STUDIA MATHEMATICA, T. LXIII. (1978)

On some parameters associated with normed lattices and on series characterisation of M-spaces

by

Y. A. ABRAMOVIČ, E. D. POSITSELSKIĬ, and L. P. YANOVSKIĬ

(Leningrad)

Abstract. Considering some parameters of normed lattices, the authors give the positive answer to the conjecture of Schlotterbeck: In every Banach lattice X, non-isomorphic to an abstract M-space in the sense of Kakutani, there exists an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that the series of absolute values $\sum_{n=1}^{\infty} |x_n|$ is divergent. Some other results are proved.

We introduce and investigate some parameters of normed lattice which are order analogues of known Macphail's constants [6]. In Theorem 2.1 we compute these parameters for classical l_p^n-spaces $(1 \leqslant p \leqslant \infty)$. Our main result, Theorem 3.1, contains some estimates for those parameters. The following fact (Theorem 3.2) follows from received estimates: In every Banach lattice non-isomorphic to an M-space, there exists an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that the series of absolute values $\sum_{n=1}^{\infty} |x_n|$ is divergent. It confirmes the conjecture of Schlotterbeck, formulated in [5]. The last Section 4 contains some open problems. In the terminology concerning normed and partially ordered normed spaces we follow [2] and [8]. The symbols l_p^n and l_p $(1 \leqslant p \leqslant \infty, n = 1, 2, \ldots)$ have their usual meaning.

1. Definitions and auxiliary results.

DEFINITION 1.1. Let X be an KN-*lineal* ($\equiv$ normed lattice). For every $k = 1, 2, \ldots$ we define

$$\psi_k(x) = \inf \frac{\sup\limits_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\|}{\left\| \sum_{i=1}^{k} |x_i| \right\|}, \tag{1.1}$$

where inf is taken over all k-tuples $\{x_1, \ldots, x_k\} \subset X$, $\sum_{i=1}^{k} |x_i| \neq 0$, and the sup is taken over all $\varepsilon_i = \pm 1$, $i = 1, \ldots, k$. Obviously, we have $1 = \psi_1(X)$

$\geqslant \psi_2(X) \geqslant \ldots \geqslant \psi_k(X) \geqslant 0$, so there exists $\lim_{k\to\infty} \psi_k(X)$. We will denote this limit by $\psi(X)$. Obviously, $0 \leqslant \psi(X) \leqslant 1$.

As was mentioned in the introduction, the constant $\psi(X)$ is the order analogue of known Macphail's constant, precisely, the l-absolutely summing constant [4],

$$\mu(X) = \inf_{\{x_i\}} \frac{\sup_{\varepsilon_i=\pm 1} \left\| \sum_i \varepsilon_i x_i \right\|}{\sum_i \|x_i\|}.$$

Let us observe that for KN-lineals l_1^n we have

$$\psi(X) = \mu(X).$$

We recall another constant introduced in [1].

Definition 1.2. *M-constant* of a KN-lineal X is the following (finite or infinite) quantity

$$P_M(X) = \sup\{\|x\|:\ x = x_1 \vee \ldots \vee x_k;\ x_i \wedge x_j = 0 \text{ for } i \neq j,\ \|x_i\| \leqslant 1,\ k = 1, 2, \ldots\}.$$

It was proved in [1] that $P_M(X) < \infty$ if and only if there exists on X an equivalent M-norm (1) $\|\cdot\|_M$ such that

$$\|\cdot\|_M \leqslant \|\cdot\| \leqslant P_M(X)\, \|\cdot\|_M. \tag{1.2}$$

Lemma 1.1. *If X is an M-space, then $\psi_k(X) = 1$ for all $k = 1, 2, \ldots$, so that $\psi(X) = 1$.*

The proof easily follows from the representation of an M-space (the Kreins–Kakutani theorem) as a normed lattice of continuous functions on some compact space.

The following lemma will be frequently used in the sequel.

