}\neq\vec{0}\Big)\leqslant m\cdot 1/n.$$

Hence $\sum_{k=1}^{\infty}\vec{X}^{(n,k)}$ is asymptotically normal, as stated by Theorem 1, if $n\to\infty$. ■

3. Weak convergence.

THEOREM 2. *Let us assume that the $\mathcal{N}$ and $\mathcal{U}$ defined above in* (D2)–(D3) *are symmetric distributions. Then*

$$Y_A(t)\Rightarrow X(t)\quad as\quad A\to\infty.$$

Our proof is based on the tightness argument, and although we do not have the "stationarity properties" of the point processes $\{y_k(t)-y(t)\}$, we use Spitzer's ideas [8]. According to Spitzer [8] and using the symmetry of our model we shall prove Theorem 2 whenever we show that for each $\delta>0$:

$$\lim_{n\to\infty}\limsup_{A\to\infty}\sum_{k=1}^{n}P\big[\sup_{t\in[t_k,t_{k+1}]}y(t)-y(t_k)>\sqrt{A}\,\delta/4\big]=0,$$

where $t_k=A(k-1)/n$, $k=1,2,\ldots,n+1$. It will be clear from the proof that we may set $\delta=1$ without loss of generality.

Notice that

$$\limsup_{A\to\infty}\sum_{k=1}^{n}P\big[\sup_{t\in[t_k,t_{k+1}]}y(t)-y(t_k)>\sqrt{A}/4\big]$$

$$\leqslant\limsup_{A\to\infty}\sum_{k=1}^{n}P\big[\sup_{t\in[t_k,t_{k+1}]}y(t)-y(t_k)>\sqrt{A}/4\text{ and }y(t_k)<(n^{1/4}-1)\cdot\sqrt{A}\big]+$$

$$+\lim_{A\to\infty}\sum_{k=1}^{n}P\big[y(t_k)\geqslant(n^{1/4}-1)\cdot\sqrt{A}\big].$$

According to Theorem 1, the second term on the right side tends exponentially to zero while n tends to infinity. We shall show that the first term on the right-hand side in the last expression is $O(n^{-1/4})$.

Let us consider the motion of our process $Y_A(t)$, $t\geqslant 0$, on the coordinate-plane $0xt$, the position $x\in\boldsymbol{R}$, and time $t\in[0,\infty)$.

Construct the intervals $I_l(A)\subset\boldsymbol{R}$

$$I_{2l-1}\equiv I_{2l-1}(A):=\big(\tfrac{1}{8}(l-1)\cdot\sqrt{A},\ \tfrac{1}{8}\cdot l\cdot\sqrt{A}\big),$$

$$I_{2l}\equiv I_{2l}(A):=\big(-\tfrac{1}{8}l\cdot\sqrt{A},\ \tfrac{1}{8}(-l+1)\cdot\sqrt{A}\big),$$

for $l=1,2,\ldots$ Let L be the smallest positive integer such that $I_{L+1}\cap(-n^{1/4}\sqrt{A},\,n^{1/4}\sqrt{A})=\emptyset$. Then $L<20n^{1/4}$. Let $K_l(t_k)$ be the number of particles in the interval I_l at time t_k. Then for every $k=1,2,\ldots,n$, $l=1,2,\ldots,L$ we have

$$\lim_{A\to\infty}8\cdot A^{-1/2}E[K_l(t_k)]=1.$$

From the Tchebyshev inequality we have

$$\lim_{A\to\infty}P\big(K_{l,k}(A)<10^{-1}A^{1/2}\big)=0$$

for every $k=1,\ldots,n$, $l=1,\ldots,L$. Therefore it remains to show that

$$\text{(1)}\quad \limsup_{A\to\infty}\sum_{k=1}^{n}P\{\sup_{t\in[t_k,t_{k+1}]}y(t)-y(t_k)>\sqrt{A}/4,$$
$$y(t_k)<(n^{1/4}-1)\sqrt{A},\ K_{(y(t_k),y(t_k)+\sqrt{A}/4)}(t_k)>\sqrt{A}/10\}=O(n^{-1/4}),$$

