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The chain rule for differentiable measures*

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Abstract. The chain rule for differentiable measures is proved. It states that if ν is an H-differentiable measure on a Banach space B and θ is a suitable transformation, then the composition $\mu = \nu \circ \theta$ is also H-differentiable and the derivative is given by $D\mu(dx) = \theta'(x)^* D\nu \circ \theta(dx) + \sum_n \langle \theta''(x)\,(\theta'(x)^{-1} e_n, \cdot), e_n \rangle \mu(dx)$, where $\{e_n;\ n = 1, 2, \ldots\}$ is an orthonormal basis of H.

1. Introduction. The notion of differentiable measure has been introduced in [5], [6], [8]. It plays an important role in Schwartz' distribution theory on Banach spaces. See, for instance, papers [1], [3], [10]. In particular, it has been shown in [10], Theorem 8, that a harmonic distribution can be represented by a smooth measure. However, in infinite dimensional spaces, there is no canonical way to represent a smooth measure by a smooth function.

In order to study distribution theory on infinite dimensional manifolds, one has to define differentiability for measures on manifolds. This obviously requires a fundamental theorem for differentiable measures, namely, the chain rule. Unlike the chain rule for differentiable functions, that for differentiable measures takes a non-trivial form and has some rather unexpected applications. For example, one can consider a Dirichlet form associated with a Borel measure on a Riemann–Wiener manifold. In case the measures is differentiable and has logarithmic derivative ([13], p. 121), we can use the chain rule to produce a self-adjoint operator associated with this Dirichlet form. This will be done in [12] and the subsequent papers. We remark that the number operator on a Riemann–Wiener manifold can be constructed in this way [11].

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2. *H*-differentiable measures. In this paper, (H, B) will denote a fixed pair of a Hilbert space H and a Banach space B with the following interpolation property: there exists a Hilbert space H_0 such that $H \subset H_0 \subset B$, the inclusion map from H into H_0 is continuous, and (H_0, B) is an abstract Wiener space (see [7] for the definition). Note that H can be finite dimensional even when B is infinite dimensional. We need the interpolation property since in the proof of Theorem 1 below we have to use two theorems on abstract Wiener spaces, i.e. [10], Theorem 1 and Theorem 3. The norm and inner product of H will be denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively.

Let U be an open subset of B. A subset A of U is said to be *properly bounded* in U if A is bounded and, in case $U \neq B$, $\operatorname{dist}(A, U^c) > 0$. $\mathscr{B}_0(U)$ will denote the collection of properly bounded Borel subsets of U. A function f from U into a Banach space K is said to be *j-times* $(j \geqslant 1)$ *H-differentiable at a point x* in U if the function $g(h) = f(x+h)$ from $(U-x)\cap H$ into K is j-times Fréchet differentiable at the origin. f is said to be *j-times H-differentiable on* U if it is j times H-differentiable at every point in U. We define the *i-th* $(1 \leqslant i \leqslant j)$ *H-derivative* $f^{(i)}(x)$ of f at x in U to be the i-th Fréchet derivative $g^{(i)}(0)$ of g at 0. Note that $f^{(i)}(x) \in L^i(H; K)$ for each x in U. Here $L^i(H; K)$ denotes the Banach space of continuous i-linear maps from $H \times \ldots \times H$ (i factors) into K.

DEFINITION 1. A *local measure* on U is a real-valued set function μ defined on $\mathscr{B}_0(U)$ such that the restriction of μ to any properly bounded open subset of U is a finite real Borel measure.

DEFINITION 2. A local measure μ on U is said to be *H-differentiable* if

(i) for any bounded uniformly continuous function f with support properly bounded in U, $\mu f(x) = \int_U f(x+y)\,\mu(dy)$ is H-differentiable at the origin and

(ii) for any sequence f_n of uniformly continuous functions converging to zero pointwise and boundedly with $\bigcup_n \operatorname{supp} f_n$ properly bounded in U, $\lim_{n\to\infty}\langle(\mu f_n)'(0), h\rangle = 0$ for all h in H.

