

if D is strongly pseudoconvex. To show it, we first note that if f is bounded holomorphic in D , then $\|T_f\| = \|f\|_\infty$. In fact, let $\zeta \in D$. Then $f(\zeta) \in \sigma(T_f)$: the spectrum of T_f , since for any $B \in L(H^2(D))$

$$(T_f - f(\zeta))B1 = (f - f(\zeta))B1 \neq 1.$$

Hence $f(D) \subset \sigma(T_f)$ and so we have

$$\|f\|_\infty = \sup_{z \in D} |f(z)| \leq \sup \{|\lambda| : \lambda \in \sigma(T_f)\} \leq \|T_f\| \leq \|f\|_\infty.$$

Hence the set $\mathcal{A} = \{T_f : f \in A(D)\}$ is a commutative closed subalgebra of $L(H^2(D))$ which is isometrically isomorphic with $A(D)$ and which contains T_{z_1}, \dots, T_{z_n} and the identity operator. Therefore, by the argument in the proof of the theorem in [4], p. 240, we have for every $\eta \in \Gamma(\mathcal{A})$: the Shilov boundary for \mathcal{A}

$$(\eta(T_{z_1}), \dots, \eta(T_{z_n})) \in \sigma_\pi(T_{z_1}, \dots, T_{z_n})$$

or equivalently for every $\xi \in \Gamma(A(D))$

$$(*) \quad (\xi(z_1), \dots, \xi(z_n)) \in \sigma_\pi(T_{z_1}, \dots, T_{z_n}).$$

However, it is known that $\Gamma(A(D)) = \partial D$ if D is strongly pseudoconvex ([1], Theorem 6.3). Hence we have $\partial D \subset \sigma_\pi$, which proves the Theorem.

Remark. As the above proof shows, for every bounded domain D in \mathbb{C}^n , (*) is still valid for $A(D)$ and also when one replaces $A(D)$ by any other closed subalgebra of $H^\infty(D)$: the Banach algebra of all bounded holomorphic functions in D , for example, $P(D)$, $R(D)$, or $H(D)$: the set of all $f \in C(\bar{D})$ which are approximated uniformly on \bar{D} by holomorphic functions on \bar{D} .

References

- [1] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries*, Trans. Amer. Math. Soc. 91 (1959), pp. 246–276.
- [2] J. Janas, *Toeplitz operators related to certain domains in \mathbb{C}^n* , Studia Math. 54 (1975), pp. 73–79.
- [3] Z. Słodkowski and W. Żelazko, *On joint spectra of commuting families of operators*, *ibid.* 50 (1974), pp. 127–148.
- [4] W. Żelazko, *On a problem concerning joint approximate point spectra*, *ibid.* 45 (1973), pp. 239–240.
- [5] W. Żelazko, *On a certain class of non-removable ideals in Banach algebras*, *ibid.* 44 (1972), pp. 87–92.

MATHEMATICAL INSTITUTE
TŌHOKU UNIVERSITY
SENDAI, JAPAN

Current address: KYOTO TECHNICAL UNIVERSITY
MATSUGASAKI, SAKYU-KU
KYOTO, JAPAN

Received April 9, 1976

(1148)

Sobolev type inequalities for $p > 0$

by

ALBERTO P. CALDERÓN (Chicago, Ill.) and
RIDGWAY SCOTT (Upton, N. Y.)*

Abstract. Sobolev type inequalities for generalized Peano derivatives with exponents p , $p > 0$, are obtained.

Certain kinds of generalized Peano derivatives (see [1] and [2] and Definition 2.2 below) have been shown to have many desirable properties that the classical Peano derivatives lack. In this paper we continue the study of such derivatives and establish Sobolev type inequalities between them. The basic results here are the estimates for the distribution functions of the $N_a^m(F)$ in Section 5, from which the desired inequalities for exponents p which are merely positive follow.

1. Notation. By x, y, z, \dots , we denote points in n -dimensional Euclidean space \mathbb{R}^n . The closed ball with center x and radius ρ will be written as $B(x, \rho)$. Given a set \mathcal{A} , let $d(x, \mathcal{A}) = \inf\{|x - y| : y \in \mathcal{A}\}$. If \mathcal{A} is (Lebesgue) measurable, let $|\mathcal{A}| = \int d\mathcal{A}$ denote the measure of \mathcal{A} ($d\mathcal{A}$ denotes Lebesgue measure).

We will deal with real or complex valued functions, and we will refer to the corresponding field as “scalars”. The term “constant” will be used to mean a positive real number. All functions are assumed to be measurable.

For an open set \mathcal{O} , let $C^\infty(\mathcal{O})$ denote the linear space of functions infinitely differentiable in \mathcal{O} , and let $C_0^\infty(\mathcal{O})$ be the subspace consisting of functions with compact support. For a function $F \in C^\infty(\mathcal{O})$ and a point $x \in \mathcal{O}$,

$$(1.1) \quad F^{(\alpha)}(x) = D^\alpha F(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} F(x).$$

As usual, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

* Work performed under the auspices of the ERDA.

For any measurable set \mathcal{M} and any positive real number p , denote by $L_p(\mathcal{M})$ the linear space of functions F such that

$$\|F\|_{p,\mathcal{M}} \equiv \left(\int_{\mathcal{M}} |F(x)|^p dx \right)^{1/p} < \infty,$$

and for $p = \infty$, make the usual modification. When the \mathcal{M} is omitted, $\mathcal{M} = \mathbf{R}^n$ is implied, e.g., $\|F\|_p \equiv \|F\|_{p,\mathbf{R}^n}$. Let \mathcal{O} be an open set. Then $L_{p,\text{loc}}(\mathcal{O})$ is defined to be the linear space of functions F such that $\varphi F \in L_p(\mathcal{O})$ for all $\varphi \in C_0^\infty(\mathcal{O})$.

For $F \in L_{1,\text{loc}}(\mathcal{O})$, we say that $D_w^\alpha F$ exists in \mathcal{O} if there is a function $G \in L_{1,\text{loc}}(\mathcal{O})$ such that

$$\int_{\mathcal{O}} F(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathcal{O}} G(x) \varphi(x) dx$$

for all $\varphi \in C_0^\infty(\mathcal{O})$. In this case, we define $D_w^\alpha F \equiv G$. Note that $D^\alpha = D_w^\alpha$ on $C^\infty(\mathcal{O})$.

Let m be a non-negative integer. For any notation of derivative (including the one to be given in the next section), $D^m F$ will denote the vector $(D^\alpha F: |\alpha| = m)$, and the notation $|D^m F(x)|$ and $\|D^m F\|_p$ will have the usual meanings ($D^0 F \equiv F$).

When m is an arbitrary real number, we denote by $[m]$ the largest integer less than m .

For a function F and $s > 0$, define $\lambda(s) = \{ |F| > s \}$. The following well-known facts will be frequently used (cf. [4]):

$$(1.2) \quad \lambda(s) \leq s^{-p} \|F\|_p^p \quad (0 < p < \infty),$$

and

$$(1.3) \quad \int_{\{|F| < t\}} |F(x)|^p dx \leq \int_0^t \lambda(s) ds^p.$$

The words "increasing" and "decreasing" will be used here in the wider sense; for example, λ is a decreasing function of s .

When the region of integration is left unspecified, the whole space \mathbf{R}^n is implied, i.e., $\int \equiv \int_{\mathbf{R}^n}$.

Let $F \in L_{1,\text{loc}}(\mathbf{R}^n)$. Define the Hardy-Littlewood maximal function

$$MF(x) = \sup_{\rho > 0} |B(x, \rho)|^{-1} \int_{B(x, \rho)} |F(y)| dy.$$

It is well known [4] that for $1 < p \leq \infty$

$$(1.4) \quad \|MF\|_p \leq c_p \|F\|_p,$$

where c_p is a constant independent of F , but depending on p .