where $K_{(a,b)}(t)$ is the number of particles in (a,b) at time t. Following Spitzer [8], for the event in the braces in (1) we get, for each k,

$$\{\ldots\}\subset\{\sup_{t\in[t_k,t_{k+1}]}Q_{B_k}(t)>\sqrt{A}/10,\ y(t_k)<(n^{1/4}-1)\sqrt{A}\},$$

where $B_k\equiv B_k(\omega):=y(t_k,\omega)+\sqrt{A}/4$, $Q_{B_k}(t):=L_{B_k}(t)-R_{B_k}(t)$ and $L_{B_k}(t)$ and R_{B_k} are equal to the number of trajectories hitting the line $x=B_k$ from the left and from the right, respectively, in the time interval $[t_k,t]$.

Let us put $g_0=0$, $g_m=[m\cdot n^{-1/4}\sqrt{A}]+1$, $m=1,\ldots,M$, $g_{-m}=-g_m$, where $[\cdot]$ is the integer part and M is the smallest positive integer such that $g_{M+1}\notin[0,n^{1/4}\sqrt{A}]$. It follows that $M<3\sqrt{n}$ for $A>2$.

Now, for every $m=0,\pm1,\ldots,\pm M$, $k=1,\ldots,n$, we easily infer that

$$\lim_{A\to\infty}P[W_m(t_k)>\sqrt{A}/40]=0,$$

where $W_m(t_k)$ is the number of particles in $[g_m,g_{m+1}]$ at time $t=t_k$. Therefore we may assume that for every $m=0,\pm1,\ldots,\pm M$, $k=1,\ldots,n$, we have $W_{m,k}<\sqrt{A}/40$. Then

$$L_{B_k(\omega)}(t)\leqslant L_{\overline{m}(\omega),k}(t)+W_{\overline{m}(\omega),k}<L_{\overline{m}(\omega),k}(t)+\sqrt{A}/40,$$
$$R_{B_k(\omega)}(t)\geqslant R_{\overline{m}(\omega),k}(t)-W_{\overline{m}(\omega),k}(t)>R_{\overline{m}(\omega),k}(t)-\sqrt{A}/40,$$

where $L_{m,k}(t)$ $(R_{m,k}(t))$ is the number of trajectories hitting the line $x=g_m$ in the time interval $[t_k,t]$ from the left (right) and $\overline{m}(\omega)=m_i$ iff $m_i\leqslant B_k(\omega)<m_{i+1}$. To show (1) it is sufficient to ascertain that

$$\text{(2)}\quad \limsup_{A\to\infty}\sum_{k=1}^{n}\sum_{m=-M}^{M}P\big[\sup_{t\in[t_k,t_{k+1}]}\big(L_{m,k}(t)-R_{m,k}(t)\big)>\sqrt{A}/20\big]<\frac{\text{const}}{\sqrt{n}}.$$

By using the Tchebyshev inequality once more, it is easy to ascertain that there exists a $c > 0$ such that, for every $k = 1, 2, \ldots, n$,

$$\lim_{A\to\infty} P\big(V_m(t_{k+1}) > c\cdot A/n\big) = 0$$

uniformly in $m = 0, \pm 1, \ldots$ (cf. the definition of g_m's, where $V_m(t_{k+1}) = L_{m,k}(t_{k+1}) + R_{m,k}(t_{k+1})$).