Lemma 1.2. *Let X be a KN-lineal; then, for every $k = 1, 2, \ldots$, $\psi_k(X) \geqslant P_M^{-1}(X)$, so*

$$\psi(X) \geqslant P_M^{-1}(X). \tag{1.3}$$

Proof. It is enough to consider the case where $P_M(X) < \infty$. Then by (1.2) and Lemma 1.1 we have

$$\psi_k(X) = \inf_{\{x_i\}} \frac{\sup_{\varepsilon_i=\pm 1} \left\| \sum \varepsilon_i x_i \right\|}{\left\| \sum |x_i| \right\|} \geqslant \frac{1}{P_M(X)} \inf_{\{x_i\}} \frac{\sup_{\varepsilon_i=\pm 1} \left\| \sum \varepsilon_i x_i \right\|_M}{\left\| \sum |x_i| \right\|_M} = \frac{1}{P_M(X)}.$$

(1) That is, the norm $\|\cdot\|_M$ satisfies the condition $\|x \vee y\|_M = \max(\|x\|_M, \|y\|_M)$, for $x, y \geqslant 0$, and consequently $(X, \|\cdot\|_M)$ is an M-space.

In the following section (see the remark after Lemma 2.2), we will obtain a stronger estimate from below for $\psi(X)$ when X is finite-dimensional KN-lineal.

Lemma 1.3. *Let X be a KN-lineal and let Y be an M-space. Let $Z = X + Y$ be a KN-lineal of all pairs $z = (x, y)$, $x \in X$, $y \in Y$, with $\|z\|_Z = \max(\|x\|, \|y\|)$ and a natural order. Then $\psi_k(Z) = \psi_k(X)$ for $k = 1, 2, \ldots$*

We omit an easy proof.

2. Finite-dimensional KN-lineals. Everywhere in this section, KN-lineal X is assumed to be finite dimensional; $\{e_1, \ldots, e_n\}$ will denote the natural basis for X. Let us recall that the cone of positive elements X_+ consists of vectors of the form $\sum_{i=1}^n a_i e_i$, $a_i \geqslant 0$. We can assume that $\|e_i\| = 1$ for all $i = 1, \ldots, n$. We will identify the vector $\sum_{i=1}^n a_i e_i$ with the sequence of coefficients $(a_1, \ldots, a_n)$.

The following remarks will be used later.

(a) It follows from (1.2) that

$$P_M(X) = \|\mathbf{1}\|, \tag{2.1}$$

where $\mathbf{1} = (1, \ldots, 1)$ is the vector with unit coordinates.

(b) For every $k = 1, 2, \ldots$ there exist vectors $\bar{x}_1, \ldots, \bar{x}_k \in X$ on which the inf in (1.1) is attained, i.e.

$$\psi_k(X) = \frac{\sup_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^k \varepsilon_i \bar{x}_i \right\|}{\left\| \sum_{i=1}^k |\bar{x}_i| \right\|}. \tag{2.2}$$

The proof of the following lemma uses easy combinatorics and is omitted.

Lemma 2.1. *Let $\dim X = n$ and $k \geqslant 2^n$. Then $\psi_k(X) = \psi(X)$.*

Our next goal is to compute $\psi(X)$ for classical spaces $X = l_p^n$. At first let us recall that for $n = 2^m$ $(m = 1, 2, \ldots)$ in n-dimensional space X there exists an orthogonal basis of Walsh's functions $u_1^n, u_2^n, \ldots, u_n^n$. The inductive construction of these functions can be described as follows:

for $n = 2$ $u_1^2 = (1, 1)$, $u_2^2 = (1, -1)$;

for $n = 2^2$ $u_1^4 = (1, 1, 1, 1)$, $u_2^4 = (1, 1, -1, -1)$,
$u_3^4 = (1, -1, 1, -1)$, $u_4^4 = (1, -1, -1, 1)$,

and so on, i.e. passing from $n = 2^m$ to 2^{m+1}, from each function u_i^n we construct two functions: $u_{2i-1}^{2^{m+1}}$ and $u_{2i}^{2^{m+1}}$, whose coordinates are coordinates of u_i^n followed by the same coordinates; once with the sign $+$ and once with the sign $-$.