2. Definitions and preliminary results.

DEFINITION 2.1. Let q be a real number such that $1 \leq q \leq \infty$, and let m be a non-negative real number. Suppose that $F \in L_{q,\text{loc}}(\mathbf{R}^n)$, and define (with the usual modification when $q = \infty$)

$$N_q^m(F, x) = \sup_{\rho > 0} \rho^{-m} \left\{ \rho^{-n} \int_{B(x, \rho)} |F(y) - P_x(y)|^q dy \right\}^{1/q}$$

if there exists a polynomial P_x of degree less than m such that the supremum is finite ($N_q^m(F, x) = +\infty$ if not). If $m = 0$, then $P_x \equiv 0$ is understood).

Remark. In [1], the cases $q = 1$ and ∞ are formally excluded, and m is restricted to be a positive integer. However, all the results used here concerning N_q^m are easily seen to be valid in these cases also. The case $m = 0$, $q = 1$, gives the usual maximal function. We now collect some simple facts about N_q^m .

PROPOSITION 2.1.

- (i) If P_x exists, it is unique.
- (ii) If c is a scalar, then $N_q^m(cF) = |c| N_q^m(F)$.
- (iii) $N_q^m(F_1 + F_2) \leq N_q^m(F_1) + N_q^m(F_2)$.
- (iv) If $q_1 \leq q_2$, then $N_{q_1}^m(F) \leq \omega^{(q_2 - q_1)/q_1 q_2} N_{q_2}^m(F)$, where ω is the measure of the unit ball in \mathbf{R}^n .

Proof. Assertion (i) is simply [1], Lemma 4. Assertions (ii)–(iii) are obvious in view of (i), and (iv) follows from Hölder's inequality. ■

PROPOSITION 2.2. Let m be a non-negative integer. Suppose that $N_q^m(F)$ is locally integrable in an open set \mathcal{O} . Then $D_w^m F$ exists in \mathcal{O} and there is a constant c depending only on m such that

$$|D_w^m F| \leq c N_q^m(F) \text{ a.e. in } \mathcal{O}.$$

Conversely, suppose that $1 < p \leq \infty$ and that $q \leq p$. Then $D_w^m F \in L_p(\mathbf{R}^n)$ implies that $N_q^m(F) \in L_p(\mathbf{R}^n)$, and

$$(2.2) \quad \|N_q^m(F)\|_p \leq c' \|D_w^m F\|_p,$$

where c' is a constant independent of F .

Proof. These results follow immediately from [1], Theorem 4, and [1], Lemma 7, respectively, with obvious modifications when $m = 0$. ■

The above results say that $\|N_q^m(F)\|_p$ gives a norm equivalent to the usual Sobolev norm provided that $q \leq p$ and $1 < p \leq \infty$ (and, in general, it gives a stronger norm). We wish to consider now the case $p \leq 1$ and to give a Sobolev type inequality. To begin with, we must introduce a new notion of derivative.

DEFINITION 2.2. Let F be a measurable function, let $x \in \mathbf{R}^n$, and let k be a non-negative integer. Then F is said to be *differentiable to order k at x* if there exists a function $\varphi \in C^\infty(\mathbf{R}^n)$ such that $\varphi \equiv 1$ in a neighborhood of x , $\varphi F \in L_{1,loc}(\mathbf{R}^n)$, and for some real number $m > k$,

$$N_1^m(\varphi F, x) < \infty.$$

In this case, we define

$$(2.3) \quad D^\alpha F(x) = D^\alpha P_x(x) \quad \text{for } |x| \leq k,$$

where P_x is the polynomial in Definition 2.1.

PROPOSITION 2.3. *The notion of derivative given in Definition 2.2 is well defined and, when it exists, coincides with the usual definition (1.1) for $F \in C^\infty(\mathcal{O})$ for any neighborhood \mathcal{O} of x .*

Proof. Suppose there exist φ_1 and φ_2 , and m_1 and m_2 , such that $N_1^{m_i}(\varphi_i F, x) < \infty$, $i = 1, 2$. Let P_i be the corresponding polynomial of degree $[m_i]$. It will be shown that

$$(2.4) \quad D^\alpha(P_1 - P_2)(x) = 0 \quad \text{for } |x| \leq \min\{[m_1], [m_2]\}.$$

Let $d_i = \text{dist}(x, \{\varphi_i \neq 1\})$, $i = 1, 2$, let $\varrho_0 = \min\{1, d_1, d_2\}$, and let $\psi \in C_0^\infty(\{|x| < \varrho_0\})$ be such that $\int \psi(x) dx = 1$. Then integration by parts gives

$$\begin{aligned} D^\alpha(P_1 - P_2)(x) &= \lim_{\varrho \downarrow 0} \varrho^{-n} \int D^\alpha(P_1 - P_2)(y) \psi((y-x)/\varrho) dy \\ &= \lim_{\varrho \downarrow 0} \varrho^{-n-|\alpha|} (-1)^{|\alpha|} \int (P_1 - P_2)(y) \psi^{(\alpha)}((y-x)/\varrho) dy. \end{aligned}$$

If $|x-y| < \varrho_0$, then $(\varphi_1 F)(y) = (\varphi_2 F)(y)$, so for $\varrho \leq 1$,

$$\begin{aligned} \int (P_1 - P_2)(y) \psi^{(\alpha)}((y-x)/\varrho) dy \\ = \sum_{i=1}^2 (-1)^i \int_{B(x,\varrho)} (\varphi_i F - P_i)(y) \psi^{(\alpha)}((y-x)/\varrho) dy. \end{aligned}$$

Since $\psi^{(\alpha)}$ is bounded, say by c ,

$$\left| \int (P_1 - P_2)(y) \psi^{(\alpha)}((y-x)/\varrho) dy \right| \leq c \sum_{i=1}^2 \varrho^{m_i+n} N_1^{m_i}(\varphi_i F, x).$$

Letting $\varrho \downarrow 0$ proves (2.4), and shows that (2.3) is unambiguous. The fact that (2.3) coincides with (1.1) for $F \in C^\infty(\mathcal{O})$ is just Taylor's theorem. ■

PROPOSITION 2.4. *Suppose that $N_q^m(F)$ is finite a.e. in an open set \mathcal{O} and that $D_w^\alpha F$ exists in \mathcal{O} for some $|a| < m$. Then for almost all $x \in \mathcal{O}$,*

$$(2.5) \quad D_w^\alpha F(x) = D^\alpha F(x).$$

Proof. Let $\varphi \in C_0^\infty(\{|x| < 1\})$ be such that $\int \varphi(x) dx = 1$. For almost all $x \in \mathcal{O}$,

$$(2.6) \quad D_w^\alpha F(x) = \lim_{\varrho \downarrow 0} \int D_w^\alpha F(y) \varphi((x-y)/\varrho) \varrho^{-n} dy,$$

because $D_w^\alpha F \in L_{1,loc}(\mathcal{O})$. We will now show that (2.5) holds for all x such that $N(F, x)$ is finite and (2.6) holds. Let x be such a point, and let P_x be the polynomial in Definition 1.1 such that $N_q^m(F, x) < \infty$. Then

$$\begin{aligned} |(D_w^\alpha F - D^\alpha F)(x)| &= |(D_w^\alpha F - D^\alpha P_x)(x)| \\ &= \left| \lim_{\varrho \downarrow 0} \int (D_w^\alpha F - D^\alpha P_x)(y) \varphi((x-y)/\varrho) \varrho^{-n} dy \right| \\ &= \left| \lim_{\varrho \downarrow 0} (-1)^{|\alpha|} \int (F - P_x)(y) \varphi^{(\alpha)}((x-y)/\varrho) \varrho^{-n-|\alpha|} dy \right| \\ &\leq \lim_{\varrho \downarrow 0} c \varrho^{m-|\alpha|} N_1^m(F, x) = 0. \quad \blacksquare \end{aligned}$$

Remark. In view of Proposition 2.2, the hypotheses of Proposition 2.4 hold whenever $N_q^m(F)$ is locally integrable in \mathcal{O} .