(Throughout the paper we shall confuse the H-derivative and the H-gradient of real-valued functions.)

Note that in Definition 2 we use uniform continuity instead of continuity which is used in [8], Definition 2. Uniform continuity is necessary in the proof of Theorem 1 below. However, as in the proof of [8], Theorem 1 it can be shown that μ is H-differentiable if and only if there exists a (unique) finitely additive set function $D\mu$ from $\mathscr{B}_0(U)$ into H such that

for each h in H, $\langle D\mu(\cdot), h\rangle$ is a local measure and

$$\langle(\mu f)'(0), h\rangle = -\int_U f(x)\,\langle D\mu(dx), h\rangle$$

for all h in H and all bounded uniformly continuous functions f with $\operatorname{supp} f$ properly bounded in U. $D\mu$ is called the *H-derivative* of μ. It follows from Pettis' theorem ([4], p. 318) that $D\mu$ is an H-valued local measure on U.

3. The chain rule. First we make the following definition (cf. [9], p. 104). A continuous bilinear map S from $H \times H$ into H is said to be *trace class type* if for each u in H, S_u is a trace class operator of H, where $\langle S_u h, k\rangle = \langle S(h,k), u\rangle$, and the linear map $u \to S_u$ is continuous from H into the Banach space $\mathscr{I}(H)$ of trace class operators of H. It follows that there is a unique vector in H, denoted by TRACE S, such that $\langle \mathrm{TRACE}\, S, u\rangle = \operatorname{trace} S_u$ for all u in H. Moreover, $\mathrm{TRACE}\, S = \sum_n S(e_n, e_n)$ for any orthonormal basis $\{e_n\}$ of H. We will denote by $\hat{S}$ the map $\hat{S}u = S_u$ from H into $\mathscr{I}(H)$. Note that $\hat{S} \in L(H; \mathscr{I}(H))$.

Let U and V be two open subsets of B. Let θ be a twice H-differentiable homeomorphism from U onto V. We assume that θ satisfies the following conditions:

(i) for each x in U, $\theta'(x) \in L(H; H)$ and is invertible, and the map $\theta'(\cdot)$ from U into $L(H;H)$ is measurable,

(ii) for each x in U, $\theta''(x) \in L^2(H; H)$ and the bilinear map $(h,k) \to \langle\theta''(x)(h,\cdot), k\rangle$ from $H \times H$ into H is trace class type, and the map $\theta''(\cdot)$ from U into $L^2(H;H)$ is measurable.

LEMMA 1. *Let $J_\theta(x)$ be the bilinear map from $H \times H$ into H defined by $J_\theta(x)(h,k) = \langle\theta''(x)\big(\theta'(x)^{-1}h, \cdot\big), k\rangle$. Then $J_\theta(x)$ is trace class type for each x in U and* $\mathrm{TRACE}\, J_\theta(\cdot)$ *from U into H is measurable.*

Proof. Let S and T denote the bilinear maps $(h,k) \to \langle\theta''(x)(h,\cdot), k\rangle$ and $(h,k) \to \langle\theta''(x)\big(\theta'(x)^{-1}h,\cdot\big), k\rangle$, respectively. It is easy to see that for each u in H,

$$S_u h = \theta''(x)(h, u), \qquad T_u h = \theta''(x)\big(\theta'(x)^{-1}h, u\big),$$

where $h \in H$. Therefore, $T_u = S_u\theta'(x)^{-1}$ as operators in $L(H;H)$. Since S is trace class type by condition (ii), this relation shows easily that T is also trace class type. The measurability of $\mathrm{TRACE}\, J_\theta(\cdot)$ follows from the fact that

$$\mathrm{TRACE}\, J_\theta(x) = \sum_n \langle\theta''(x)\big(\theta'(x)^{-1}e_n, \cdot\big), e_n\rangle.$$

THEOREM 1. (The chain rule.) *Suppose θ is a twice H-differentiable homeomorphism from U onto V satisfying the above conditions* (i) *and* (ii).