In the case where $X = l_2^n$ $(n = 2^m)$, we have

$$\sup_{\varepsilon_i=\pm1} \Big\| \sum_i \varepsilon_i u_i^n \Big\| = \Big\| \sum_i u_i^n \Big\| = n.$$

THEOREM 2.1. *For spaces l_p^n $(1 \leqslant p \leqslant \infty)$ the following estimates are true:*

(1) *if $2 \leqslant p \leqslant \infty$, then $\psi(l_p^n) = P_M^{-1}(l_p^n) = n^{-1/p}$,*

(2) *if $1 \leqslant p < 2$, then $cn^{-1/2} \leqslant \psi(l_p^n) \leqslant n^{-1/2}$, where c is some positive constant independent of p and n.*

Proof. Let $2 \leqslant p \leqslant \infty$. By (2.1), $P_M(l_p^n) = n^{1/p}$, so, by Lemma 1.2, $\psi(l_p^n) \geqslant n^{-1/p}$. Let us prove the opposite inequality. Observe that $\|\cdot\|_{l_p^n} \leqslant \|\cdot\|_{l_2^n}$ since $p \geqslant 2$.

Let us start with the case $n = 2^m$. Let $u_1^n, \ldots, u_n^n$ denote 2^m Walsh's functions. Then for $k \geqslant n$ we have

$$\psi_k(l_p^n) \leqslant \psi_n(l_p^n) \leqslant \frac{\sup\limits_{\varepsilon_i=\pm1} \| \sum \varepsilon_i u_i^n \|_{l_p^n}}{\| \sum |u_i^n| \|_{l_p^n}} = \frac{\sup\limits_{\varepsilon_i=\pm1} \| \sum \varepsilon_i u_i^n \|_{l_p^n}}{n \|\mathbf{1}\|_{l_p^n}}$$

$$\leqslant \frac{\sup\limits_{\varepsilon_i=\pm1} \| \sum \varepsilon_i u_i^n \|_{l_2^n}}{n \|\mathbf{1}\|_{l_p^n}} = \frac{1}{\|\mathbf{1}\|_{l_p^n}} = P_M^{-1}(l_p^n) = n^{-1/p},$$

so $\psi(l_p^n) = \lim \psi_k(l_p^n) \leqslant P_M^{-1}(l_p^n) = n^{-1/p}$ what proves the desired equality in the case $n = 2^m$.

Let now n be arbitrary. We can choose the natural number m such that $2^{m-1} < n \leqslant 2^m$. Let us consider the space $Z = l_p^n + l_\infty^{2^m - n}$. By Lemma 1.3, $\psi_k(Z) = \psi_k(l_p^n)$. But the dimension of Z is 2^m, so Z contains 2^m Walsh's functions. As above, we can prove that, for $k \geqslant 2^m$, $\psi_k(Z) \leqslant P_M^{-1}(l_p^n)$. This proves equality (1).

To prove the case (2) we start with an observation that

$$\|\cdot\|_{l_p^n} \leqslant n^{1/p - 1/2} \|\cdot\|_{l_2^n}.$$

If $n = 2^m$ and $u_1^n, \ldots, u_n^n$ are Walsh's functions in l_p^n and $k \geqslant n$, we have

$$\psi_k(l_p^n) \leqslant \psi_n(l_p^n) \leqslant \frac{\sup\limits_{\varepsilon_i=\pm1} \| \sum \varepsilon_i u_i^n \|_{l_p^n}}{\| \sum |u_i^n| \|_{l_p^n}} \leqslant \frac{n^{1/p-1/2} \sup\limits_{\varepsilon_i=\pm1} \| \sum \varepsilon_i u_i^n \|_{l_2^n}}{\| \sum |u_i^n| \|_{l_p^n}}$$

$$= \frac{n^{1/p-1/2} n}{n \|\mathbf{1}\|_{l_p^n}} = \frac{n^{1/p-1/2}}{n^{1/p}} = n^{-1/2},$$

so, for $n = 2^m$, $\psi(l_p^n) \leqslant n^{-1/2}$.

Using Lemma 1.3, like in the proof of case (1), we get this inequality for arbitrary n.

