Therefore $f(t) \leqslant I/g(t)$ for any $t > 0$. Integrating, we get

$$\int_0^\infty f(t)\,d(-g) = \int_0^\infty f(t)^\gamma f(t)^{1-\gamma}\,d(-g)$$
$$\leqslant \int_0^\infty f(t)^\gamma I^{1-\gamma} g(t)^{\gamma-1}\,d(-g)$$
$$= I^{1-\gamma}\frac{1}{\gamma}\int_0^\infty f(t)^\gamma\,d(-g^\gamma) = \frac{1}{\gamma}I. \ ■$$

7. Proof of Theorem II. In view of part (iv) of Proposition 2.1, it suffices to assume that $q = 1$, and in view of Theorem I, it suffices to assume that $m = k+\varepsilon$, $\varepsilon > 0$. The proof follows that of Theorem I with minor modifications. Define

$$\lambda(s) = |\{N_q^m(F) > s\}| \quad \text{and} \quad \mu(s) = \big|\{|D^k F| > s\}\big|.$$

As in Lemma 5.1, decompose $F = F_1+F_2$; since $F_1 \equiv 0$ in $\mathcal{O}^c$,

$$\mu(s) \leqslant |\mathcal{O}| + \big|\{|D^k F_2| > s\}\big|.$$

From (5.2), it follows that

$$\big|\{|D^k F_2| > s\}\big| \leqslant cs^{-u}\|N_q^m(F_2)\|_p^u,$$

so that Lemma 5.1 yields

$$\mu(s) \leqslant \lambda(t) + cs^{-u}\Big(\int_0^t \lambda(\sigma)\,d\sigma^p\Big)^{u/p}.$$

Using this replacement for Theorem III, the result now follows as in the proof of Theorem I.

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Received May 6, 1976 (1155)

STUDIA MATHEMATICA. T. LXII. (1978)

A characterization of $H^p(\boldsymbol{R}^n)$ in terms of atoms

by

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Abstract. Distributions in $H^p(\boldsymbol{R}^n)$, where $p \leqslant 1$, are represented as weighted sums of atoms.

§ 1. Introduction. Let $H^p(\boldsymbol{R}^n)$ denote the space of functions u, harmonic in the upper half-space $R_+^{n+1} = \{(x, y) = (x_1, \ldots, x_n, y)\colon y > 0\}$, whose non-tangential maximal function $u^*(x_0) = \sup\limits_{|x-x_0|<y} |u(x, y)|$ is in the Lebesgue space $L^p(\boldsymbol{R}^n)$. Give $H^p(\boldsymbol{R}^n)$ the "norm" $\|u\|_{H^p} = \|u^*\|_p$. C. Fefferman and E. M. Stein [4] have shown that if $u \in H^p(\boldsymbol{R}^n)$, then $\lim\limits_{t\to 0} u(\cdot, t) = f$ exists in the sense of tempered distributions and that u is uniquely determined by f. We will denote also by $H^p(\boldsymbol{R}^n)$ the space of boundary distributions of functions in $H^p(\boldsymbol{R}^n)$. R. Coifman [2] has exhibited an explicit representation for $f \in H^p(\boldsymbol{R}^1)$, $0 < p \leqslant 1$, by means of a purely real variables construction. Here we modify Coifman's construction in order to obtain such a representation for $H^p(\boldsymbol{R}^n)$, $n \geqslant 1$.

Let $0 < p \leqslant 1$ and define a *p-atom* to be a function b on $\boldsymbol{R}^n$ which is supported on a cube Q in $\boldsymbol{R}^n$ with sides parallel to the axes and which satisfies

(i) $|b(x)| \leqslant |Q|^{-1/p}$, where $|Q|$ is the volume of Q

and

(ii) $\int b(x)x^\alpha\,dx = 0$, where α is a multi-index of order $|\alpha| \leqslant N = [n(1/p-1)]$, the integer part of $n(1/p-1)$. We then have:

THEOREM. *A distribution f is in $H^p(\boldsymbol{R}^n)$, $0 < p \leqslant 1$, if and only if there exist a sequence of p-atoms b_i and a sequence of non-negative real numbers λ_i such that*

$$f = \sum_{i=0}^\infty \lambda_i b_i \tag{1.1}$$

in the sense of distributions and

$$A\|f\|_{H^p}^p \leqslant \sum_{i=0}^\infty \lambda_i^p \leqslant B\|f\|_{H^p}^p, \tag{1.2}$$

where A, B are constants which depend only on n and p.