Let ν be an H-differentiable local measure on V and $\mu = \nu\circ\theta$. Assume that the following conditions are also satisfied:

(iii) *$\theta(A)\in\mathscr{B}_0(V)$ for all $A\in\mathscr{B}_0(U)$,*

(iv) *over every properly bounded subset of U: $\theta'(\cdot)$ and $\theta'(\cdot)^{-1}$ are bounded in operator norm, $\theta'(\cdot)$ is Bartle $D\nu\circ\theta$-integrable and $\hat{J}_\theta$ taking values in $L(H;\mathscr{I}(H))$ is Bochner μ-integrable.*

Then μ is an H-differentiable local measure on U and its H-derivative is given by

$$D\mu(dx) = \theta'(x)^* D\nu\circ\theta(dx) + \big(\mathrm{TRACE}\, J_\theta(x)\big)\mu(dx),$$

*where * denotes the adjoint and $J_\theta(x)$ is defined by*

$$J_\theta(x)(h,k) = \big\langle \theta''(x)\big(\theta'(x)^{-1}h,\cdot\big),k\big\rangle \qquad h,k\in H,$$

so that

$$\mathrm{TRACE}\, J_\theta(x) = \sum_n \big\langle \theta''(x)\big(\theta'(x)^{-1}e_n,\cdot\big)e_n\big\rangle.$$

for any orthonormal basis $\{e_n\}$ of H.

Remarks. (1) See [4], p. 112 for Bochner μ-integrability and [2], p. 341 for Bartle $D\nu\circ\theta$-integrability.

(2) Suppose that ν is a finite real Borel measure on V (instead of a local measure); then μ is a finite real Borel measure on U. In this case (iii) need not be assumed. If (iv) holds for every Borel subset A of U, then we have a stronger conclusion, i.e. $D\mu$ is an H-valued vector measure on U.

Proof. Let f be bounded, Lip-1 w.r.t. B-norm and H-differentiable with support properly bounded in U such that f' is bounded and Lip-1 from U into H. Let $\mu f(x) = \int_U f(x+y)\mu(dy)$, which is defined on some B-open ball W with center at the origin such that $W+\operatorname{supp} f = \{x+y;\ x\in W, y\in\operatorname{supp} f\}$ is properly bounded in U. It is easy to see that μf is H-differentiable on W and its H-derivative at x in W is given by

$$\langle(\mu f)'(x),h\rangle = \int_U \langle f'(x+y),h\rangle\mu(dy) = \int_V \big\langle f'\big(x+\theta^{-1}(z)\big),h\big\rangle\nu(dz), \qquad h\in H.$$

Define $g(z) = f\big(x+\theta^{-1}(z)\big)$, $z\in V$. Then $\operatorname{supp} g$ is properly bounded in V by condition (iii), and $\langle g'(z),h\rangle = \big\langle f'\big(x+\theta^{-1}(z)\big),\ \theta'\big(\theta^{-1}(z)\big)^{-1}h\big\rangle$ for h in H. Therefore, $g'(z) = \big[\theta'\big(\theta^{-1}(z)\big)^{-1}\big]^* f'\big(x+\theta^{-1}(z)\big)$ and so $f'\big(x+\theta^{-1}(z)\big) = \theta'\big(\theta^{-1}(z)\big)^* g'(z)$. Hence

$$\langle(\mu f)'(x),h\rangle = \int_V \big\langle g'(z),\theta'\big(\theta^{-1}(z)\big)h\big\rangle\nu(dz) = \int_V \sum_n \langle g'(z),e_n\rangle\big\langle\theta'\big(\theta^{-1}(z)h\big),e_n\big\rangle\nu(dz),$$

