STUDIA MATHEMATICA, T. LXII. (1978)

The trace of Sobolev and Besov spaces if $0 < p < 1$

by

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Abstract. We determine the trace of Sobolev and Besov spaces if $0 < p < 1$ and $s > 1/p + (n-1)(1/p-1)$.

0. Introduction. The trace of these spaces if $1 < p < \infty$ has been known to be a Besov space for a long time. (See the discussion in [5].) The case $p = 1$ has been covered by Strichartz [6]. The purpose of this note is to extend these results to the case $0 < p < 1$. The only new thing is that we have to impose the restriction $s > 1/p + (n-1)(1/p-1)$ rather than $s > 1/p$. Our method of proof, which works for the entire range $0 < p < \infty$, seems to be new and is in a way simpler than those previously known. For technical reasons we prefer to work with the homogeneous spaces rather than with the inhomogeneous ones, but the result is valid in the latter case too.

Finally, I take this opportunity to express my deep gratitude to Professor Jaak Peetre for proposing the problem and for his kind interest and advice.

1. Preliminaries. We begin by defining the spaces to be studied. Let $\{\varphi_\nu\}_{\nu\in Z}$ be a sequence of testfunctions such that

$$(1.1)\qquad \begin{aligned} &\varphi_\nu \in \mathcal{S}(\boldsymbol{R}^n),\\ &\operatorname{supp}\hat{\varphi}_\nu = \{2^{\nu-1} \leqslant |\xi| \leqslant 2^{\nu+1}\},\\ &|\hat{\varphi}_\nu(\xi)| \geqslant C_\varepsilon > 0 \quad \text{if} \quad 2^\nu(2-\varepsilon)^{-1} \leqslant |\xi| \leqslant 2^\nu(2-\varepsilon),\\ &|D_\alpha \hat{\varphi}_\nu(\xi)| \leqslant C_\alpha |\xi|^{-|\alpha|} \quad \text{for every multiindex } \alpha. \end{aligned}$$

$\mathcal{S}(\boldsymbol{R}^n)$ is as usual the space of rapidly decreasing functions on $\boldsymbol{R}^n$ and $\mathcal{S}'(\boldsymbol{R}^n)$ the dual space of tempered distributions.

DEFINITION 1.1. Let s be real, $0 < p, q \leqslant \infty$. The *Besov space* $\dot{B}_p^{sq}(\boldsymbol{R}^n)$ is the space of all $f \in \mathcal{S}'(\boldsymbol{R}^n)$ such that

$$\|f\|_{\dot{B}_p^{sq}(\boldsymbol{R}^n)} \equiv \Big(\sum_\nu [2^{\nu s}\|\varphi_\nu * f\|_{L_p(\boldsymbol{R}^n)}]^q\Big)^{1/q} < \infty.$$

DEFINITION 1.2. Let s be real, $0<p<\infty$, and let $I^s=(-\Delta)^{s/2}$. The *Riesz potential space* $\dot{P}_p^s(\boldsymbol{R}^n)$ is the space of all $f\in\mathscr{S}'(\boldsymbol{R}^n)$ such that

$$\|f\|_{\dot{P}_p^s(\boldsymbol{R}^n)}\equiv\|I^sf\|_{H_p(\boldsymbol{R}^n)}<\infty.$$

(All properties of the Hardy space $H_p(\boldsymbol{R}^n)$ we will use are given in Lemmas 1.1 and 1.2 below.)

We also need the following space:

DEFINITION 1.3. Let m be a positive integer, $0<s<m$, $0<p,q<\infty$. The space $\dot{\boldsymbol{B}}_p^{sq}(\boldsymbol{R}^n)$ is the completion of C^∞ in the quasi-norm

$$\|f\|_{\dot{\boldsymbol{B}}_p^{sq}(\boldsymbol{R}^n)}\equiv\sum_{i=1}^n\Big(\int_0^\infty[h^{-s}\|\Delta_{he_i}^mf\|_{L_p(\boldsymbol{R}^n)}]^q\,dh/h\Big)^{1/q}$$

with the mth order differences

$$\Delta_{he_i}^mf(x)=\sum_{\nu=0}^m(-1)^m\binom{m}{\nu}f(x+\nu he_i),\qquad (e_1=(1,0,\ldots,0),\text{ etc.}).$$

Remark 1.1. $\dot{B}_p^{sq}(\boldsymbol{R}^n)$ is independent of the sequence $\{\varphi_\nu\}_{\nu\in\boldsymbol{Z}}$ chosen.