C. Fefferman obtained this theorem by non-constructive means in case $p=1$. In fact, he showed that it is equivalent to the duality between H^1 and BMO. A constructive proof for $p\leqslant 1$ was obtained in the case of martingales by C. Herz [5].

Before proceeding I should like to express my deep gratitude to Steven Krantz and especially to John Garnett without whose help and encouragement I could not have undertaken this work.

§ 2. Proof of Theorem—the easy direction. We begin by proving the left inequality of (1.2). Let b be a p-atom. It is no loss of generality to assume that b is supported on a cube Q centered at the origin with side of length l. Let $\varphi\geqslant 0$ be an infinitely differentiable function with compact support in the unit ball of $\boldsymbol{R}^n$, $\{x\colon |x|<1\}$, and with $\int_{\boldsymbol{R}^n}\varphi=1$. Then, if $\varphi_\varepsilon(t)=\varepsilon^{-n}\varphi(t/\varepsilon)$,

$$|\varphi_\varepsilon * b| = \Big|\int_{\boldsymbol{R}^n}\varphi_\varepsilon(x-t)b(t)\,dt\Big|\leqslant |Q|^{-1/p}. \tag{2.1}$$

Let $2Q$ denote the cube with the same center as Q but with twice the diameter, and fix $x\notin 2Q$. Notice that then $|x|/2\leqslant |x-t|\leqslant 2|x|$ for $t\in Q$. Thus $\varphi_\varepsilon(x-t)=0$ if $\varepsilon\leqslant |x|/2$. Fix $\varepsilon>|x|/2$. Let P_φ be the Taylor polynomial approximation to φ expanded about x/ε of degree N. Then

$$|\varphi_\varepsilon * b| = \left|\frac{1}{\varepsilon^n}\int_Q\left[\varphi\left(\frac{x-t}{\varepsilon}\right)-P_\varphi\left(-\frac{t}{\varepsilon}\right)\right]b(t)\,dt\right|$$
$$\leqslant \frac{C}{\varepsilon^n}\int_Q\left|\frac{t}{\varepsilon}\right|^{N+1}|b(t)|\,dt\leqslant \frac{C|Q|^{1-1/p}l^{N+1}}{|x|^{N+1+n}}.$$

(Here C is a "constant" which may change but will depend only on n and p.) It follows that

$$\int_{x\notin 2Q}[\sup_{\varepsilon>0}|\varphi_\varepsilon * b|]^p\,dx\leqslant C|Q|^{p-1}l^{p(N+1)}\int_{|x|>l}\frac{dx}{|x|^{(N+1+n)p}}\leqslant C.$$

Putting this together with (2.1) yields $\sup_{\varepsilon>0}|\varphi_\varepsilon * b|\in L^p$ and $\|\sup_{\varepsilon>0}|\varphi_\varepsilon * b|\|_p\leqslant C$. From the maximal function characterizations of $H^p(\boldsymbol{R}^n)$ [4] it follows that if $\sum_j\lambda_j^p<\infty$ and if b_j are p-atoms, then $\sum\lambda_j b_j\in H^p$ and $\|\sum\lambda_j b_j\|_{H^p}^p\leqslant C\sum\lambda_j^p$.

§ 3. The converse. The proof of the converse will be presented in four steps. This section will contain the preliminaries. In the following section we will complete the proof for the case $p=1$. Section 5 will contain the modifications necessary to extend to $p<1$, and in Section 6 we will tie up some loose ends.

At the outset we assume $f\in L^1\cap H^p$, which is dense in H^p (see [4]). We will remove the restriction by a limiting argument later. The "grand" maximal function is defined as follows: Put

$$\mathscr{A}_{N_0}=\Big\{\varphi\in C^{N_0}(\boldsymbol{R}^n)\colon\ |||\varphi|||_{N_0}=\int_{\boldsymbol{R}^n}(1+|x|)^{N_0}\sum_{|\beta|\leqslant N_0}|D^\beta\varphi(x)|\,dx\leqslant 1\Big\},$$

$$f^*_{N_0}(x)=\sup_{\varphi\in\mathscr{A}_{N_0}}\ \sup_{|x-y|<10t}|\varphi_t * f(y)|.$$

Fefferman and Stein have shown in [4] that if $f\in L^1$ and if N_0 is chosen large enough, depending only on n and p, then $f\in H^p$ if and only if $f^*_{N_0}\in L^p$. Moreover, there are constants A, B, independent of f, such that

$$A\|f\|_{H^p}\leqslant \|f^*_{N_0}\|_p\leqslant B\|f\|_{H^p}.$$

(It is easily seen that, if $f\in L^1\cap H^p$, $f^*_{N_0}$ coincides with the function f^* of [4].)