where $\{e_n\}$ is an orthonormal basis of H. Let Q denote the support of f. Then $Q-x$ is properly bounded for any x in W. Let $|\mu|$ denote the total variation of μ. Then

$$\begin{aligned}
&\sum_n\int_U |\langle g'(\theta(y)),e_n\rangle|\,|\langle\theta'(y)h,e_n\rangle|\,|\mu|(dy)\\
&= \sum_n\int_U |\langle[\theta'(y)^{-1}]^* f'(x+y),e_n\rangle|\,|\langle\theta'(y)h,e_n\rangle|\,|\mu|(dy)\\
&= \sum_n\int_{Q-x} |\langle[\theta'(y)^{-1}]^* f'(x+y),e_n\rangle|\,|\langle\theta'(y)h,e_n\rangle|\,|\mu|(dy)\\
&\leqslant \sum_n\Big\{\int_{Q-x}\langle[\theta'(y)^{-1}]^* f'(x+y),e_n\rangle^2|\mu|(dy)\Big\}^{1/2}\Big\{\int_{Q-x}\langle\theta'(y)h,e_n\rangle^2|\mu|(dy)\Big\}^{1/2}\\
&\leqslant \Big\{\sum_n\int_{Q-x}\langle[\theta'(y)^{-1}]^* f'(x+y),e_n\rangle^2|\mu|(dy)\Big\}^{1/2}\Big\{\sum_n\int_{Q-x}\langle\theta'(y)h,e_n\rangle^2|\mu|(dy)\Big\}^{1/2}\\
&= \Big\{\int_{Q-x}|[\theta'(y)^{-1}]^* f'(x+y)|^2|\mu|(dy)\Big\}^{1/2}\Big\{\int_{Q-x}|\theta'(y)h|^2|\mu|(dy)\Big\}^{1/2}\\
&\leqslant |h|\sup_{y\in Q}|f'(y)|\sup_{y\in Q-x}\|\theta'(y)^{-1}\|\sup_{y\in Q-x}\|\theta'(y)\|\,|\mu|(Q-x),
\end{aligned}$$

which is finite by condition (iv) and the boundedness of f'. Therefore, we can interchange integration and summation in the expression of $\langle(\mu f)'(x),h\rangle$ to get

$$\langle(\mu f)'(x),h\rangle = \sum_n\int_V\langle g'(z),e_n\rangle\big\langle\theta'\big(\theta^{-1}(z)\big)h,e_n\big\rangle\nu(dz) = \sum_n\int_V\langle g'(z),e_n\rangle\varrho_n(dz),$$

where $\varrho_n(dz) = \big\langle\theta'\big(\theta^{-1}(z)\big)h,e_n\big\rangle\nu(dz)$ is defined on some open subset of V containing the support of g. It is easy to see that ϱ_n is a local measure and, by [8], Theorem 3,

$$\begin{aligned}\langle D\varrho_n(dz),k\rangle &= \big\langle\theta'\big(\theta^{-1}(z)\big)h,e_n\big\rangle\langle D\nu(dz),k\rangle+\\
&\quad+\Big\langle\theta''\Big(\theta^{-1}(z)\ \big(\theta'\big(\theta^{-1}(z)\big)^{-1}k,h\big),e_n\Big\rangle\nu(dz), \qquad k\in H.\end{aligned}$$

Apply the integration by parts formula ([8], Theorem 2) to obtain

$$\begin{aligned}\langle(\mu f)'(x),h\rangle = -\sum_n\int_V g(z)\Big\{&\big\langle\theta'\big(\theta^{-1}(z)\big)h,e_n\big\rangle\langle D\nu(dz),e_n\rangle+\\
&+\Big\langle\theta''\big(\theta^{-1}(z)\big)\Big(\theta'\big(\theta^{-1}(z)\big)^{-1}e_n,h\Big),e_n\Big\rangle\nu(dz)\Big\}.\end{aligned}$$