Thus it remains to prove

$$\limsup_{A\to\infty} 3\sqrt{n}\sum_{k=1}^{n} P\Big[\sup_{t\in[t_k,t_{k+1}]} Q_{0,k}(t) > \sqrt{A}/30,\ V_0(t_{k+1}) \leqslant c\cdot A/n\Big] \leqslant \mathrm{const}/\sqrt{n}, \tag{3}$$

where $Q_{0,k}(t) := L_{0,k}(t) - R_{0,k}(t)$. In the further considerations for $\omega \in \Omega$ and $n > 0$ we shall neglect "collisions" from x_{-n} and x_n if $x_{-n} > 0$ or $x_n < 0$. Hence, Theorem 2, in order to complete the proof of it suffices to prove that

$$P\Big[\sup_{t\in[t_k,t_{k+1}]} Q_{0,k}(t) > \sqrt{A}/30,\ V_0(t_{k+1}) \leqslant cA/n\Big] \leqslant \mathrm{const}/n^2, \tag{4}$$

the universal constant depending on c only.

We shall show that, for every $A > 0$, $k = 1, 2, \ldots, n$, and $N \leqslant Ac/n$,

$$P\Big(\sup_{t\in[t_k,t_{k+1}]} Q_{0,k}(t) > \sqrt{A}/30 \,|\, V_0(t_{k+1}) = N, \tau_1, \ldots, \tau_N\Big) \leqslant \frac{\mathrm{const}}{n^2},$$

where $(\tau_1, \ldots, \tau_N)(\omega)$ are all the moments of "collisions" of the lines $x_k + v_k\cdot t$ with the line $x = 0$ in the time interval $[t_k, t_{k+1})$.

The random variables

$\tau_i^{(\nu)} := \varphi^{(\nu)}(\tau_i)$ where

$$\varphi^{(\nu)}(\tau) = t_k + [(\tau - t_k)2^\nu n/A]\cdot A/2^\nu n, \qquad i = 1, \ldots, N,$$

where $\nu = 1, 2, \ldots$, approximate the variables $\tau_1, \ldots, \tau_N$ in the following sense: The σ-algebras spanned by $\tau_1, \ldots, \tau_N$, reduced to the event $V_0(t_{k+1}) = N$, are P-equivalent to the limit $\sigma\big(\bigcup_{\nu=1}^{\infty} F_\nu\big)$, where F_ν is spanned by $(\tau_1^{(\nu)}, \ldots, \tau_N^{(\nu)})$, for $\nu = 1, 2, \ldots$

Now, according to the Doob theorem on conditional distributions it remains to show that for every positive integer ν:

$$P\Big(\sup_{t\in[t_k,t_{k+1}]} Q_{0,k}(t) > \sqrt{A}/30 \mid \tau_1^{(\nu)} = r_1^{(\nu)}, \ldots, \tau_N^{(\nu)} = r_N^{(\nu)}\Big) \leqslant \frac{\mathrm{const}}{n^2}. \tag{5}$$

Fix ν and $r_1^{(\nu)}, \ldots r_N^{(\nu)}$ — points from the νth diadic partition. The event $\omega \in \Omega_N\big((r_1^{(\nu)}, n_1^{(\nu)}), \ldots, (r_N^{(\nu)}, n_N^{(\nu)})\big)$, where $n_i^{(\nu)} \geqslant 1$ and $i = 1, 2, \ldots,$ means

that

$$x_{n_i}(\omega) + tv_{n_i}(\omega) = 0$$

or

$$x_{-n_i}(\omega) + tv_{-n_i}(\omega) = 0,$$

for some $t \in \Big[r_i^{(\nu)}, r_i^{(\nu)} + \dfrac{A}{n}\cdot 2^{-\nu}\Big]$. Then for $\omega \in \Omega_N\big((r_1^{(\nu)}, n_1^{(\nu)}), \ldots, (r_N^{(\nu)}, n_N^{(\nu)})\big)$ we may introduce the following order in $\{1, \ldots, N\}$:

$$i \prec k \equiv r_i^{(\nu)} < r_k^{(\nu)} \text{ or } [r_i^{(\nu)} = r_k^{(\nu)} \text{ and } (|n_i| < |n_k| \text{ or } n_i < 0 < n_k = -n_i)].$$