Let $\Omega_k=\{x\colon f^*_{N_0}(x)>2^k\}$ for $k=0,\pm1,\pm2,\ldots$ The Whitney decomposition theorem (see Stein [7]) provides us with closed dyadic cubes Q_j^k with the following properties:

(3.1)

(a) $\Omega_k=\bigcup_{j=1}^\infty Q_j^k$, $k=0,\pm1,\ldots$

(b) The interiors of the cubes Q_i^k and Q_j^k are disjoint whenever $i\neq j$.

(c) Put $d_j^k=$ diameter of Q_j^k. Then $d_j^k\leqslant$ distance $(Q_j^k,\boldsymbol{R}^n\setminus\Omega_k)\leqslant 4d_j^k$.

(d) If $l>k$, then for each j there is an i such that $Q_j^l\subseteq Q_i^k$.

Let $(Q_j^k)^*$ be the cube with the same center as Q_j^k but with $9/8$ the diameter. We now present a lemma which will provide us with partitions of unity over the Ω_k's:

LEMMA 3.2. *Let N_0 be fixed as above. Let $\{Q_j^k\}$ be the Whitney decomposition of Ω_k described above. Then for each $k=0,\pm1,\ldots$ and $j=1,2,\ldots$ there is a function $\varphi_j^k\in C^{N_0}$ with the following properties:*

(i) *$0\leqslant\varphi_j^k\leqslant 1$ and φ_j^k has compact support equal to $(Q_j^k)^*$.*

(ii) *If $|\alpha|\leqslant N_0$, then $|D^\alpha\varphi_j^k|\leqslant A_\alpha(d_j^k)^{-|\alpha|}$. Also, there are constants A, B, depending only on n, such that*

$$A(d_j^k)^n\leqslant\int_{\boldsymbol{R}^n}\varphi_j^k\leqslant B(d_j^k)^n.$$

(iii) $$\sum_{j=1}^\infty\varphi_j^k=\chi_{\Omega_k}\equiv\begin{cases}1 & \text{if } x\in\Omega_k,\\ 0 & \text{if } x\notin\Omega_k.\end{cases}$$

(iv) *If* $|\alpha| \leqslant N_0$, *then*

$$|D^\alpha(\varphi_i^k \varphi_j^{k+1})| \leqslant B_\alpha (d_j^{k+1})^{-|\alpha|-n} \int_{\mathbf{R}^n} \varphi_i^k \varphi_j^{k+1}.$$

(v) *If* $\varphi_i^k \varphi_j^{k+1} \not\equiv 0$, *then there are* 2^n *cubes*, Q_l, *and a constant* A *such that for some* l,

$$\varphi_i^k(x)\varphi_j^{k+1}(x) \geqslant \frac{A}{|Q_j^{k+1}|} \int_{\mathbf{R}^n} \varphi_i^k \varphi_j^{k+1} \quad \textit{for all } x \in Q_l.$$

We will indicate the proof of this rather technical lemma later.

§ 4. The case $p = 1$. With Coifman [2] we write, for each integer k,

$$(4.1)\qquad f = \Big(f\chi_{\Omega_k^c} + \sum_{j=1}^{\infty} m_j^k \varphi_j^k\Big) + \sum_{j=1}^{\infty} (f - m_j^k)\varphi_j^k = g_k + \sum_{j=1}^{\infty} (f - m_j^k)\varphi_j^k,$$

where $m_j^k = \dfrac{1}{\int \varphi_j^k} \int f\varphi_j^k$. Then

$$(4.2)\qquad |m_j^k| \leqslant C 2^k.$$

To see this pick $y_j^k \notin \Omega_k$ such that $|x - y_j^k| < 10 d_j^k$ for all $x \in (Q_j^k)^*$. Put

$$\Phi_j^k(x) = \frac{(d_j^k)^n}{\int \varphi_j^k} \varphi_j^k(y_j^k - d_j^k x).$$