Recall that $g(z) = f\big(x+\theta^{-1}(z)\big)$ and let $y = \theta^{-1}(z)$. Then

$$\begin{aligned}\langle(\mu f)'(x),h\rangle = -\sum_n\int_U f(x+y)\Big\{&\langle\theta'(y)h,e_n\rangle\langle D\nu\circ\theta(dy),e_n\rangle+\\
&+\big\langle\theta''(y)\big(\theta'(y)^{-1}e_n,h\big),e_n\big\rangle\mu(dy)\Big\}.\end{aligned}$$

If H is finite dimensional, we can obviously interchange summation and integration. Suppose that H is infinite dimensional and let P_n be the orthogonal projection onto the span of $\{e_1, \ldots, e_n\}$. Then

$$\sum_n \int_U f(x+y)\langle \theta'(y)h, e_n\rangle \langle D\nu\circ\theta(dy), e_n\rangle = \lim_{n\to\infty}\int_U \langle P_n f(x+y)\theta'(y)h, D\nu\circ\theta(dy)\rangle.$$

By [2], Theorem 4 and Theorem 10, for any bounded Bartle $D\nu\circ\theta$-integrable function F with values in H, we have

$$\int \langle F(y), D\nu\circ\theta(dy)\rangle = \lim_{n\to\infty}\int \langle P_n F(y), D\nu\circ\theta(dy)\rangle.$$

Therefore,

$$\sum_n \int_U f(x+y)\langle \theta'(y)h, e_n\rangle\langle D\nu\circ\theta(dy), e_n\rangle = \int_U f(x+y)\langle\theta'(y)h, D\nu\circ\theta(dy)\rangle = \int_U f(x+y)\langle \theta'(y)^* D\nu\circ\theta(dy), h\rangle.$$

Moreover,

$$\sum_n \int_U f(x+y)\langle \theta''(y)\big(\theta'(y)^{-1}e_n, h\big), e_n\rangle \mu(dy)$$
$$= \operatorname{trace}\int_U f(x+y)\big(J_\theta(y)\big)_h\mu(dy) = \int_U f(x+y)\operatorname{trace}\big(J_\theta(y)\big)_h\mu(dy)$$
$$= \int_U f(x+y)\langle \mathrm{TRACE}\, J_\theta(y), h\rangle\mu(dy)$$
$$= \int_U f(x+y)\langle \big(\mathrm{TRACE}\, J_\theta(y)\big)\mu(dy), h\rangle,$$

where $(J_\theta(y))_h$ denotes the operator such that $\langle (J_\theta(y))_h u, v\rangle = \langle (J_\theta(y)\times\times(u,v), h\rangle$. Here we have used the integrability of $\hat{J}_\theta$. Therefore, we have shown that

$$\langle (\mu f)'(x), h\rangle = -\int_U f(x+y)\langle \theta'(y)^* D\nu\circ\theta(dy) + \big(\mathrm{TRACE}\, J_\theta(y)\big)\mu(dy), h\rangle$$

holds for any bounded, Lip-1, H-differentiable function f with support properly bounded in U such that f' is bounded, Lip-1 from U into H, and for all x in W.

Now let f be any bounded uniformly continuous function with support properly bounded in U. By [10], Theorem 1 and Theorem 3, there exists a sequence $\{f_k\}$ of bounded Lip-1 functions converging uniformly to f such that $\bigcup_k \operatorname{supp} f_k$ is properly bounded in U and f_k, $k = 1, 2, \ldots$, are H-differentiable with bounded Lip-1 derivatives. It is easy to see that there exists an B-open ball W_0 with center at the origin such that $W_0 + (\operatorname{supp} f \cup \bigcup_{k=1}^\infty \operatorname{supp} f_k)$ is properly bounded in U. We have shown that μf_k, $k = 1, 2, \ldots$, are H-differentiable and for h in H,

$$\langle (\mu f_k)'(x), h\rangle = -\int_U f_k(x+y)\langle\theta'(y)^* D\nu\circ\theta(dy) + \big(\mathrm{TRACE}\, J_\theta(y)\big)\mu(dy), h\rangle.$$