The numbers $r_1^{(\nu)}, \ldots, r_N^{(\nu)}$ ordered by $\prec$ are denoted by $\tilde{r}_1^{(\nu)}, \ldots, \tilde{r}_N^{(\nu)}$. Hence it remains to prove that

$$P\big[\max_{1\leqslant i\leqslant N} S_i > \sqrt{A}/30 \mid \Omega_N\big((r_1^{(\nu)}, n_1^{(\nu)}), \ldots (r_N^{(\nu)}, n_N^{(\nu)})\big)\big] \leqslant \frac{1}{n^2}\cdot\mathrm{const},$$

where S_i is a sum of random variables $z_j = \pm 1$, $1 \leqslant j \leqslant i$, where z_j equals $+1$ iff the "collision" in $\tilde{r}_j^{(\nu)}$ with $x = 0$ arises from the left and -1 if it arises from the right.

Now we shall use the following theorem of Billingsley [2] for sums of (possibly dependent) random variables:

If for every $h > 0$

$$P(|\hat{S}_j - \hat{S}_i| \geqslant h) \leqslant h^{-4}(j-i)^2,$$

then

$$P\big(\max_{0\leqslant i\leqslant l} |\hat{S}_i| \geqslant h\big) \leqslant Kh^{-4}l^2,$$

where K is a universal constant.

If in the chain $z_{i+1}, \ldots, z_j$ there are two "collisions" beginning from particles x_n and x_{-n}, then, obviously, $|S_j - S_i|$ is not changed and may be neglected.

Finally we obtain at the remaining points $\tilde{r}_\mu$, $i+1 \leqslant \mu \leqslant j$, independent random variables ± 1 with probability $1/2$. Hence

$$P[|S_j - S_i| \geqslant h] \leqslant (i-j)^2 h^{-4}.$$

Applying Billingsley's theorem, we have

$$P\big[\max_{1\leqslant i\leqslant l} |S_i| \geqslant A/30\big] \leqslant K\cdot 30^4 n^{-2}.$$

Thus the proof of Theorem 2 is complete.

4. Discussion of the process $X(t)$.

Example 1. Assume the distribution $\mathscr{U}$ to be a uniform distribution on $[-a, a]$; then we have

$$E[X(t)\cdot X(s)] = \frac{1}{4}a\Big[\min(t, s) + \frac{1}{3}\cdot\frac{\min^2(t, s)}{\max(t, s)}\Big]$$

$$\text{for} \quad (t, s) \neq (0, 0),\ 0 \leqslant t, s \leqslant 1.$$

Proof. Let $0 \leqslant s \leqslant t \leqslant 1$; $t \neq 0$. Then

$$E[X(t)\cdot X(s)] = \frac{1}{2}\,as - \frac{1}{2a^2}\cdot\int_0^a\int_0^a \min(tu, sv)\,du\,dv$$

$$= \frac{1}{2}\,as - \frac{1}{2a^2}\left(\int_0^a\int_0^{\frac{s}{t}v} tu\,du\,dv + \int_0^a\int_{\frac{s}{t}v}^a sv\,du\,dv\right) = a\left(\frac{1}{2}s - \frac{1}{2}\left(\frac{s}{2} - \frac{s^2}{6t}\right)\right).$$

Example 2. Assume the distribution $\mathscr{U}$ to be normal $\mathscr{U} = N(0, \sigma)$. Then we have

$$E[X(t)\cdot X(s)] = \frac{\sigma}{\sqrt{2\pi}}\left(\sqrt{t^2+s^2} - |t-s|\right).$$

Proof.

$$P\big(\min(t|u|, s|v|) < x\big) = P(t|u| < x) + P(s|v| < x) - P(t|u| < x)\cdot P(s|v| < x)$$

and then $\min(t|u|, s|v|)$ has the density

$$\frac{\sqrt{2}}{\sqrt{\pi}t\sigma}\exp\left(-\frac{x^2}{2t^2\sigma^2}\right) + \frac{\sqrt{2}}{\sqrt{\pi}s\sigma}\exp\left(\frac{-x^2}{2s^2\sigma^2}\right) -$$

$$- \frac{2}{\pi ts\sigma^2}\left\{\exp\left(-\frac{x^2}{2t^2\sigma^2}\right)\int_0^x \exp\left(-\frac{y^2}{2s^2\sigma^2}\right)dy + \right.$$

$$\left. + \exp\left(\frac{-x^2}{2s^2\sigma^2}\right)\int_0^x \exp\left(\frac{-y^2}{2t^2\sigma^2}\right)dy\right\}.$$