Then, by Lemma 3.2, for some constant C, $C\Phi_j^k \in \mathscr{A}_{N_0}$. Thus

$$|m_j^k| = |(\Phi_j^k)_{d_j^k} * f(y_j^k)| \leqslant C f_{N_0}^*(y_j^k) \leqslant C 2^k.$$

From (4.2) and (4.1) we have $|g_k| \leqslant C 2^k$ for each k and, consequently, $g_k \to 0$ as $k \to -\infty$. On the other hand, $f - g_k$ is supported on Ω_k, and hence $g_k \to f$ as $k \to \infty$ a.e. Thus we find

$$(4.3)\qquad f = \sum_{k=-\infty}^{\infty} g_{k+1} - g_k \quad \text{a.e.}$$

Let us examine

$$(4.4)\qquad g_{k+1} - g_k = \sum_{i=1}^{\infty} \Big[(f - m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1}\Big] + \sum_{j=1}^{\infty} \Big[\sum_{i=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} - (f - m_j^{k+1})\varphi_j^{k+1}\Big]$$

$$= \sum_{i=1}^{\infty} \beta_i^k + \sum_{j=1}^{\infty} \gamma_j^k,$$

where

$$m_{ij}^{k+1} = \frac{1}{\int \varphi_i^k \varphi_j^{k+1}} \int f \varphi_i^k \varphi_j^{k+1}$$

when this makes sense and is zero otherwise. Using (iv) of Lemma 3.2, the argument leading to (4.2) shows that $|m_{ij}^{k+1}| \leqslant C 2^{k+1}$. Now a moment's reflection yields $|\beta_i^k| \leqslant C 2^{k+1}$ and $|\gamma_j^k| \leqslant C 2^{k+1}$. Notice also that $\int \beta_i^k = 0$ and $\int \gamma_j^k = 0$ and that β_i^k and γ_j^k are supported on cubes. Thus we may write $\beta_i^k = \lambda_i^k a_i^k$ and $\gamma_j^k = \mu_j^k b_j^k$, where $\lambda_i^k = C|Q_i^k| 2^{k+1}$ and $\mu_j^k = C|Q_j^{k+1}| 2^{k+1}$ and a_i^k, b_j^k are 1-atoms. Because of (4.3) and (4.4), we have

$$(4.5)\qquad f = \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} (\lambda_i^k a_i^k + \mu_i^k b_i^k).$$

Finally,

$$\sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} (\lambda_i^k + \mu_i^k) = C \sum_{k=-\infty}^{\infty} 2^{k+1} \sum_{i=1}^{\infty} (|Q_i^k| + |Q_i^{k+1}|)$$

$$= C \sum_{k=-\infty}^{\infty} 2^{k-1} |\Omega_k| + C \sum_{k=-\infty}^{\infty} 2^k |\Omega_{k+1}|$$

$$\leqslant C \int_0^\infty |\{x\colon f_{N_0}^*(x) > \lambda\}|\, d\lambda$$

$$= C\|f_{N_0}^*\|_1 \leqslant C\|f\|_{H^1}.$$

§ 5. The case $p < 1$. The case $p < 1$ requires a modification: We must replace the "mean values" m_j^k and m_{ij}^{k+1} by certain polynomials defined below. Once this is done we will, except for the proof of Lemma 3.2, have proved the theorem completely for $f \in L^1(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$. We will pass to the general case via a limiting argument.

Define P_j^k, for each j and k, to be the unique polynomial with the property that

$$\int_{\mathbf{R}^n} (x - x_j^k)^\alpha P_j^k(x) \varphi_j^k(x)\, dx = \int_{\mathbf{R}^n} (x - x_j^k)^\alpha f(x) \varphi_j^k(x)\, dx$$

for $|\alpha| \leqslant N = [n(1/p - 1)]$. Here x_j^k is the center of the cube Q_j^k. Also, if $\varphi_i^k \varphi_j^{k+1} \not\equiv 0$, define P_{ij}^{k+1} in a like manner by