3. Main results. We now wish to prove a Sobolev type inequality for N_q^m . However, the L_p spaces are ill suited for measuring $N_q^m(F)$. For suppose F is not a polynomial of degree less than m . Then there is a ball $B = B(y, R)$ such that

$$\inf_P \|F - P\|_{q,B} = c_0 > 0,$$

where the infimum is over all polynomials P of degree less than m . Therefore, for all $x \in \mathbf{R}^n$,

$$(3.1) \quad N_q^m(F, x) \geq c_0(|x-y| + R)^{-m-n/q}.$$

Thus $N_q^m(F)$ is in $L_p(\mathbf{R}^n)$ only if $m+n/q > n/p$. The spaces $L_{p_1} + L_{p_2}$ discussed below are better suited for our purposes.

DEFINITION 3.1. Let p_1 and p_2 be positive real numbers. We say that $f \in L_{p_1} + L_{p_2}(\mathbf{R}^n)$ if there exist $f_i \in L_{p_i}(\mathbf{R}^n)$, $i = 1, 2$, such that $f = f_1 + f_2$. For such f , define

$$\|f\|_{p_1;p_2} = \inf\{\|f_1\|_{p_1} + \|f_2\|_{p_2} : f = f_1 + f_2, f_i \in L_{p_i}(\mathbf{R}^n), i = 1, 2\}.$$

PROPOSITION 3.1. *Let $f, g \in L_{p_1} + L_{p_2}(\mathbf{R}^n)$ and let c be a scalar. Then*

$$(i) \quad \|cf\|_{p_1;p_2} = |c| \|f\|_{p_1;p_2}$$

and

$$(ii) \quad \|f+g\|_{p_1;p_2} \leq \max\{1, 2^{(1/\min\{p_1,p_2\})-1}\} (\|f\|_{p_1;p_2} + \|g\|_{p_1;p_2}).$$

Proof. Part (i) is obvious, and part (ii) follows from the inequality $\|f+g\|_p \leq \max\{1, 2^{(1/p)-1}\} (\|f\|_p + \|g\|_p)$, $p > 0$. For $1 > p > 0$, this follows from the inequalities

$$(a+b)^p \leq a^p + b^p \quad \text{and} \quad (a+b)^{1/p} \leq 2^{1/p-1} (a^{1/p} + b^{1/p})$$

for $a, b \geq 0$ and $0 < p \leq 1$, namely,

$$\begin{aligned} \|f+g\|_p &\leq \left(\int (|f|+|g|)^p dx \right)^{1/p} \\ &\leq \left(\int (|f|^p+|g|^p) dx \right)^{1/p} \\ &\leq 2^{1/p-1} (\|f\|_p + \|g\|_p). \quad \blacksquare \end{aligned}$$

From the definition of $\|\cdot\|_{p_1, p_2}$, it is obviously symmetric in p_1, p_2 , i.e., $\|f\|_{p_1, p_2} = \|f\|_{p_2, p_1}$. The following proposition says that $\|f\|_{p_1, p_2}$ measures f in $L_{\max(p_1, p_2)}(\mathbf{R}^n)$, where it is small and in $L_{\min(p_1, p_2)}(\mathbf{R}^n)$, where it is large.

PROPOSITION 3.2. *Let $f \in L_{p_1} + L_{p_2}(\mathbf{R}^n)$. Let $\lambda(s)$ be the distribution of f , i.e., $\lambda(s) = |\{ |f| > s \}|$, and let*

$$I = \int_0^1 \lambda(s) ds^{\max(p_1, p_2)} + \int_1^\infty \lambda(s) ds^{\min(p_1, p_2)}.$$

Then

$$I \leq 2^{\max(p_1, p_2)} (\|f\|_{p_1, p_2}^{p_1} + \|f\|_{p_1, p_2}^{p_2})$$

and

$$\|f\|_{p_1, p_2} \leq I^{1/p_1} + I^{1/p_2}.$$

Proof. For simplicity, let us assume that $p_1 \geq p_2$. Define $D_1 = \{x: |f(x)| \leq 1\}$ and $D_2 = D_1^c$. An integration by parts shows that

$$(3.2) \quad I = - \int_0^1 s^{p_1} d\lambda(s) - \int_1^\infty s^{p_2} d\lambda(s) = \int_{D_1} |f|^{p_1} dx + \int_{D_2} |f|^{p_2} dx.$$

Let $\varepsilon > 0$ and $f_i \in L_{p_i}(\mathbf{R}^n)$, $i = 1, 2$, be such that $f = f_1 + f_2$ and

$$\|f_1\|_{p_1} + \|f_2\|_{p_2} < \|f\|_{p_1, p_2} + \varepsilon.$$

Define $E_1 = \{x: |f_1(x)| > |f_2(x)|\}$ and $E_2 = E_1^c$. Then

$$\int_{E_1} |f|^{p_1} dx \leq 2^{p_1} \int_{E_1} |f_1|^{p_1} dx \leq 2^{p_1} (\|f\|_{p_1, p_2} + \varepsilon)^{p_1}$$

and

$$\int_{E_2} |f|^{p_2} dx \leq 2^{p_2} \int_{E_2} |f_2|^{p_2} dx \leq 2^{p_2} (\|f\|_{p_1, p_2} + \varepsilon)^{p_2}.$$

Therefore

$$(3.3) \quad \int_{E_1} |f|^{p_1} dx + \int_{E_2} |f|^{p_2} dx \leq 2^{p_1} [(\|f\|_{p_1, p_2} + \varepsilon)^{p_1} + (\|f\|_{p_1, p_2} + \varepsilon)^{p_2}].$$

Since $|f| \leq 1$ in D_1 , $|f| > 1$ in D_2 , and $p_1 \geq p_2$, we have

$$\int_{E_1 \cap D_2} |f|^{p_1} dx \geq \int_{E_1 \cap D_2} |f|^{p_2} dx \quad \text{and} \quad \int_{E_2 \cap D_1} |f|^{p_2} dx \geq \int_{E_2 \cap D_1} |f|^{p_1} dx.$$

Therefore

$$\begin{aligned} I &= \int_{D_1} |f|^{p_1} dx + \int_{D_2} |f|^{p_2} dx \\ &= \int_{E_1 \cap D_1} |f|^{p_1} dx + \int_{E_2 \cap D_1} |f|^{p_1} dx + \int_{E_1 \cap D_2} |f|^{p_2} dx + \int_{E_2 \cap D_2} |f|^{p_2} dx \\ &\leq \int_{E_1 \cap D_1} |f|^{p_1} dx + \int_{E_2 \cap D_1} |f|^{p_2} dx + \int_{E_1 \cap D_2} |f|^{p_1} dx + \int_{E_2 \cap D_2} |f|^{p_2} dx \\ &= \int_{E_1} |f|^{p_1} dx + \int_{E_2} |f|^{p_2} dx. \end{aligned}$$

Combined with (3.3), this proves the first half of the proposition, since $\varepsilon > 0$ was arbitrary.