To prove the theorem, we have to establish the inequality $cn^{-1/2} \leqslant \psi(l_p^n)$, for some $c > 0$. As we remarked $\psi(l_1^n) = \mu(l_1^n)$, and the Macphail constants $\mu(l_1^n)$ satisfy the inequality $\mu(l_1^n) \geqslant cn^{-1/2}$ (cf. [4]), where c is a constant independent of n. To finish the proof it is enough to prove the following

LEMMA 2.2. *Let X be an n-dimensional* KN-*lineal. Then $\psi(X) \geqslant \psi(l_1^n) \geqslant cn^{-1/2}$.*

Proof. By remark (b) (cf. (2.2)) and Lemma 2.1, there exist $\bar{x}_1, \ldots, \bar{x}_k \in X$ such that

$$\psi(X) = \frac{\sup\limits_{\varepsilon_i=\pm1} \| \sum \varepsilon_i \bar{x}_i \|}{\| \sum |\bar{x}_i| \|}.$$

Obviously, we can assume that $\|z\| = 1$, where $z = \sum |\bar{x}_i|$. We can construct a hyperplane supporting the unite ball of X at point z. We assume that this hyperplane intersects all coordinate axes (the general case can be reduced to this one by an easy approximation). By $\|\cdot\|_1$ we denote the new monotone norm on X, determined by the condition that the positive part of the new unit ball coincides with the set of positive elements in X lying below our hyperplane. It is clear that $\|\cdot\| \geqslant \|\cdot\|_1$ and that $(X, \|\cdot\|_1)$ is order isometric to l_1^n. We have $\|z\| = \|z\|_1 = 1$, so

$$\psi(X) = \sup_{\varepsilon_i=\pm1} \Big\| \sum \varepsilon_i \bar{x}_i \Big\| \geqslant \sup_{\varepsilon_i=\pm1} \Big\| \sum \varepsilon_i \bar{x}_i \Big\|_1 \geqslant \psi(l_1^n) \geqslant cn^{-1/2}.$$

Remark. Lemmas 1.2 and 2.2 imply the stronger estimate from below for n-dimensional KN-lineal X:

$$\psi(X) \geqslant \max\big(cn^{-1/2}, P_M^{-1}(X)\big).$$

3. Characterisation of M-spaces. Now we prove our main theorem. It shows the relation between $\psi(X)$ and $P_M(X)$.

THEOREM 3.1. *For an arbitrary* KN-*lineal X we have*

$$P_M^{-1}(X) \leqslant \psi(X) \leqslant P_M^{-1/2}(X).$$

In particular, if $P_M(X) = \infty$, then $\psi(X) = 0$.

Proof. We need to show only the right-hand side inequality. Let us start with finite-dimensional KN-lineal X, $\dim X = n$ and let $\{e_j\}_{j=1}^n$ be the natural basis in X. We will assume $\|e_j\| = 1$, $j = 1, 2, \ldots, n$.

By Λ we denote the n-dimensional Lorentz space with the norm defined by

$$\|x\|_\Lambda = \sum_{i=1}^{p} x_i^* + \big(P_M(X) - p\big) x_{p+1}^*,$$

where $p = [P_M(X)]$ is the integer part of $P_M(X)$, and $\{x_i^*\}$ is the decreasing rearrangement of absolute values of coordinates of the vector x.

One can show that $\|\cdot\|_A$ is the strongest lattice norm on X satisfying $\|e_j\| = 1$ for $j = 1, 2, \ldots, n$ and $\|\mathbf{1}\| = P_M(X)$. To see this, let us observe that for the strongest norm $\|\ \|'$ satisfying the above conditions, the set of extreme points of the unit ball coincides with the set of extreme points of the unit ball of $\|\cdot\|_A$. This implies $\|\cdot\| = \|\cdot\|'$.

An easy calculation shows that

$$\max_{0\neq x\in X} \frac{\|x\|_A}{\|x\|_{l_2^n}} = \sqrt{p+(P_M(X)-p)^2},$$

so (using the fact that $(P_M(X)-p)^2 \leqslant P_M(X)-p$) we infer

$$\|x\|_A \leqslant P_M^{1/2}(X)\,\|x\|_{l_2^n}.$$

Without loss of generality we can assume that $n = 2^m$ (see Lemma 1.3), so we can consider Walsh's functions $u_1^n, \ldots, u_n^n$ in X. Let us recall that $|u_i^n| = \mathbf{1}$ and $P_M(X) = \|\mathbf{1}\|$.