$$\int_{\mathbf{R}^n} (x - x_j^{k+1})^\alpha P_{ij}^{k+1}(x) \varphi_i^k(x) \varphi_j^{k+1}(x)\, dx = \int_{\mathbf{R}^n} (x - x_j^{k+1})^\alpha f(x) \varphi_i^k(x) \varphi_j^{k+1}(x)\, dx$$

for $|\alpha| \leqslant N$. If $\varphi_i^k \varphi_j^{k+1} \equiv 0$, put $P_{ij}^{k+1} = 0$. C. Fefferman, N. M. Riviere, and Y. Sagher use the polynomials P_j^k in [3], and there it is shown that if $x \in (Q_j^k)^*$, then $|P_j^k(x)| \leqslant C 2^k$. We modify their proof to show that $|P_{ij}^{k+1}(x)| \leqslant C 2^{k+1}$ on $(Q_j^{k+1})^*$. Fix i, j and k with $\varphi_i^k \varphi_j^{k+1} \not\equiv 0$. We note that by a dilation and a translation of $\mathbf{R}^n$ we may assume that Q_j^{k+1} has side of length 1 and center at the origin. Also, for simplicity, we denote by φ the function $\frac{1}{\int \varphi_i^k \varphi_j^{k+1}} \varphi_i^k \varphi_j^{k+1}$.

Let $\pi_1, \ldots, \pi_L$ be an orthonormal basis for the Hilbert space of polynomials of degree no larger than N with the norm

$$\|P\|_\varphi^2 = \int_{R^n} |P(x)|^2 \varphi(x)\, dx.$$

Then the polynomials $\{\pi_l\}$ have coefficients bounded above by a constant independent of φ. To see this we begin the construction of the π_l with $\pi_1 = 1$. By the Gram–Schmidt orthonormalization process

$$\pi_m(x) = \frac{x^{a_m} - \sum_{l=1}^{m-1} \pi_l \int x^{a_m} \pi_l \varphi\, dx}{\left\| x^{a_m} - \sum_{l=1}^{m-1} \pi_l \int x^{a_m} \pi_l \varphi\, dx \right\|_\varphi}. \tag{5.1}$$

If $j = 1, \ldots, m-1$ and if a_j^l is the coefficient of x^{a_j} in π_l, then

$$a_j^m = \frac{-\sum_{l=j}^{m-1} a_j^l \left(\int x^{a_m} \pi_l \varphi dx\right)}{\left\| x^{a_m} - \sum_{l=1}^{m-1} \pi_l \left(\int x^{a_m} \pi_l \varphi dx\right) \right\|_\varphi}. \tag{5.2}$$

Using (v) of Lemma 3.2 we see that for some r, $1 \leqslant r \leqslant 2^n$,

$$\left\| x^{a_m} - \sum_{l=1}^{m-1} \pi_l \left(\int x^{a_m} \pi_l \varphi dx\right) \right\|_\varphi^2 \geqslant A \int_{Q_r} \left| x^{a_m} - \sum_{l=1}^{m-1} \pi_l \left(\int x^{a_m} \pi_l \varphi dx\right) \right|^2 dx$$
$$\geqslant A \int_{Q_r} |x^{a_m} - R_{m-1}^r|^2 dx \geqslant C.$$

Here R_{m-1}^r is the projection of x^{a_m} into the Hilbert space generated by $\{1, x^{a_2}, \ldots, x^{a_{m-1}}\}$ with the norm $\| \ \|_{\chi_{Q_r}}$. By induction on m, (5.2) shows $|a_j^m| \leqslant C$ for $j = 1, \ldots, m$. The argument used to prove (4.2) now shows

$$\left| \int_{R^n} f(x) \pi_l(x) \varphi(x)\, dx \right| \leqslant C 2^{k+1}. \tag{5.3}$$

From (5.3) and the identity

$$P_{ij}^{k+1} = \sum_{l=1}^{L} \left(\int_{R^n} f(x) \pi_l(x) \varphi(x)\, dx \right) \pi_l$$

it follows that $|P_{ij}^{k+1}| \leqslant C 2^{k+1}$ on $(Q_j^{k+1})^*$.