Hence

$$\min(t, s)\,Ev - \tfrac{1}{2}E\min(t|u|, s|v|)$$

$$= \frac{-|t-s|\sigma}{\sqrt{2\pi}} + \frac{1}{\pi ts\sigma^2}\int_0^\infty x\left\{\exp\left(\frac{-x^2}{2t^2\sigma^2}\right)\int_0^x \exp\left(\frac{-y^2}{2s^2\sigma^2}\right)dy + \right.$$

$$\left. + \exp\left(\frac{-x^2}{2s^2\sigma^2}\right)\int_0^x \exp\left(\frac{-y^2}{2t^2\sigma^2}\right)dy\right\}dx$$

$$= \frac{-|t-s|\sigma}{\sqrt{2\pi}} + \frac{1}{\pi ts}(t^2+s^2)\int_0^\infty \exp\left(\frac{-x^2}{2t^2\sigma^2}\right)\exp\left(\frac{-x^2}{2s^2\sigma^2}\right)dx$$

$$= \frac{\sigma}{\sqrt{2\pi}}\left(\sqrt{t^2+s^2} - |t-s|\right).$$

It is natural to ask whether the limit process $X(t)$ is Markovian. The answer turns out to be generally negative. Namely we have

Theorem 3. *The process $X(t)$ is Markovian if and only if there exist $a > 0$, $\beta < 0$ and non-negatives p, q, r, $p+q+r = 1$, such that*

$$(6)\qquad v = \begin{cases} 0 & \text{with probability } p, \\ a & \text{with probability } q, \\ \beta & \text{with probability } r. \end{cases}$$

In this case the covariance is

$$\tfrac{1}{2}E|v|\cdot(1+p)\cdot\min(t, s).$$

Then $X(t)$ is simply a Brownian motion.

Proof. If $X(t)$ is Markovian, then for every k and $a > 1$

$$E\left[X\left(\frac{1}{a^k}\right)\cdot X\left(\frac{1}{a^{k-1}}\right)\right]\cdot E\left[X\left(\frac{1}{a^{k-1}}\right)\cdot X(1)\right] = E\left(X\left(\frac{1}{a^k}\right)\cdot X(1)\right]\cdot E\left[X\left(\frac{1}{a^{k-1}}\right)\right]$$

(cf. Feller [5]). But this means that for every $k = 1, 2, \ldots$

$$\{E|v| - E[\min(v^-, au^-) + \min(v^+, au^+)]\}\times$$

$$\times\{E|v| - E[\min(v^-, a^{k-1}u^-) + \min(v^+, a^{k-1}u^+)]\}$$

$$= \{E|v| - E[\min(v^-, a^k u^-) + \min(v^+, a^k u^+)]\}\times$$

$$\times\{E|v| - E[\min(v^-, u^-) + \min(v^+, u^+)]\}.$$

Given a k approaching infinity, we have for every $a > 1$

$$E[\min(v^-, u^-) + \min(v^+, u^+)] = E[\min(v^-, au^-) + \min(v^+, au^+)].$$

But this may be fulfilled only if (6) is satisfied.

The process $X(t)$ can be realized on $C_{[0,1]}$. In a "symmetric case" it is a consequence of Theorem 2. But even in the general case we may prove more:

$$\lim_{\delta\to 0^+}\ \sup_{|t_1-t_2|\leqslant\delta}\frac{|x(t_1)-x(t_2)|}{\left(|t_1-t_2|\,\left|\log|t_1-t_2|\right|\right)^{1/2}} \leqslant C$$

with probability one (cf. Ciesielski [3]).