Obviously, on W_0, μf_k converges uniformly to μf and $(\mu f_k)'$ converges uniformly to

$$-\int_U f(\cdot+y)\{\theta'(y)^* D\nu\circ\theta(dy) + \big(\mathrm{TRACE}\, J_\theta(y)\big)\mu(dy)\}.$$

Therefore, μf is H-differentiable on W_0 and

$$\langle (\mu f)'(x), h\rangle = -\int_U f(x+y)\langle\theta'(y)^* D\nu\circ\theta(dy) + \big(\mathrm{TRACE}\, J_\theta(y)\big)\mu(dy), h\rangle.$$

In particular, for $x = 0$, we have

$$\langle (\mu f)'(0), h\rangle = -\int_U f(y)\langle\theta'(y)^* D\nu\circ\theta(dy) + \big(\mathrm{TRACE}\, J_\theta(y)\big)\mu(dy), h\rangle.$$

This shows that μ is an H-differentiable local measure and its H-derivative is given by

$$D\mu(dy) = \theta'(y)^* D\nu\circ\theta(dy) + \big(\mathrm{TRACE}\, J_\theta(y)\big)\mu(dy).$$

4. Logarithmic derivative. In this section we assume that H is dense in B. Let i be the inclusion map from H into B. Then i^* is injective from B into H^*. We may identify B^* with $i^*(B^*)$ and also, by the Riesz representation theorem, identify H^* with H. Thus we have $B^* \subset H \subset B$. It is easy to see that B^* is dense in H with respect to H-topology and hence there exists an orthonormal basis $\{e_n\}$ of H such that $e_n \in B^*$ for all n. Let $(\cdot, \cdot)$ denote the natural pairing of B and B^*; then $(h, k) = \langle h, k\rangle$ for all h in H and k in B^*.

Let $\|\cdot\|$ denote the norm of B and $\|\cdot\|_*$ the norm of B^*. We will use $\|T\|_{X,Y}$ to denote the operator norm of a bounded operator T in $L(X, Y)$.

Lemma 2. *Suppose $T \in L(H, H)$ and $T(B^*) \subset B^*$. Then the adjoint T^* of T extends uniquely by continuity to a bounded operator $(T^*)^\sim$ from B into itself, i.e. $(T^*)^\sim \in L(B, B)$. Moreover, $\|(T^*)^\sim\|_{B,B} = \|T\|_{B^*,B^*}$ and $(x, Ty) = \big((T^*)^\sim x, y\big)$ for any $x \in B$ and $y \in B^*$.*

Proof. First note that, by the closed graph theorem, $T \in L(B^*, B^*)$. Let $x \in H$; then

$$\|T^*x\| = \sup_{\|y\|_*=1}|(T^*x, y)| = \sup_{\|y\|_*=1}|\langle T^*x, y\rangle| = \sup_{\|y\|_*=1}|\langle x, Ty\rangle|$$
$$\leqslant \|x\|\sup_{\|y\|_*=1}\|Ty\|_* \leqslant \|x\|\,\|T\|_{B^*,B^*}.$$

Therefore, T^* extends uniquely to a bounded operator $(T^*)^{\sim}$ of B and $\|(T^*)^{\sim}\|_{B,B} \leqslant \|T\|_{B^*,B^*}$. Similar computation as above shows that $\|T\|_{B^*,B^*} \leqslant \|(T^*)^{\sim}\|_{B,B}$. The last assertion is obvious.

DEFINITION 3. Let μ be a local measure on an open subset U of B. A Borel subset N of U is said to be *μ-negligible* if $\mu(N \cap A) = 0$ for all A in $\mathscr{B}_0(U)$. Two Borel measurable functions f and g defined on U are said to be *equal a.e.* $[\mu]$ if the set $\{x \in U;\, f(x) \neq g(x)\}$ is μ-negligible.