The proof used in the case $p = 1$ now yields, with obvious modifications, the result when $f \in H^p \cap L^1$. Let $f \in H^p$ be arbitrary. Since $H^p \cap L^1$

is dense in H^p, we may choose $f_n \in H^p \cap L^1$ such that $f_n \to f$ in H^p as $n \to \infty$, $\|f_1\|_{H^p}^p \leqslant \frac{3}{2} \|f\|_{H^p}^p$, and $\|f_n - f_{n-1}\|_{H^p}^p \leqslant 2^{-n} \|f\|_{H^p}^p$ $(n > 1)$. Putting $g_1 = f_1$ and $g_n = f_n - f_{n-1}$ $(n > 1)$, we have $f = \sum_{n=1}^\infty g_n$. Let $g_n = \sum_{j=1}^\infty \lambda_j^n b_j^n$ be the decomposition into p-atoms of g_n whose existence was proved above. Then $f = \sum_{n=1}^\infty \sum_{j=1}^\infty \lambda_j^n b_j^n$ and

$$\sum_{n=1}^\infty \sum_{j=1}^\infty |\lambda_j^n|^p \leqslant C \sum_{n=1}^\infty \|g_n\|_{H^p}^p \leqslant C \|f\|_{H^p}^p.$$

We now have the decomposition (1.1) for $f \in H^p$ except the series converges to f in H^p norm. The corollary below shows this convergence also takes place in the sense of distributions. This completes the proof of the theorem.

If φ is a testing function denote by $P_Q(\varphi)$ the unique polynomial of degree $N = [n(1/p - 1)]$ with the property that

$$\int_Q x^a \varphi(x)\, dx = \int_Q x^a P_Q(\varphi)(x)\, dx$$

for $|a| \leqslant N$. As in Coifman [2] we note the following

COROLLARY. *If φ is a testing function, then*

$$\left| \int_{R^n} f(x) \varphi(x)\, dx \right| \leqslant C \|f\|_{H^p}^p \sup_Q |Q|^{-1/p} \int_Q |\varphi - P_Q(\varphi)|\, dx.$$

Proof. It is true for p-atoms.

We note that, by results of G. N. Meyers [6] and of S. Campanato [1],

$$\sup_Q |Q|^{-1/p} \int_Q |\varphi - P_Q(\varphi)|\, dx$$

is a norm on the Lipschitz space Λ_a, $a = n(1/p - 1)$, equivalent to the usual norm, if $p < 1$. If $p = 1$ it is the BMO norm. The corollary thus implies the duality results

$$[H^p(\mathbf{R}^n)]^* = \begin{cases} \Lambda_a, & p < 1, \\ \mathrm{BMO}, & p = 1. \end{cases}$$

§ 6. Proof of Lemma 3.2. Fix N_0. Put

$$l(x) = \begin{cases} x^{N_0}, & 0 \leqslant x < \frac{7}{8}, \\ 0, & x \leqslant 0, \\ 1, & x \geqslant 1. \end{cases}$$

Extend l to be defined on all of $\mathbf{R}$ so that l is non-decreasing and infinitely differentiable except at $x = 0$. Let $m(x) = l(16x + 9)$. Finally, let

$$\Phi(x) = \begin{cases} m(x), & x \leqslant -\frac{1}{2}, \\ 1, & -\frac{1}{2} \leqslant x \leqslant \frac{1}{2}, \\ m(-x), & \frac{1}{2} \leqslant x. \end{cases}$$

If I is any interval let φ_I be Φ adjusted to I; i.e. $\varphi_I = \Phi\left(\frac{x-x_0}{l_0}\right)$, where x_0 is the center of I and l_0 is its length. Then it is clear that

$$(6.1)\qquad |\varphi_I^{(k)}| \leqslant A_k |I|^{-k}, \quad 0 \leqslant k \leqslant N_0,$$

and $|I| \leqslant \int \varphi_I$. Also φ_I has compact support equal to $I^* = \frac{9}{8} I$.

In what follows S^0 will denote the interior of the set S.