To prove the second inequality, define

$$f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and let $f_2 = f - f_1$. Then as in (3.2),

$$\|f_1\|_{p_1} = \left(\int_{D_1} |f|^{p_1} dx \right)^{1/p_1} \leq I^{1/p_1}$$

and

$$\|f_2\|_{p_2} = \left(\int_{D_2} |f|^{p_2} dx \right)^{1/p_2} \leq I^{1/p_2}.$$

Consequently,

$$\|f\|_{p_1, p_2} \leq \|f_1\|_{p_1} + \|f_2\|_{p_2} \leq I^{1/p_1} + I^{1/p_2}. \quad \blacksquare$$

Remark. If $N_q^m(F) \in L_{p_1} + L_{p_2}(\mathbf{R}^n)$ for $p_1, p_2 > 0$, then $N_q^m(F)$ is finite a.e., and hence $D^k F$ is defined a.e. for $k < m$. This will be used implicitly in the statement of the following theorems.

THEOREM I. *Let m, a, v_1, v_2 , and q be real numbers such that*

- (i) $0 < a \leq m$,
- (ii) $0 < v_1, v_2 < n/\max\{a, m - [m]\}$ and
- (iii) $1 \leq q < n$ if $n > 1$ and $q = 1$ if $n = 1$.

Let $F \in L_{q, \text{loc}}(\mathbf{R}^n)$ be such that $N_q^m(F) \in L_{v_1} + L_{v_2}(\mathbf{R}^n)$ and such that, for j an integer less than m and not less than the integral part of $m - a$, $|\{D^j F| > \varepsilon\}| < \infty$ for all $\varepsilon > 0$. Then $N_q^{m-a}(F) \in L_{w_1} + L_{w_2}(\mathbf{R}^n)$, where $1/w_i = 1/v_i - a/n$, $i = 1, 2$, and there is a constant c independent of F such that

$$\|N_q^{m-a}(F)\|_{w_1, w_2} \leq C \|N_q^m(F)\|_{v_1, v_2}.$$

The proof of Theorem I is given in Section 6, following some preliminaries. The following is the main result of the paper.

THEOREM II. *Let m be a positive real number, let k be a non-negative real number less than m , let v_1 and v_2 be real numbers such that $0 < v_1, v_2 < n/(m-k)$, and let q be in the range $1 \leq q \leq \infty$. Then there is a constant c such that, if $F \in L_{q, \text{loc}}(\mathbf{R}^n)$ satisfies $|\{ |D^j F| > \varepsilon \}| < \infty$ for all $\varepsilon > 0$ and all j in the range $k \leq j < m$, then*

$$\|D^k F\|_{w_1, w_2} \leq c \|N_q^m(F)\|_{v_1, v_2},$$

where $1/w_i = 1/v_i - (m-k)/n$, $i = 1, 2$.

Theorem II follows immediately from Theorem I in the case $w_1, w_2 \geq 1$ (see Proposition 2.2), since then $D^k F$ coincides with $D_q^k F$. However, the general case requires a separate proof, given in Section 7.

The condition $|\{ |D^j F| > \varepsilon \}| < \infty$ in Theorems I and II is clearly necessary: let F be a polynomial of degree less than m ; then $N_q^m(F) = 0$, but $N_q^k(F)$ (resp. $D^k F$) need not be zero for any real number (resp. integer) $k \leq [m]$. The condition is satisfied, e.g., if F has compact support.

4. Results concerning N_q^m for $0 < m \leq 1$. To study N_q^m for $0 < m \leq 1$, we introduce a generalization of the function $F^\#$ studied in [3].

DEFINITION 4.1. Let $0 \leq a \leq 1$ and let $F \in L_{1, \text{loc}}(\mathbf{R}^n)$. For $x \in \mathbf{R}^n$, define

$$F_a^\#(x) = \sup_{\rho > 0} \rho^{-a-n} \int_{B(x, \rho)} |F(y) - m_\rho| dy,$$

where m_ρ is the average of F over the ball $B(x, \rho)$:

$$m_\rho = |B(x, \rho)|^{-1} \int_{B(x, \rho)} F(y) dy.$$

In [3], the case $a = 0$ was considered; a function F is said to have bounded mean oscillation if $\|F_0^\#\|_\infty < \infty$. We begin with some elementary properties. In the following propositions, $F \in L_{1, \text{loc}}(\mathbf{R}^n)$, B is a bounded, measurable subset of \mathbf{R}^n , and $m_B = m_B(F)$ denotes the average of F over B : $m_B = |B|^{-1} \int_B F(y) dy$.

PROPOSITION 4.1. *For any scalar c ,*

$$|B|^{-1} \int_B |F(y) - m_B| dy \leq 2 |B|^{-1} \int_B |F(y) - c| dy.$$

Proof. Clearly.

$$|B|^{-1} \int_B |F(y) - m_B| dy \leq |B|^{-1} \int_B |F(y) - c| dy + |m_B - c|.$$

By the definition of m_B ,

$$\begin{aligned} |m_B - c| &= \left| |B|^{-1} \int_B F(y) dy - c \right| = |B|^{-1} \left| \int_B (F(y) - c) dy \right| \\ &\leq |B|^{-1} \int_B |F(y) - c| dy. \quad \blacksquare \end{aligned}$$

COROLLARY. $F_a^\# \leq 2N_1^a(F)$.

PROPOSITION 4.2. *Suppose that B and B_1 are bounded measurable subsets of \mathbf{R}^n such that $B \subset B_1$ and $2^{-n}|B_1| \leq |B| < \infty$. Then*

$$|B|^{-1} \int_B |F(y) - m_B| dy \leq 2^{n+1} |B_1|^{-1} \int_{B_1} |F(y) - m_{B_1}| dy.$$

Proof. By the previous proposition,

$$|B|^{-1} \int_B |F(y) - m_B| dy \leq 2 |B|^{-1} \int_B |F(y) - m_{B_1}| dy.$$

Using the hypotheses on B and B_1 ,

$$\begin{aligned} |B|^{-1} \int_B |F(y) - m_{B_1}| dy &\leq |B|^{-1} \int_{B_1} |F(y) - m_{B_1}| dy \\ &\leq 2^n |B_1|^{-1} \int_{B_1} |F(y) - m_{B_1}| dy. \quad \blacksquare \end{aligned}$$

PROPOSITION 4.3. $|F_a^\#| \leq 2F_a^\#$.

Proof. Proposition 4.1 implies that

$$\begin{aligned} |B|^{-1} \int_B \left| |F(y) - m_B(|F|)| \right| dy &\leq 2 |B|^{-1} \int_B \left| |F(y)| - |m_B(F)| \right| dy \\ &\leq 2 |B|^{-1} \int_B |F(y) - m_B(F)| dy. \quad \blacksquare \end{aligned}$$

PROPOSITION 4.4. $(F_1 + F_2)_a^\# \leq (F_1)_a^\# + (F_2)_a^\#$ and, for any scalar c , $(cF)_a^\# = |c| F_a^\#$ and $(F+c)_a^\# = F_a^\#$.

Proof. Obvious. \blacksquare

PROPOSITION 4.5. *Let F be real valued and let $c \in \mathbf{R}^1$. Define*

$$(F \vee c)(x) = \max\{F(x), c\} \quad \text{and} \quad (F \wedge c)(x) = \min\{F(x), c\}.$$

Then $(F \vee c)_a^\# \leq \frac{3}{2} F_a^\#$ and $(F \wedge c)_a^\# \leq \frac{3}{2} F_a^\#$.

