STUDIA MATHEMATICA, T. LXII. (1978)

A remark to a paper of Janas "Toeplitz operators related to certain domains in C^n"

by

KÔZÔ YABUTA (Sandai, Japan)

Abstract. We shall improve the main result by Janas [2]. He showed that the C^*-algebra generated by Toeplitz operators in strongly pseudoconvex domain D in C^n is, modulo compact operators, isometrically isomorphic with the space of all continuous functions on a set $\sigma \subset \partial D$. We shall show that this set σ is exactly ∂D. Hence one sees that a Toeplitz operator with continuous $N\times N$ matrix valued symbol φ is a Fredholm operator if and only if $\det\varphi(z) \neq 0$ for $z \in \partial D$.

In this note we remark that the assumption $A(D) = P(D)$ posed by Janas in [2], Theorems 2.1 and 3.1, can be removed. Let $L(H)$ be the algebra of all bounded linear operators in a complex Hilbert space H and let $\mathcal{K}(H)$ be the ideal of all compact operators in H. Let D be a bounded domain in the n-dimensional complex euclidian space C^n and $C(\bar{D})$ be the Banach algebra of all continuous functions on $\bar{D}$ and $A(D)$ the Banach subalgebra of $C(\bar{D})$ of all $f \in C(\bar{D})$ which are holomorphic in D. Denote by $P(D)$ the Banach algebra of all $f \in C(\bar{D})$, which are uniformly approximated by analytic polynomials on $\bar{D}$. Let $L^2(D)$ be the space of functions $f\colon D \to C$ which are square integrable with respect to the Lebesgue measure dV in C^n, and by $H^2(D)$ the space of all $f \in L^2(D)$, which are holomorphic in D. Denote by P the orthogonal projection from $L^2(D)$ onto $H^2(D)$. For a $\varphi \in L^\infty(dV)$ the Teoplitz operator $T_\varphi\colon H^2(D) \to H^2(D)$ is defined by $T_\varphi f = P(\varphi f)$. Let $\mathcal{C}$ be a C^*-algebra generated by Toeplitz operators T_φ $(\varphi \in C(\bar{D}))$. What we shall show is the following

THEOREM. *If D is a strongly pseudoconvex domain in C^n, then $\mathcal{C}/\mathcal{K}(H^2(D))$ is isometrically isomorphic with $C(\partial D)$, more precisely, there exists a *-homomorphism ϱ from $\mathcal{C}$ onto $C(\partial D)$ such that the sequence*

$$(0) \to \mathcal{K}(H^2(D)) \xrightarrow{i} \mathcal{C} \xrightarrow{\varrho} C(\partial D) \to (0)$$

is exact, where i denotes the inclusion map, $\varrho(T_\varphi) = \varphi|_{\partial D}$.

Proof. Janas proved this theorem under the additional assumption $A(D) = P(D)$. He proved that the theorem is valid if ∂D is replaced by $\sigma_\pi = \sigma_\pi(T_{z_1}, \ldots, T_{z_n})$: the joint approximate point spectrum for $T_{z_1}, \ldots$ $\ldots, T_{z_n}$. He also showed $\sigma_\pi \subset \partial D$. Hence it is enough to show $\sigma_\pi \supset \partial D$

if D is strongly pseudoconvex. To show it, we first note that if f is bounded holomorphic in D, then $\|T_f\| = \|f\|_\infty$. In fact, let $\zeta \in D$. Then $f(\zeta) \in \sigma(T_f)$: the spectrum of T_f, since for any $B \in L(H^2(D))$

$$\big(T_f - f(\zeta)\big)B1 = \big(f - f(\zeta)\big)B1 \neq 1.$$

Hence $f(D) \subset \sigma(T_f)$ and so we have

$$\|f\|_\infty = \sup_{z\in D}|f(z)| \leqslant \sup\{|\lambda|:\ \lambda \in \sigma(T_f)\} \leqslant \|T_f\| \leqslant \|f\|_\infty.$$