Suppose that μ is a positive H-differentiable local measure on an open subset U of B such that, for each h in H, $\langle D\mu(\cdot), h\rangle$ is absolutely continuous with respect to μ. We can take an increasing sequence $\{U_n\}$ of properly bounded open subsets of U such that $\bigcup_n U_n = U$ and apply the Radon–Nikodym theorem to each U_n. In this way, we get a Borel measurable function ξ_h defined on U such that for all $A \in \mathscr{B}_0(U)$

$$\langle D\mu(A), h\rangle = \int_A \xi_h(x)\mu(dx).$$

It is easy to see that ξ_h is uniquely defined up to a.e. $[\mu]$ in the sense of Definition 3 above. ξ_h will be denoted by $d\langle D\mu, h\rangle/d\mu$ and called the *logarithmic derivative of μ in the direction h*.

DEFINITION 4. A positive H-differentiable local measure μ on an open subset U of B is said to have *logarithmic derivative* if it has logarithmic derivative in every direction h of H and there exists a Borel measurable function ξ from U into B such that

(1) $\|\xi\|$ is μ-integrable over every properly bounded subset of U, and

(2) for each k in B^*, $d\langle D\mu, k\rangle/d\mu = (\xi, k)$ a.e. $[\mu]$.

Suppose that ξ and η are two Borel measurable functions with the above property. Let $\{e_n\}$ be an orthonormal basis of H such that $e_n \in B^*$ for all n. Then, for each n, $(\xi, e_n) = (\eta, e_n)$ a.e. $[\mu]$. Hence there exists a μ-negligible set N_n such that $(\xi(x), e_n) = (\eta(x), e_n)$ for all x in N_n^c. Let $N = \bigcup_n N_n$. Then N is also μ-negligible. If $x \in N^c$, then $(\xi(x), e_n) = (\eta(x), e_n)$ for all n and so $\xi(x) = \eta(x)$. Therefore, $\xi = \eta$ a.e. $[\mu]$. Thus ξ in Definition 4 is uniquely determined up to a.e. $[\mu]$. ξ will be denoted by $dD\mu/d\mu$ and called the *logarithmic derivative* of μ. For example, when (H, B) is an abstract Wiener space, let p_t be the Wiener measure with mean 0 and variance $t > 0$. It has been shown in the example on [8], p. 193, that $\langle Dp_t(dx), h\rangle = -t^{-1}(x, h)p_t(dx)$, $h \in B^*$. Hence p_t has logarithmic derivative

$$\frac{dDp_t}{dp_t}(x) = -t^{-1}x.$$

The terminology for $dD\mu/d\mu$ is motivated by the following finite dimensional example. Let $H = B = R^n$ and $\mu(dx) = w(x)dx$, where w is a positive continuously differentiable function and dx is the Lebesgue measure on R^n. It is easy to show that $dD\mu/d\mu = (\log w)'$.

To state the next theorem, we assume the following approximation property on (H, B): there exists an orthonormal basis $\{e_n\} \subset B^*$ of H such that if $P_n x = (x, e_1)e_1 + \ldots + (x, e_n)e_n$ for x in B then $P_n x \to x$ as $n \to \infty$ for every x in B. It follows from the Uniform Boundedness Principle that $\sup_n \|P_n\|_{B,B} < \infty$.

THEOREM 2. *Suppose that B has the above approximation property. Let μ, ν, and θ be given as in Theorem 1 so that the conditions in Theorem 1 are satisfied. Suppose that, for each x in U, $\theta'(x)(B^*) \subset B^*$, $\theta'(\cdot)$ is measurable from U into $L(B^*, B^*)$, and $\|\theta'(\cdot)\|_{B^*,B^*}$ is bounded on every properly bounded subset of U.*