Proof. The previous two propositions suffice since

$$F \vee c = \frac{1}{2}(F+c) + \frac{1}{2}|F-c| \quad \text{and} \quad F \wedge c = \frac{1}{2}(F+c) - \frac{1}{2}|F-c|. \quad \blacksquare$$

PROPOSITION 4.6. *Suppose that $a < n$ (since $a \leq 1$ this is automatic when $n > 1$), that p is in the range $1 < p < n/a$ and that $F_a^\# \in L_p(\mathbf{R}^n)$. Then $F_0^\# \in L_r(\mathbf{R}^n)$ for $1/r = 1/p - a/n$, and there is a constant v independent*

of F such that

$$\|F_0^\# \|_r \leq c \|F_a^\# \|_p.$$

Proof. Let $x \in \mathbf{R}^n$, $\varrho > 0$, and $z \in B(x, \varrho)$. Define $m_\varrho = m_{B(x, \varrho)}$ and $m'_\varrho = m_{B(z, 2\varrho)}$. Proposition 4.2 implies that

$$|B(x, \varrho)|^{-1} \int_{B(x, \varrho)} |F(y) - m_\varrho| dy \leq 2^{n+1} |B(z, 2\varrho)|^{-1} \int_{B(z, 2\varrho)} |F(y) - m'_\varrho| dy.$$

Therefore

$$\varrho^{-n} \int_{B(x, \varrho)} |F(y) - m_\varrho| dy \leq 2^{n+1+a} \varrho^a F_a^\#(z).$$

Integrating over $B(x, \varrho)$ and using the fact that $\varrho \geq |x - z|$ for $z \in B(x, \varrho)$, one obtains

$$\omega \varrho^{-n} \int_{B(x, \varrho)} |F(y) - m_\varrho| dy \leq 2^{n+1+a} \int_{B(x, \varrho)} |x - z|^{a-n} F_a^\#(z) dz,$$

where $\omega = |B(0, 1)|$. Since $\varrho > 0$ was arbitrary, it follows that

$$F_0^\#(x) \leq c I^a(F_a^\#)(x),$$

where I^a is the Riesz potential of order a and c is a constant. Using a well-known inequality for Riesz potentials [4], the result follows. ■

PROPOSITION 4.7 (cf. [3]). Assume that $|\{F > s\}| < \infty$ for all $s > 0$ and that $F_0^\# \in L_p(\mathbf{R}^n)$ for p in the range $1 < p < \infty$. Then $F \in L_p(\mathbf{R}^n)$, and there is a constant c independent of F such that

$$\|F\|_p \leq c \|F_0^\# \|_p.$$

Proof. First assume that F is non-negative, bounded, and supported on a set of finite measure.

LEMMA 4.1. Given $\delta > 0$, $s > 0$, and a ball B containing a point x_0 such that $MF(x_0) \leq s$, then, if $\omega = |B(0, 1)|$,

$$|\{F > (1+2^n)s\} \cap \{F_0^\# < 2^{-n-1}\delta s\} \cap B| < \delta |B| \omega^{-1}.$$

Proof. Let B_1 be the ball with radius twice that of B and with center x_0 . Then $B \subset B_1$ and $|B_1| = 2^n |B|$, and since $MF(x_0) \leq s$,

$$\int_B F \leq \int_{B_1} F \leq s |B_1| = 2^n s |B|.$$

That is, $m_B(F) \leq 2^n s$. Let E be the set given by

$$E = \{F > (1+2^n)s\} \cap \{F_0^\# < 2^{-n-1}\delta s\} \cap B.$$

Then since $E \subset B$ and $F > (1+2^n)s \geq s + m_B(F)$ on E ,

$$|B|^{-1} \int_B |F(y) - m_B| dy \geq |B|^{-1} \int_E |F(y) - m_B| dy \geq |B|^{-1} s |E|.$$

Let $z \in \{F_0^\# < 2^{-n-1}\delta s\} \cap B$, and let B_z be the ball with center z and with radius twice that of B . Then Proposition 4.2 implies that

$$|E| |s| |B|^{-1} \leq 2^{n+1} |B_z|^{-1} \int_{B_z} |F(y) - m_{B_z}| dy \leq 2^{n+1} F_0^\#(z) < \delta s \omega^{-1}. \blacksquare$$

LEMMA 4.2. For any $\delta > 0$ and $s > 0$,

$$|\{F > (1+2^n)s\} \cap \{F_0^\# \leq 2^{-n-1}\delta s\}| \leq 5^n \delta |\{MF > s\}| \omega^{-1}.$$

Proof. Let \mathcal{B} be the class of all (closed) balls B such that the interior of B is contained in $\{MF > s\}$, but the closure is not. Since $\{MF > s\}$ is an open set (the maximal function is lower semi-continuous), the union of \mathcal{B} contains $\{MF > s\}$. Since F is integrable, $\{MF > s\}$ has finite measure, hence the radii of the balls in \mathcal{B} is bounded. Therefore, (cf. [4], p. 10) we may take a disjoint subfamily $\{B_i\}$ of \mathcal{B} such that, if \bar{B}_i is the ball concentric to B_i with five times its radius, then

$$\{MF > s\} \subset \bigcup_i \bar{B}_i.$$

By changing F on a set of measure zero, we may assume that $F \leq MF$ everywhere, so $\{F > s\} \subset \bigcup_i \bar{B}_i$ also. Let E be the set given by

$$E = \{F > (2^n + 1)s\} \cap \{F_0^\# \leq 2^{-n-1}\delta s\}.$$

Since $E \subset \bigcup_i \bar{B}_i$, we have $E = \bigcup_i E_i$, where $E_i = E \cap \bar{B}_i$. Every ball \bar{B}_i contains a point not in $\{MF > s\}$ because the closure of B_i is contained in \bar{B}_i . By the previous lemma,

$$|E_i| = |E \cap \bar{B}_i| \leq \delta |\bar{B}_i| \omega^{-1} \leq \delta 5^n |B_i| \omega^{-1}.$$

Therefore, $|E| \leq \sum_i |E_i| \leq \delta 5^n \sum_i |B_i| \omega^{-1} \leq \delta 5^n |\{MF > s\}| \omega^{-1}$. ■

Returning to the proof of Proposition 4.7, Lemma 4.2 shows that, for any $s > 0$ and for any $\delta > 0$,

$$(4.1) \quad |\{F > (1+2^n)s\}| \leq |\{F_0^\# > 2^{-n-1}\delta s\}| + 5^n \delta |\{MF > s\}| \omega^{-1}.$$

Integrating with respect to ds^p and applying (1.3) and (1.4) this gives

$$\|F\|_p \leq c_1(\delta) \|F_0^\# \|_p + c_2 \delta \|MF\|_p \leq c_1(\delta) \|F_0^\# \|_p + c_3 \delta \|F\|_p.$$

Thus $(1 - c_3 \delta) \|F\|_p \leq c_1(\delta) \|F_0^\# \|_p$, and the result follows by taking δ sufficiently small, provided that F is positive, bounded and supported in a set of finite measure.

Now consider the case of general F . In view of Proposition 4.3, it suffices to assume that $F \geq 0$. Define

$$F_n = (F \vee 1/n) \wedge n - 1/n.$$

Then $\{F_n > 0\} = \{F > 1/n\}$, and so by our assumption on F , the support

of F_n has finite measure. Since F_n is bounded, the first part of the proof yields

$$\|F_n\|_p \leq c \| (F_n)_0^\# \|_p \quad \text{for all } n = 1, 2, \dots$$

By Propositions 4.4 and 4.5, $(F_n)_0^\# \leq \frac{2}{4} F_0^\#$. Therefore,

$$\|F_n\|_p \leq c \|F_0^\#\|_p \quad \text{for all } n.$$

The conclusion follows from the monotone convergence theorem if we let n tend to infinity. ■

PROPOSITION 4.8. Let m and p be real numbers such that $0 < m \leq 1$, $m < n$ and $1 < p < n/m$. Assume that $F \in L_{1,\text{loc}}(\mathbf{R}^n)$ is such that $|\{ |F| > \varepsilon \}| < \infty$ for all $\varepsilon > 0$ and such that $N_1^m(F) \in L_p(\mathbf{R}^n)$, $1 < p < n/m$. Then $F \in L_r(\mathbf{R}^n)$ for $1/r = 1/p - m/n$, and

$$\|F\|_r \leq c \|N_1^m(F)\|_p,$$

where c is independent of F .