Using the fact that $\|x\| \leqslant \|x\|_A \leqslant P_M^{1/2}(X)\,\|x\|_{l_2^n}$, we get

$$\begin{aligned}\psi(X) \leqslant \psi_n(X) &\leqslant \frac{\sup_{\varepsilon_i=\pm 1}\|\sum \varepsilon_i u_i^n\|}{\|\sum |u_i^n|\,\|} = (nP_M(X))^{-1}\sup_{\varepsilon_i}\Big\|\sum \varepsilon_i u_i^n\Big\| \\ &\leqslant (nP_M(X))^{-1}\sup_{\varepsilon_i}\Big\|\sum \varepsilon_i u_i^n\Big\|_A \\ &\leqslant \frac{1}{nP_M(X)}\,P_M^{1/2}(X)\sup_{\varepsilon_i}\Big\|\sum \varepsilon_i u_i^n\Big\|_{l_2^n} \leqslant \frac{P_M^{1/2}(X)n}{nP_M(X)} = P_M^{-1/2}(X),\end{aligned}$$

so $\psi(X) \leqslant P_M^{-1/2}(X)$ and the theorem is proved for finite-dimensional X. Now we will reduce the general case to the previous one.

Let $P_M(X) < \infty$. Then for every $\varepsilon > 0$ there exist $x_1, \ldots, x_n$ such that $\|x_i\| = 1$, $x_i \wedge x_j = 0$ for $i \neq j$ and $\|x_1 \vee x_2 \vee \ldots \vee x_n\| > P_M(X) - \varepsilon$. Let X_n be the linear span of $x_1, \ldots, x_n$. Since x_i's are pairwise disjoint, we have that X_n is KN-sublineal in X, thus $\psi(X_n) \leqslant P_M^{-1/2}(X_n)$. But $P_M(X_n) = \|x_1 \vee x_2 \vee \ldots \vee x_n\|$ and $\psi(X) \leqslant \psi(X_n)$ so $\psi(X) \leqslant (P_M(X)-\varepsilon)^{-1/2}$. Since ε was arbitrary, we have $\psi(X) \leqslant P_M^{-1/2}(X)$.

If $P_M(X) = \infty$, the analogous proof shows that $\psi(X) = 0$. This proves the theorem.

Remark. The inequalities $P_M^{-1}(X) \leqslant \psi(X) \leqslant P_M^{-1/2}(X)$ are exact as follows from Theorem 2.1: we put $X = l_1^n$ for the right-hand side inequality and $X = l_\infty^n$ for the left-hand side one.

DEFINITION 3.1. We say that KN-*lineal satisfies condition* (J) if there exists an unconditionally convergent series $\sum x_n$ in X such that $\sum |x_n|$ is divergent.

There is a clear connection between condition (J) and the well-known Dvoretzky–Rogers theorem saying that in every infinite-dimensional Banach space there exists an unconditionally convergent but not absolutely convergent series. In fact, for the space l_1 condition (J) is equivalent to the Dvoretzky–Rogers theorem. For l_p, $1 < p < 2$, Jameson observed that Dvoretzky–Rogers theorem implies condition (J). Apparently these are all the connections. Jameson [5] have shown that l_2 satisfies (J) and asked whether l_p for $p > 2$ also satisfies (J). Moreover, at the end of paper [5] is quoted the conjecture of Schlotterbeck saing that condition (J) is fulfilled in every KN-lineal, not isomorphic to an M-space.

Using Theorem 3.1, we can easily confirm this conjecture.

THEOREM 3.2. *For Banach* KN-*lineal* X *the following conditions are equivalent*:

(1) X *admits an equivalent* M-*norm*;

(2) $P_M(X) < \infty$;

(3) $\psi(X) > 0$;

(4) X *does not satisfy condition* (J).

Proof. As we remarked in the introduction, the equivalence of (1) and (2) was shown by one of the authors in [1]. Condition (2) is equivalent to (3) by Theorem 3.1. The equivalence of (3) and (4) easily follows from definitions, as was observed in [5].

Remark. If, analogously to condition (J), we introduce the condition (J_C), which requires the existence of an unconditionally Cauchy series $\sum x_n$ such that $\sum |x_n|$ is not Cauchy, we can prove a theorem similar to Theorem 3.2 for arbitrary KN-lineals (not necessarily norm complete).

4. Additional remarks. (1) The alternative proof of Theorem 3.2 can be obtained by using results of Fremlin [3]. It can be found in the book of Schaefer [7].

(2) It is interesting to find the procedure for exact evaluation of $\psi_k(X)$, at least for classical l_p^n spaces, $1 \leqslant p < 2$. It is also interesting to compute $\psi(X)$ (and maybe also $\psi_k(X)$) for finite-dimensional Marcinkiewicz spaces $M(C)$ and Lorentz spaces $\Lambda(C)$.