Then, if ν is positive and has logarithmic derivative, μ is also positive and has logarithmic derivative given by

$$\frac{dD\mu}{d\mu}(x) = \big(\theta'(x)^*\big)^{\sim}\,\frac{dD\nu}{d\nu}\big(\theta(x)\big) + \mathrm{TRACE}\, J_\theta(x).$$

Remark. The assumption that B has the approximation property is a technical one and can be dropped as follows. Suppose there exists a Banach space B_1 such that $B_1 \subset B$, (H, B_1) is a pair of spaces with the interpolation property and has the approximation property, and $dD\nu/d\nu$ takes values in B_1 a.e. $[\nu]$. Then the theorem remains true with the obvious modification, i.e. replace B in the conditions with B_1. If ν is a Wiener measure, such a space B_1 exists by [9], p. 66.

Proof. We need only to prove that for any $A \in \mathscr{B}_0(V)$ and any $k \in B^*$,

$$\int_A \langle D\nu(dy),\, \theta'\big(\theta^{-1}(y)\big)k\rangle = \int_A \left(\Big(\theta'\big(\theta^{-1}(y)\big)^*\Big)^{\sim}\left(\frac{dD\nu}{d\nu}(y)\right), k\right)\nu(dy).$$

Let P_n be given by the approximation property of (H, B), $P_n x = (x, e_1)e_1 + \ldots + (x, e_n)e_n$ $x \in B$. As in the proof of Theorem 1

$$\begin{aligned}\int_A \langle D\nu(dy),\, \theta'\big(\theta^{-1}(y)\big)k\rangle &= \sum_n \int_A \langle \theta'\big(\theta^{-1}(y)\big)k, e_n\rangle \langle D\nu(dy), e_n\rangle\\ &= \sum_n \int_A \langle \theta'\big(\theta^{-1}(y)\big)k, e_n\rangle \frac{d\langle D\nu, e_n\rangle}{d\nu}(y)\nu(dy)\\ &= \sum_n \int_A \langle \theta'\big(\theta^{-1}(y)\big)k, e_n\rangle \left(\frac{dD\nu}{d\nu}(y), e_n\right)\nu(dy)\\ &= \lim_{n\to\infty} \int_A \left(P_n\frac{dD\nu}{d\nu}(y),\, \theta'\big(\theta^{-1}(y)\big)k\right)\nu(dy).\end{aligned}$$

Let $a = \sup_n \|P_n\|_{B,B}$ and $b = \sup_{y\in A} \|\theta'(\theta^{-1}(y))\|_{B^*,B^*}$. Then

$$\begin{aligned}\left|\left(P_n\frac{dD\nu}{d\nu}(y),\, \theta'(\theta^{-1}(y))k\right)\right| &\leqslant \left\|P_n\frac{dD\nu}{d\nu}(y)\right\| \|\theta'(\theta^{-1}(y))k\|_* \\ &\leqslant \|P_n\|_{B,B}\left\|\frac{dD\nu}{d\nu}(y)\right\| \|\theta'(\theta^{-1}(y))\|_{B^*,B^*}\|k\|_* \\ &\leqslant ab\,\|k\|_*\left\|\frac{dD\nu}{d\nu}(y)\right\|.\end{aligned}$$

Moreover, as $n\to\infty$,

$$\left(P_n\frac{dD\nu}{d\nu}(y),\, \theta'(\theta^{-1}(y))k\right) \to \left(\frac{dD\nu}{d\nu}(y),\, \theta'(\theta^{-1}(y))k\right).$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty}\int_A\left(P_n\frac{dD\nu}{d\nu}(y),\, \theta'(\theta^{-1}(y))k\right)\nu(dy) &= \int_A\left(\frac{dD\nu}{d\nu}(y),\, \theta'(\theta^{-1}(y))k\right)\nu(dy) \\ &= \int_A\left(\left(\theta'(\theta^{-1}(y))^*\right)^{\sim}\left(\frac{dD\nu}{d\nu}(y)\right),\, k\right)\nu(dy).\end{aligned}$$

This completes the proof.

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