SUBLEMMA 6.2. *Let I and I' be closed diadic intervals such that $|I'| \leqslant 4|I|$. Let $\varphi = \varphi_I$ and $\psi = \varphi_{I'}$. Suppose $|I| = 2^{-k}$ and $|I'| = 2^{-k-m}$. Then, if $0 \leqslant l \leqslant N_0$,*

$$(6.3)\qquad |(\varphi\psi)^{(l)}| \leqslant B_k (2^{k+m})^{l+1} \int_{\boldsymbol{R}} \varphi\psi.$$

Proof. If $\varphi\psi \equiv 0$, (6.3) is trivial. Suppose then $\varphi\psi \not\equiv 0$. The proof of (6.3) in this case is based on the geometry of dyadic intervals. Notice that the endpoints of I^* are dyadic boundary points. Thus, if I' is small, $|I'| \leqslant \frac{1}{16}|I|$, and $I' \cap I^* \neq \emptyset$, then either $I' \subseteq I^*$ or $I' \cap I^*$ is an endpoint of I^*. The only way an endpoint of I^* can be in $(I')^\circ$ is for I' to be large: $|I'| \geqslant \frac{1}{8}|I|$.

Case I. If $(I')^\circ$ contains an endpoint of I^*, then as we have seen, $\frac{1}{8}|I| \leqslant |I'| \leqslant 4|I|$. There are only finitely many such I'. Choose the constants B_k so large that (6.3) holds for all of them.

Now notice that it is no loss of generality to assume $|I| = 1$; i.e. $k = 0$. The estimate (6.3) is obtained most easily for small I' by dealing with three cases. The easiest of these cases is when I' lies deeply inside I^*.

Case II. $I' \subset \frac{17}{16} I$ and $|I'| \leqslant \frac{1}{16}|I|$. First we notice that there is a constant C such that $\varphi \geqslant C$ on $(I')^*$. Thus $\int \varphi\psi \geqslant C \int \psi \geqslant C 2^{-m}$. The derivatives are estimated using the Leibniz rule and (6.1).

Case III. $(I')^\circ \cap (I^*)^\circ = \emptyset$. That is, I' just touches an endpoint of I^*. The estimates are straightforward using (6.1) to estimate the derivatives of ψ and computing the derivatives of φ directly.

Case IV. $I' \subset I^* \setminus (\frac{17}{16} I)^\circ$. This case is done similarly to the last case again using the fact that φ is known exactly in a neighborhood of the endpoints of I^*.

The extension of Sublemma 6.2 to $\boldsymbol{R}^n$ is immediate: If $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$ define

$$\Phi(x) = \Phi_n(x) = \Phi(x_1) \ldots \Phi(x_n).$$

If Q is a cube with center x_0 and side l_0 let φ_Q be Φ adjusted to Q: $\varphi_Q(x) = \Phi\left(\frac{x-x_0}{l_0}\right)$.

SUBLEMMA 6.4. *Let Q and Q' be closed dyadic cubes in $\boldsymbol{R}^n$ such that $\operatorname{diam}(Q') \leqslant 4 \operatorname{diam} Q$. Let $\varphi = \varphi_Q$ and $\psi = \varphi_{Q'}$. Suppose Q has side of length 2^{-k} and Q' has side of length 2^{-k-m}. Then, if $0 \leqslant |a| \leqslant N_0$,*

$$(6.5)\qquad |D^a(\varphi\psi)| \leqslant B_a (2^{k+m})^{|a|+n} \int_{\boldsymbol{R}^n} \varphi\psi.$$

Now fix Ω_k as in the theorem and let $\{Q_j^k\}$ be the Whitney decomposition of Ω_k in (3.1). Let $\psi_j^k = \varphi_{Q_j^k}$. From the above it is clear that statements (i), (ii), and (iv) of Lemma (3.2) hold for $\{\psi_j^k\}$. Statement (v) also holds if $Q_l = \frac{1}{2} Q_l'$, $l = 1, 2, \ldots, 2^n$, where the Q_l' are the corners of $(Q_j^{k+1})^* \setminus Q_j^{k+1}$.

Recall that no point $x \in \Omega_k$ can be in more than L of the cubes $(Q_j^k)^*$, where L is a large constant depending only on n. (See [7].) Thus

$$(6.6)\qquad 1 \leqslant \sum_{j=1}^{\infty} \psi_j^k(x) \leqslant L, \quad x \in \Omega_k.$$

Define $\varphi_j^k = \psi_j^k / \sum_{i=1}^{\infty} \psi_i^k$. Properties (i), (ii), (iii) and (v) of Lemma (3.2) are immediate consequences of the corresponding properties of the ψ_j^k and (6.6). Property (iv) requires some work but is straightforward. Details are left to the interested reader.

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Received May 6, 1976 (1156)