Then we may ask whether there exists a "velocity" process of the limit motion, i.e. whether the trajectories of the process $X(t)$ are differentiable. The answer is negative and we can show a little more:

Theorem 4. *We have*

$$P\left[\overline{\lim_{h\searrow 0}}\,\frac{|X(t+h)-X(t)|}{h} = +\infty,\ \text{for each } 0 < t < 1\right] = 1.$$

We use the following theorem of Kawada and Kono [7].

If there exists a positive integer q such that

$$\lim_{h\searrow 0}\left(\frac{h}{g(h)}\right)^{q}\frac{1}{h}=0,$$

where $g(h)$ is a negative, even, non-decreasing function such that

$$E[X(t+h)-X(t)]^2\geqslant g^2(h),$$

and if there exists also a positive integer p such that

$$\overline{\lim_{h\searrow 0}}\sup_{|t-s|\geqslant ph}\left|E\left(\Delta_h X(t)\cdot\Delta_h X(s)\right)\right|<\frac{1}{2q},$$

where

$$\Delta_h X(t)=\frac{X(t+h)-X(t)}{\{E[X(t+h)-X(t)]^2\}^{1/2}},$$

then

$$P\left[\overline{\lim_{h\searrow 0}}\frac{|X(t+h)-X(t)|}{h}=+\infty\ \text{for every}\ t_1\leqslant t<t_2\right]=1.$$

Proof of Theorem 4. Let $0<T<1$. We verify this assumption for $q=3$ and $p=1$ on the interval

$$A_k=[T+k\beta,\,T+(k+1)\beta)\cap(0,3/2),$$

where $\beta>0$ is chosen in such a way that:

$$\int\limits_{\substack{T/(T+\beta)<\frac{u^-}{v^-}<(T+2\beta)/T\\ u^-\neq v^-}} v^-P\,du\,P\,dv<\frac{E(v^-)}{12}$$

and

$$\int\limits_{\substack{T/(T+\beta)<\frac{u^+}{v^+}<(T+2\beta)/T\\ u^+\neq v^+}} v^+P\,du\,P\,dv<\frac{P(v^+)}{12}.$$

Since

$$E[X(t+h)-X(t)]^2\geqslant \tfrac{1}{2}E|v|\cdot h\quad \text{for}\quad h>0,$$

we may set $g(h)=\sqrt{\frac{1}{2}E|v|\cdot h}$. Hence it remains to show that for every k such that $A_k\neq\emptyset$, we have

$$\sup_{\substack{t,s\in A_k\\ h\leqslant|t-s|}}|E\{[X(t+h)-X(t)]\cdot[X(s+h)-X(s)]\}|<\frac{E|v|\cdot h}{12}.$$

But for $h\leqslant|t-s|$ we have

$$\begin{aligned}
&|E\{[X(t+h)-X(t)]\,[X(s+h)-X(s)]\}|\\
&\quad\leqslant \tfrac{1}{2}\big|E\big[\min(tu^-,sv^-)+\min\big((t+h)u^-,(s+h)v^-\big)-\\
&\qquad\qquad -\min\big((t+h)u^-,sv^-\big)-\min\big(tu^-,(s+h)v^-\big)\big]\big|+\\
&\quad\ +\tfrac{1}{2}\big|E\big[\min(tu^+,sv^+)+\min\big((t+h)u^+,(s+h)v^+\big)-\\
&\qquad\qquad -\min\big((t+h)u^+,sv^+\big)-\min\big(tu^+,(s+h)v^+\big)\big]\big|.
\end{aligned}$$