(3) Instead of Walsh functions used in proofs of Sections 2 and 3 one can use another set of functions, namely the set K of all vertices of the unite ball of l_∞^n.

(4) It would be interesting to find those sets of vectors on which $\psi(X)$ is attained (for finite-dimensional X). Probably, the set K is good for symmetric spaces.

The authors are thankful to V. Gejler and F. Wojtaszczyk for attention and for the help in translation of this paper into English.

References

[1] Y. A. Abramovič, *Some theorems on normed structures*, Vestnik L. G. U. 13 (1971), pp. 5–11 (in Russian).
[2] M. M. Day, *Normed linear spaces*, Springer-Verlag, Berlin 1973.
[3] D. Fremlin, *A characterisation of L-spaces*, Indag. Math. 36 (1974), pp. 270–275.
[4] Y. Gordon, *On p-absolutely summing constants of Banach spaces*, Israel J. Math. 7 (1969), pp. 151–163.
[5] G. J. O. Jameson, *Unconditional convergence in partially ordered linear spaces*, Math. Ann. 200 (1973), pp. 227–233.
[6] M. S. Macphail, *Absolute and unconditional convergence*, Bull. Amer. Math. Soc. 53 (1947), pp. 121–123.
[7] H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin 1974.
[8] B. Z. Vulikh, *Introduction to the theory of partially ordered spaces*, Wolters-Noordhoff, Groningen 1967.

Received February 17, 1976 (1122)

STUDIA MATHEMATICA, T. LXIII. (1978)

О ветвлении и устойчивости периодических решений дифференциальных уравнений с неаналитической правой частью

П. Г. АЙЗЕНГЕНДЛЕР, М. М. ВАЙНБЕРГ (Москва)

Резюме. Авторами был предложен метод для решения задачи Пуанкаре и выяснения вопроса об устойчивости решений этой задачи в аналитическом случае (Доклады АН СССР 165.2 (1965), стр. 255–257; 176.1 (1967), стр. 9–12; 179.5 (1968), стр. 1015–1018). Полное доказательство всех предложений данных работ было дано в монографии М. М. Вайнберга и В. А. Треногина (*Теория ветвления решений нелинейных уравнений*, Наука, Москва 1969). Здесь рассматривается неаналитический случай задачи Пуанкаре в вещественном банаховом пространстве. Предлагается метод для нахождения числа решений и их асимптотического представления. Для иллюстрации предлагаемого метода приводятся примеры. Идея метода заключается в том, что если функция, действующая в банаховом пространстве, не является аналитической, но дифференцируема по Фреше n раз, то ее можно представить в виде суммы полинома степени n и остатка. В статье показано, как в этом случае можно находить число решений и асимптотику. В том случае, когда пространство конечномерно, исследуется вопрос об устойчивости решений.

1. Рассмотрим уравнение

$$\frac{dx}{dt} = Ax + \lambda F(t, x, \lambda) + \lambda Q(t, x, \lambda), \tag{1}$$

где $\lambda \geqslant 0$ — малый параметр; A — линейный ограниченный оператор, действующий в вещественном банаховом пространстве E; $F(t, x, \lambda) = \sum_{i+k=0}^{n} F_{ik}(t) x^i \lambda^k$ — полином в смысле Фреше с непрерывным и T-периодическими коэффициентами; Q — периодическая по t функция с периодом T, непрерывная и ограниченная на некотором множестве

$$K = \{(t, x, \lambda):\ t \in \boldsymbol{R}^1,\ \|x\| \leqslant \varrho_0,\ \lambda \in [0, \varrho_0],\ \varrho_0 > 0 - \text{const}\}$$

и удовлетворяющая равномерно по t условию

$$\|Q(t, x, \lambda)\| = o\big((\|x\| + \lambda)^n\big) \quad \text{при} \quad \|x\| + \lambda \to 0. \tag{2}$$

Определение. Функция $x(t, \lambda)$: $\boldsymbol{R}^1 \times [0, \bar\varrho_0] \to E$, где $0 < \bar\varrho_0 \leqslant \varrho_0$, называется *малым T-периодическим решением* уравнения (1), если выполнены условия: 1) она дифференцируема и T-периодична по t при