Proof. Apply Corollary to Proposition 4.1 and Propositions 4.6 and 4.7.

5. Inequalities for the distribution function of $N_q^m(F)$. The proof of Theorem I is based on the following:

THEOREM III. Let m , a , p , and q be positive real numbers such that $0 < a \leq m - [m]$ and

(i) when $n/(m - [m]) > 1$, then $1 \leq q < n/(m - [m])$, $1 < p < n/(m - [m])$, and $q \leq p$, and

(ii) when $n = m - [m] = 1$, then $p = q = 1$.

Let $F \in L_{q,\text{loc}}(\mathbf{R}^n)$ and suppose there is a set \mathcal{A} with finite measure such that $N_q^m(F) \in L_p(\mathcal{A}^c)$. Suppose that, for all $\varepsilon > 0$, $|\{x \in \mathcal{A}^c: |D^{[m]}F(x)| > \varepsilon\}| < \infty$. Define, for $s > 0$,

$$\lambda(s) = |\{N_q^m(F) > s\}| \quad \text{and} \quad \mu(s) = |\{N_q^{m-a}(F) > s\}|.$$

Then there are constants c_1 and c_2 , independent of F , such that

(iii) when $n/a > 1$, then, for all $s > 0$ and $t > 0$,

$$\mu(s) \leq \lambda(t) + c_1 s^{-a} t^a \lambda(t)^{1+aq/n} + c_2 s^{-u} \left(\int_0^t \lambda(\sigma) d\sigma \right)^{u/p},$$

where $1/u = 1/p - a/n$, and

(iv) when $a = n = 1$ (and $p = q = 1$), then

$$\mu(s) \leq \lambda(t) + c_1 s^{-1} t \lambda(t)^2 \quad \text{for} \quad s \geq c_2 \int_0^t \lambda(\sigma) d\sigma.$$

Proof. In the following, c will denote various constants independent of F , not necessarily the same at each occurrence. We begin by recalling the following result ([1], Theorem 5).

LEMMA 5.1. Let $m > 0$, $1 \leq q \leq p < \infty$, $F \in L_{q,\text{loc}}(\mathbf{R}^n)$, and suppose there is a set \mathcal{A} having finite measure such that $N_q^m(F) \in L_p(\mathcal{A}^c)$. Let $t > 0$ and define $\mathcal{O} = \{N_q^m(F) > t\}$. Then \mathcal{O} is open and has finite measure, and F can be decomposed as $F = F_1 + F_2$, where

- (i) $F_1 \equiv 0$ in \mathcal{O}^c ,
- (ii) if $\delta(y) = \text{dist}(y, \mathcal{O}^c)$, then $\int |F_1(x)|^q \delta(x)^{-qm} dx \leq ct^a |\mathcal{O}|$, and
- (iii) $\int N_q^m(F_2, x)^p dx \leq c(|\mathcal{O}|^{p^2} + \int_{\mathcal{O}^c} N_q^m(F, x)^p dx)$.

Using the splitting above, we first consider $N_q^{m-a}(F_1)$. Since $\text{supp } F_1 \subset \mathcal{O}$, part (i) of Proposition 2.1 implies that, for $y \in \mathcal{O}^c$,

$$N_q^{m-a}(F_1, y) \leq \left\{ \int_{\mathcal{O}} |F_1(x)|^q |x-y|^{-(n+(m-a)a)} dx \right\}^{1/a}.$$

Integrating over \mathcal{O}^c , one obtains from Fubini's theorem that

$$\int_{\mathcal{O}^c} N_q^{m-a}(F_1, y)^a dy \leq c \int_{\mathcal{O}} |F_1(x)|^q \delta(x)^{-(m-a)a} dx.$$

For each $x \in \mathcal{O}$, the set \mathcal{O} contains a ball of radius $\delta(x)$ with center x , hence $|\mathcal{O}| \geq c\delta(x)^n$. Thus part (ii) of Lemma 5.1 implies that

$$\int_{\mathcal{O}^c} N_q^{m-a}(F_1, y)^a dy \leq c |\mathcal{O}|^{aq/n} \int_{\mathcal{O}} |F_1(x)|^q \delta(x)^{-ma} dx \leq ct^a |\mathcal{O}|^{1+aq/n}.$$

Recalling that $|\mathcal{O}| = \lambda(t)$ and using (1.2), we have

$$(5.1) \quad |\{N_q^{m-a}(F_1) > s/2\}| \leq |\mathcal{O}| + |\{N_q^{m-a}(F_1) > s/2\} \cap \mathcal{O}^c| \\ \leq \lambda(t) + c_1 s^{-a} t^a \lambda(t)^{1+aq/n}.$$

Now consider $N_q^{m-a}(F_2)$. We begin by estimating $N_q^{[m]}(F_2)$.

LEMMA 5.2. Let a be a multi-index such that $|a| = [m]$. Then

$$|D^a F(x) - D^a F(y)| \leq c(N_q^m(F, x) + N_q^m(F, y)) |x-y|^{m-[m]},$$

where c is a constant independent of F , x , and y .

Proof. Apply [1], Lemma 5, with $E = \{x, y\}$. ■

Applying Lemma 5.2. to F_2 and integrating yields

$$\int_{E(x,e)} |D^a F_2(x) - D^a F_2(y)| dy \leq c e^{m-[m]+n} (N_q^m(F_2, x) + M N_q^m(F_2)(x)).$$

Dividing by $e^{m-[m]+n}$ and taking the supremum over e , we find that

$$N_1^{m-[m]}(D^a F_2, x) \leq c(N_q^m(F_2, x) + M N_q^m(F_2)(x)).$$

Let us now assume that $n/(m - [m]) > 1$. Then Proposition 4.8 and (1.4) imply that

$$(5.2) \quad \|D^a F_2\|_{\bar{u}} \leq c \|N_a^m(F_2)\|_p,$$

where $1/\bar{u} = 1/p - (m - [m])/n$.

In view of part (iii) of Lemma 5.1 and Proposition 2.4, $D^a F_2 = D_w^a F_2$, and since a was arbitrary, Proposition 2.2 implies that

$$(5.3) \quad \|N_a^{[m]}(F_2)\|_{\bar{u}} \leq c \|N_a^m(F_2)\|_p.$$

PROPOSITION 5.1. *Let $m > 0$, $0 < \theta < 1$, and $k = \theta m + (1 - \theta)[m]$. There is a constant c depending only on $[m]$ and n such that*

$$N_a^k(F) \leq N_a^m(F)^\theta (N_a^{[m]}(F) + c |D^{[m]}F|)^{1-\theta}.$$

Proof. Let $x \in \mathbf{R}^n$ and let P be the polynomial of degree $[m]$ in the definition of $N_a^m(F, x)$ and $N_a^k(F, x)$ (same polynomial for both). Let \bar{P} be the polynomial of degree $[m] - 1$ such that $P - \bar{P}$ is homogeneous of degree $[m]$. Then, for $\rho > 0$,

$$\rho^{-k} \left\{ \rho^{-n} \int_{B(x, \rho)} |F - P|^a \right\}^{1/a} = (\rho^{-m} \{ \dots \})^\theta (\rho^{-[m]} \{ \dots \})^{1-\theta}.$$