Hence the set $\mathscr{A} = \{T_f:\ f \in A(D)\}$ is a commutative closed subalgebra of $L(H^2(D))$ which is isometrically isomorphic with $A(D)$ and which contains $T_{z_1}, \ldots, T_{z_n}$ and the identity operator. Therefore, by the argument in the proof of the theorem in [4], p. 240, we have for every $\eta \in \Gamma(\mathscr{A})$: the Shilov boundary for $\mathscr{A}$

$$\big(\eta(T_{z_1}), \ldots, \eta(T_{z_n})\big) \in \sigma_\pi(T_{z_1}, \ldots, T_{z_n})$$

or equivalently for every $\xi \in \Gamma\big(A(D)\big)$

$$(*) \qquad \big(\xi(z_1), \ldots, \xi(z_n)\big) \in \sigma_\pi(T_{z_1}, \ldots, T_{z_n}).$$

However, it is known that $\Gamma\big(A(D)\big) = \partial D$ if D is strongly pseudoconvex ([1], Theorem 6.3). Hence we have $\partial D \subset \sigma_\pi$, which proves the Theorem.

Remark. As the above proof shows, for every bounded domain D in C^n, $(*)$ is still valid for $A(D)$ and also when one replaces $A(D)$ by any other closed subalgebra of $H^\infty(D)$: the Banach algebra of all bounded holomorphic functions in D, for example, $P(D)$, $R(D)$, or $H(D)$: the set of all $f \in C(\bar{D})$ which are approximated uniformly on $\bar{D}$ by holomorphic functions on $\bar{D}$.

References

[1] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries*, Trans. Amer. Math. Soc. 91 (1959), pp. 246–276.

[2] J. Janas, *Toeplitz operators related to certain domains in* C^n, Studia Math. 54 (1975), pp. 73–79.

[3] Z. Słodkowski and W. Żelazko, *On joint spectra of commuting families of operators*, ibid. 50 (1974), pp. 127–148.

[4] W. Żelazko, *On a problem concerning joint approximate point spectra*, ibid. 45 (1973), pp. 239–240.

[5] W. Żelazko, *On a certain class of non-removable ideals in Banach algebras*, ibid. 44 (1972), pp 87–92.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN

Current address: KYOTO TECHNICAL UNIVERSITY
MATSUGASAKI, SAKYO-KU
KYOTO, JAPAN

Received April 9, 1976 (1148)

STUDIA MATHEMATICA, T. LXII. (1978)

Sobolev type inequalities for $p > 0$

by

ALBERTO P. CALDERÓN (Chicago, Ill.) and
RIDGWAY SCOTT (Upton, N. Y.)*

Abstract. Sobolev type inequalities for generalized Peano derivatives with exponents p, $p > 0$, are obtained.

Certain kinds of generalized Peano derivatives (see [1] and [2] and Definition 2.2 below) have been shown to have many desirable properties that the classical Peano derivatives lack. In this paper we continue the study of such derivatives and establish Sobolev type inequalities between them. The basic results here are the estimates for the distribution functions of the $N_a^m(F)$ in Section 5, from which the desired inequalities for exponents p which are merely positive follow.

1. Notation. By $x, y, z, \ldots$, we denote points in n-dimensional Euclidean space $\boldsymbol{R}^n$. The closed ball with center x and radius ϱ will be written as $B(x, \varrho)$. Given a set $\mathscr{A}$, let $d(x, \mathscr{A}) = \inf\{|x-y|:\ y \in \mathscr{A}\}$. If $\mathscr{A}$ is (Lebesgue) measurable, let $|\mathscr{A}| = \int_{\mathscr{A}} dx$ denote the measure of $\mathscr{A}$ (dx denotes Lebesgue measure).

We will deal with real or complex valued functions, and we will refer to the corresponding field as "scalars". The term "constant" will be used to mean a positive real number. All functions are assumed to be measurable.

For an open set $\mathcal{O}$, let $C^\infty(\mathcal{O})$ denote the linear space of functions infinitely differentiable in $\mathcal{O}$, and let $C_0^\infty(\mathcal{O})$ be the subspace consisting of functions with compact support. For a function $F \in C^\infty(\mathcal{O})$ and a point $x \in \mathcal{O}$,

$$(1.1) \qquad F^{(\alpha)}(x) = D^\alpha F(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} F(x).$$

As usual, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

* Work performed under the auspices of the ERDA.