We restrict our attention to the first integral. If $(u^-=v^-)$ or $(tu^-\geqslant(s+h)v^-)$ or $(sv^-\geqslant(t+h)u^-)$, then we have 0 under the integral. Hence it remains to show that for every k

$$\lim_{h\searrow 0}\sup_{\substack{t,s\in A_k\\ h\leqslant|t-s|}}\int\limits_{\substack{\frac{t}{s}u^-<v^-<\frac{t+h}{s}u^-\\ u^-\neq v^-}}\left[\min\big((t+h)u^-,(s+h)v^-\big)-sv^-\right]P\,dv\,P\,du\leqslant\frac{Ev^-}{12}.$$

But this follows immediately from our assumptions, because the expression W under the integral satisfies $W\leqslant h\cdot v^-$ and

$$\left\{(u^-,v^-):\ \left(\frac{t}{s}<\frac{v^-}{u^-}<\frac{t+h}{s},\ u^-\neq v^-,\ t,s\in A_k,\ |t-s|\geqslant h\right)\right\}$$
$$\subset\left\{(u^-,v^-):\ \left(\frac{T}{T+\beta}<\frac{v^-}{u^-}<\frac{T+2\beta}{T},\ u^-\neq v^-\right)\right\}.$$

Then Theorem 4 is true for $T<t<1$ and hence for $0<t<1$.

The assumption $E|v|<\infty$ is essential. In fact, if $E|v|=+\infty$, then it is easy to see that each trajectory $x_k(t)$ collides infinitely many times in each time interval.

Let us now consider the model of Harris and Spitzer (see [6], [8], [9]), which may be described as follows:

(S1) $x_0=0$;

(S2) $\xi_k=x_k-x_{k-1}$ *are exponential random variables with the mean one for* $k=0,\pm1,\pm2,\ldots$;

(S3) v_k *are identically distributed with* $E[v_k]=0$, *and* $E|v_k|=1$;

(S4) $\xi_0,v_0,\xi_1,v_1,\xi_{-1},v_{-1},\xi_2,\ldots$ *are independent random variables.*

Here, x_k is a system of particles in the macroscopic equilibrium (the origin $x_0=0$ is included in the system) and, what is more, Spitzer proved that if we define $w_k(t)=\big(y_k(t)-y(t),\,v_k(t)\big)$, then $w_k(t)$ is a random Poisson measure in a phase space for each $t>0$. Also for this model Spitzer proved that $Y_A(t)\Rightarrow W(t)$ as $A\to\infty$, where $Y_A(t)$ is defined as in Theorem 1 and $W(t)$ is a standard Brownian motion.

Now let us consider the model D, in which $x_k=k$ and $P\{v_k=+1\}=P\{v_k=-1\}=1/2$. Then, according to Theorem 2, $Y_A(t)\Rightarrow 1/2\,W(t)$

as $A \to \infty$. On the other hand, $y(At)$ may be treated as a symmetric continuous random walk (i.e. $2y(k/2)$ is an ordinary simple random walk), where changes occur at the moments $\pm 1/2$, ± 1, $\pm 3/2$ ... Weak convergence is now a consequence of Donsker's theorem on the weak convergence of random walks to the Wiener process. Therefore our model may be viewed as a generalization of Donsker's theorem and this is why we call it "model D". Now, if we specify in the model D the distribution $\mathcal{U}$, assuming that it is absolutely continuous, then the assumptions of Dobrushin's theorem are satisfied (see Stone [10], Th. 5). Therefore it is plausible that via the macroscopic equilibrium, suggested by Dobrushin's theorem, the corresponding process of the null particle, in this special case, should converge to the Wiener process. This is apparently why Spitzer [9] writes "Now there should be no great difference between a particle system initially on the integers, and one which is initially distributed as a Poisson system. The intuitive idea is in fact supported by the theorems of Dobrushin and Stone". Our theorems on model D show that at this point the intuition fails. It is essential that in Dobrushin's theorem the convergence to a Poisson distribution holds only for conditionally compact sets but, if A tends to infinity, the length of the relevant intervals tends to infinity as well.

This is the reason why the passage to the limit with $Y_A(t)$, $A \to \infty$, cannot be done in two steps: first passing to the Poisson system and then passing to the Brownian motion.

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