The first term is $\leq N_a^m(F, x)$, and the second is estimated as follows:

$$\begin{aligned} & \rho^{-[m]} \left\{ \rho^{-n} \int_{B(x, \rho)} |F - P|^a \right\}^{1/a} \\ & \leq \rho^{-[m]} \left\{ \rho^{-n} \int_{B(x, \rho)} |F - \bar{P}|^a \right\}^{1/a} + \rho^{-[m]} \left\{ \rho^{-n} \int_{B(x, \rho)} |\bar{P} - P|^a \right\}^{1/a} \\ & \leq N_a^{[m]}(F, x) + c |D^{[m]}F(x)|. \quad \blacksquare \end{aligned}$$

Setting $k = m - a$, we obtain $N_a^{m-a}(F_2) \leq N_a^m(F_2)^\theta N_a^{[m]}(F_2)^{1-\theta}$, and estimates (5.2) and (5.3) combine with Proposition 5.1 to yield

$$(5.4) \quad \|N_a^{m-a}(F_2)\|_{\bar{u}} \leq c \|N_a^m(F_2)\|_p,$$

in view of Hölder's inequality. Now part (iii) of Lemma 5.1 and (1.3) imply that

$$\begin{aligned} \|N_a^{m-a}(F_2)\|_{\bar{u}} & \leq c \left(|\mathcal{C}| t^p + \int_{\mathcal{C}} N_a^m(F)^p \right)^{1/p} \\ & \leq c \left(\lambda(t) t^p + \int_0^t \lambda(\sigma) d\sigma^p \right)^{1/p}. \end{aligned}$$

Since λ is decreasing,

$$\lambda(t) t^p = \lambda(t) \int_0^t d\sigma^p \leq \int_0^t \lambda(\sigma) d\sigma^p.$$

Therefore (1.2) implies that

$$(5.5) \quad |\{N_a^{m-a}(F_2) > s/2\}| \leq c_2 s^{-u} \left(\int_0^t \lambda(\sigma) d\sigma^p \right)^{u/p}.$$

In view of part (iii) of Proposition 2.1, (5.1) and (5.5) suffice to prove the theorem in the case $n/(m - [m]) > 1$.

When $n = m - [m] = 1$, we may use the elementary Sobolev inequality $\|D^{[m]}F_2\|_\infty \leq \|D^m F_2\|_1$ together with Proposition 2.2 to get

$$(5.6) \quad \|N_a^{[m]}(F_2)\|_\infty \leq c \|N_a^m(F_2)\|_1.$$

For the case $a < 1$, the proof follows the previous lines, with (5.3) replaced by (5.6), and completes the proof of conclusion (iii) of the theorem. When $a = 1$, the steps leading to (5.5) show that

$$\|N_a^{m-a}(F_2)\|_\infty \leq c \int_0^t \lambda(\sigma) d\sigma.$$

Since $|\{f > c\}| = 0$ if $c \geq \|f\|_\infty$, part (iv) of the theorem follows from (5.1). ■

6. Proof of Theorem I. It suffices to assume that $0 < a \leq m - [m]$, as the general case follows by repeated application of this case. Define, for $s > 0$,

$$\lambda(s) = |\{N_a^m(F) > s\}| \quad \text{and} \quad \mu(s) = |\{N_a^{m-a}(F) > s\}|.$$

We shall assume that $v_2 \leq v_1$. In view of Proposition 3.2, it suffices to show that there is a constant c such that,

$$\text{if } \int_0^\infty \lambda(s) ds^{v_1} < 1, \quad \text{then } \int_0^\infty \mu(s) ds^{v_2} \leq c.$$

(Here and below, the expression s^{v_i} means s^{v_1} when $s \leq 1$ and s^{v_2} when $s \geq 1$, and σ^{v_i}, t^{v_i} , etc., are similarly defined.) To apply Theorem III, let $\mathcal{A} = \{N_a^m(F) > 1\}$ and let p be chosen so that, when $n > 1$, $\max(v_i, v_i) < p < n/(m - [m])$, and, when $n = 1$, $p = 1$. Since λ is decreasing and integrable with respect to ds^{v_1} , $|\mathcal{A}| = \lambda(1) < \infty$; since $p > v_i$,

$$\int_{\mathcal{A}} N_a^m(F, x)^p dx \leq \int_0^1 \lambda(s) ds^{v_1} \leq \frac{p}{v_1} \int_0^1 \lambda(s) ds^{v_1} < \infty.$$

Thus Theorem III may be applied, and in the case $n/a > 1$, conclusion (iii) of Theorem III is

$$(6.1) \quad \mu(s) \leq \lambda(t) + c_1 s^{-a} t^a \lambda(t)^{1+aq/n} + c_2 s^{-u} \left(\int_0^1 \lambda(\sigma) d\sigma^p \right)^{u/p}$$

for any $s, t > 0$. Since $\int_0^\infty \lambda(s) ds^{v_i} \leq 1$, λ is decreasing, and $v_1 \geq v_2$, we conclude that $\lambda(s) \leq s^{-v_1}$. Thus (6.1) becomes

$$\mu(s) \leq \lambda(t) (1 + c_1 (s^{-1} t^{1-av_i/n})^q) + c_2 s^{-u} \left(\int_0^t \lambda(\sigma) d\sigma^p \right)^{u/p}.$$

For each $s > 0$, let $t = s^{w_i/v_i}$; then $s^{-1} t^{1-av_i/n} = 1$. Therefore, as $w_i < u$,

$$\begin{aligned} \int_0^\infty \mu(s) ds^{w_i} &\leq (1 + c_1) \int_0^\infty \lambda(s^{w_i/v_i}) ds^{w_i} + \\ &+ c_2 \max_{i=1,2} \{w_i/(u-w_i)\} \int_0^\infty \left(\int_0^{s^{w_i/v_i}} \lambda(\sigma) d\sigma^p \right)^{u/p} d(-s^{w_i-u}). \end{aligned}$$

For the first term, the change of variables $\sigma^{v_i} = s^{w_i}$ yields

$$\int_0^\infty \lambda(s^{w_i/v_i}) ds^{w_i} = \int_0^\infty \lambda(\sigma) d\sigma^{v_i} \leq 1.$$

For the second, we apply the following:

LEMMA 6.1. *Let f be a non-negative, increasing function on $]0, \infty[$ and let g be a continuous, decreasing function on $]0, \infty[$ such that $\lim_{t \rightarrow \infty} g(t) = 0$. Then, for any γ such that $0 < \gamma < 1$,*

$$\int_0^\infty f(s) d(-g) \leq \frac{1}{\gamma} \left(\int_0^\infty f(s)^\gamma d(-g^\gamma) \right)^{1/\gamma}.$$

Apply this with $f(s) = \left(\int_0^{s^{w_i/v_i}} \lambda(\sigma) d\sigma^p \right)^{u/p}$, $g(s) = s^{w_i-u}$, and $\gamma = p/u$.

Then, with the aid of Fubini's theorem,

$$\begin{aligned} \int_0^\infty \left(\int_0^{s^{w_i/v_i}} \lambda(\sigma) d\sigma^p \right)^{u/p} d(-s^{w_i-u}) &\leq \frac{u}{p} \left[\int_0^\infty \int_0^{s^{w_i/v_i}} \lambda(\sigma) d\sigma^p d(-s^{(w_i-u)(p/u)}) \right]^{u/p} \\ &= \frac{u}{p} \left[\int_0^\infty \left(\int_{\sigma^{v_i/w_i}}^1 d(-s^{(w_i-u)(p/u)}) \right) \lambda(\sigma) d\sigma^p \right]^{u/p} \\ &= \frac{u}{p} \left[\frac{p}{v_1} \int_0^1 \lambda(\sigma) d\sigma^{v_1} + \frac{p}{v_2} \int_1^\infty \lambda(\sigma) d\sigma^{v_2} \right]^{u/p} \\ &\leq \frac{u}{p} [p \max_{i=1,2} \{v_i^{-1}\}]^{u/p}. \end{aligned}$$

Thus in the case $n/a > 1$, Theorem I is proved, with

$$c = 1 + c_1 + c_2 \frac{u}{p} [p \max_{i=1,2} \{v_i^{-1}\}]^{u/p} \max_{i=1,2} \{w_i/(u-w_i)\}.$$

Now suppose that $n = a = 1$. Conclusion (iv) of Theorem III is that

$$\mu(s) \leq \lambda(t) + c_1 s^{-1} t \lambda(t)^2 \quad \text{for } s \geq c_2 \int_0^t \lambda(\sigma) d\sigma.$$

Again, $\lambda(t) \leq t^{-v_i}$, so that

$$\mu(s) \leq \lambda(t) (1 + c_1 s^{-1} t^{1-v_i}) \quad \text{for } s \geq c_2 t^{1-v_i} / (1 - v_i)$$

(recall that condition (ii) of Theorem 1 guarantees that $1 > v_i$ in this case). Let $c_3 = \max\{1, c_2/(1-v_i)\}$ and define t by the equation $s = c_3 t^{1-v_i} = c_3 t^{v_i/w_i}$. Then

$$\int_0^\infty \mu(s) ds^{w_i} \leq (1 + c_1) \int_0^\infty \lambda((s/c_3)^{w_i/v_i}) ds^{w_i}.$$

Note that $(s/c_3)^{w_i/v_i} \neq (s^{w_i/v_i})/c_3^{w_i/v_i}$, but

$$\begin{aligned} \int_0^\infty \lambda((s/c_3)^{w_i/v_i}) ds^{w_i} &= \int_0^1 \lambda((s/c_3)^{w_1/v_1}) ds^{w_1} + \int_1^{c_3} \lambda((s/c_3)^{w_1/v_1}) ds^{w_2} + \int_{c_3}^\infty \lambda((s/c_3)^{w_2/v_2}) ds^{w_2} \\ &\leq \int_0^{c_3} \lambda((s/c_3)^{w_1/v_1}) ds^{w_1} + \int_{c_3}^\infty \lambda((s/c_3)^{w_2/v_2}) ds^{w_2} \\ &= c_3^{w_1} \int_0^1 \lambda(\sigma) d\sigma^{v_1} + c_3^{w_2} \int_1^\infty \lambda(\sigma) d\sigma^{v_2} \leq c_3^{w_1}. \end{aligned}$$

Thus the theorem is complete, with $c = (1 + c_1) \max_{i=1,2} \{c_3^{w_i}\}$ in the case that $a = n = 1$. ■

Proof of Lemma 6.1. It suffices to consider the case when

$$I \equiv \left(\int_0^\infty f(s)^\gamma d(-g^\gamma) \right)^{1/\gamma}$$

is finite. Then for any $t > 0$,

$$I^\gamma \geq \int_t^\infty f(s)^\gamma d(-g^\gamma) \geq f(t)^\gamma \int_t^\infty d(-g^\gamma) = (f(t)g(t))^\gamma.$$

Therefore $f(t) \leq I/g(t)$ for any $t > 0$. Integrating, we get

$$\begin{aligned} \int_0^\infty f(t) d(-g) &= \int_0^\infty f(t)^\gamma f(t)^{1-\gamma} d(-g) \\ &\leq \int_0^\infty f(t)^\gamma I^{1-\gamma} g(t)^{\gamma-1} d(-g) \\ &= I^{1-\gamma} \frac{1}{\gamma} \int_0^\infty f(t)^\gamma d(-g^\gamma) = \frac{1}{\gamma} I. \blacksquare \end{aligned}$$

7. Proof of Theorem II. In view of part (iv) of Proposition 2.1, it suffices to assume that $q = 1$, and in view of Theorem I, it suffices to assume that $m = k + \varepsilon$, $\varepsilon > 0$. The proof follows that of Theorem I with minor modifications. Define

$$\lambda(s) = |\{N_q^m(F) > s\}| \quad \text{and} \quad \mu(s) = |\{|D^k F| > s\}|.$$

As in Lemma 5.1, decompose $F = F_1 + F_2$; since $F_1 \equiv 0$ in \mathcal{O}^c ,

$$\mu(s) \leq |\mathcal{O}| + |\{|D^k F_2| > s\}|.$$

From (5.2), it follows that

$$|\{|D^k F_2| > s\}| \leq cs^{-u} \|N_q^m(F_2)\|_p^u,$$

so that Lemma 5.1 yields

$$\mu(s) \leq \lambda(t) + cs^{-u} \left(\int_0^t \lambda(\sigma) d\sigma^\nu \right)^{u/p}.$$

Using this replacement for Theorem III, the result now follows as in the proof of Theorem I.

References

- [1] A. P. Calderón, *Estimates for singular integral operators in terms of maximal functions*, *Studia Math.* 44 (1972), pp. 167-186.
- [2] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, *Studia Math.*, 20 (1961), pp. 171-225.
- [3] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, *Acta Math.* 129 (1972), pp. 137-193.
- [4] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO, CHICAGO, ILL.
and
APPLIED MATHEMATICS DEPARTMENT
BROOKHAVEN NATIONAL LABORATORY, UPTON, N.Y.

Received May 6, 1976

(1155)

A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms

by

ROBERT H. LATTEK (Los Angeles, Calif.)

Abstract. Distributions in $H^p(\mathbf{R}^n)$, where $p < 1$, are represented as weighted sums of atoms.

§ 1. Introduction. Let $H^p(\mathbf{R}^n)$ denote the space of functions u , harmonic in the upper half-space $\mathbf{R}_+^{n+1} = \{(x, y) = (x_1, \dots, x_n, y): y > 0\}$, whose non-tangential maximal function $u^*(x_0) = \sup_{|x-x_0| < y} |u(x, y)|$ is in the Lebesgue space $L^p(\mathbf{R}^n)$. Give $H^p(\mathbf{R}^n)$ the "norm" $\|u\|_{H^p} = \|u^*\|_p$. C. Fefferman and E. M. Stein [4] have shown that if $u \in H^p(\mathbf{R}^n)$, then $\lim_{t \rightarrow 0} u(\cdot, t) = f$ exists in the sense of tempered distributions and that u is uniquely determined by f . We will denote also by $H^p(\mathbf{R}^n)$ the space of boundary distributions of functions in $H^p(\mathbf{R}^n)$. R. Coifman [2] has exhibited an explicit representation for $f \in H^p(\mathbf{R}^n)$, $0 < p \leq 1$, by means of a purely real variables construction. Here we modify Coifman's construction in order to obtain such a representation for $H^p(\mathbf{R}^n)$, $n \geq 1$.

Let $0 < p \leq 1$ and define a p -atom to be a function b on \mathbf{R}^n which is supported on a cube Q in \mathbf{R}^n with sides parallel to the axes and which satisfies

(i) $|b(x)| \leq |Q|^{-1/p}$, where $|Q|$ is the volume of Q

and

(ii) $\int b(x) x^\alpha dx = 0$, where α is a multi-index of order $|\alpha| \leq N = [n(1/p - 1)]$, the integer part of $n(1/p - 1)$. We then have:

THEOREM. *A distribution f is in $H^p(\mathbf{R}^n)$, $0 < p \leq 1$, if and only if there exist a sequence of p -atoms b_i and a sequence of non-negative real numbers λ_i such that*

$$(1.1) \quad f = \sum_{i=0}^\infty \lambda_i b_i$$

in the sense of distributions and

$$(1.2) \quad A \|f\|_{H^p} \leq \sum_{i=0}^\infty \lambda_i^p \leq B \|f\|_{H^p}^p,$$

where A, B are constants which depend only on n and p .