Remark 1.2. We emphasize that we in all these homogeneous spaces work modulo polynomials.

Next we collect some properties of the spaces that will be called for later. (Cf. [2] and [3].)

LEMMA 1.1.

(i) *If s is real and $0<p,q<\infty$, then $\mathscr{S}_0=\{f\in\mathscr{S}:\ 0\notin\operatorname{supp}\hat{f}\}$ is dense in $\dot{P}_p^s$ and $\dot{B}_p^{sq}$.*

(ii) *We have*

$$\dot{B}_p^{sp}\to\dot{P}_p^s\to\dot{B}_p^{s2}\quad\text{if}\quad 0<p\leqslant 2,$$
$$\dot{B}_p^{s2}\to\dot{P}_p^s\to\dot{B}_p^{sp}\quad\text{if}\quad 2\leqslant p<\infty.$$

(iii) *Concerning real interpolation we have*

$$(\dot{P}_p^{s_0},\dot{P}_p^{s_1})_{\theta q}=\dot{B}_p^{sq},$$
$$(\dot{B}_p^{s_0q_0},\dot{B}_p^{s_1q_1})_{\theta q}=\dot{B}_p^{sq},$$

if $s=(1-\theta)s_0+\theta s_1$ $(0<\theta<1;\ s_0\neq s_1)$.

(iv) *If $s>\max(0,n(1/p-1))$ and $0<p,q\leqslant\infty$, then*

$$\dot{B}_p^{sq}=\dot{\boldsymbol{B}}_p^{sq}$$

with equivalent quasi-norms.

Let $u(x,t)$ be a harmonic function in $\boldsymbol{R}_+^{n+1}$. Define the *maximal*

operator M_a by

$$M_au(x)=\sup_{t>0}\sup_{|x-y|<at}|u(y,t)|,\quad a>0.$$

The following lemma is a result of Fefferman–Stein (see [1] or [3]).

LEMMA 1.2. *If $f\in H_p(\boldsymbol{R}^n)$, $0<p<\infty$, and $u(x,t)$ is the Poisson integral of f, then*

$$\|M_au\|_{L_p(\boldsymbol{R}^n)}\approx\|f\|_{H_p(\boldsymbol{R}^n)}$$

and

$$\|M_a(t^m\partial_i^mu)\|_{L_p(\boldsymbol{R}^n)}\leqslant C\|f\|_{H_p(\boldsymbol{R}^n)}\qquad(i=1,\ldots,n)$$

for a fixed but arbitrary $a>0$.

Here and in what follows $\partial_i=\partial/\partial x_i$.

2. The trace. Let us denote a point $x\in\boldsymbol{R}^n$ by $x=(x',x_n)$, where $x'\in\boldsymbol{R}^{n-1}$ and $x_n\in\boldsymbol{R}^1$. Identify $\boldsymbol{R}^{n-1}$ with the hyperplane $x_n=0$ in $\boldsymbol{R}^n$ and consider the trace operator

$$\mathrm{Tr}:\ \mathscr{S}_0(\boldsymbol{R}^n)\to\mathscr{S}(\boldsymbol{R}^{n-1})$$

defined by

$$\mathrm{Tr}f(x')=f(x',0).$$

Our main result is the following theorem:

THEOREM 2.1. *Let $0<p<\infty$, $0<q\leqslant\infty$ and $s>1/p+\max(0,(n-1)(1/p-1))$. Then the trace operator can be extended so that*

$$\mathrm{Tr}:\ \dot{B}_p^{sq}(\boldsymbol{R}^n)\to\dot{B}_p^{s-1/p,q}(\boldsymbol{R}^{n-1}),\tag{2.1}$$

$$\mathrm{Tr}:\ \dot{P}_p^{s}(\boldsymbol{R}^n)\to\dot{B}_p^{s-1/p,p}(\boldsymbol{R}^{n-1}).\tag{2.2}$$

Conversely, there is an operator Sr

$$\mathrm{Sr}:\ \dot{B}_p^{s-1/p,q}(\boldsymbol{R}^{n-1})\to\dot{B}_p^{s,q}(\boldsymbol{R}^n),\tag{2.3}$$

$$\mathrm{Sr}:\ \dot{B}_p^{s-1/p,p}(\boldsymbol{R}^{n-1})\to\dot{P}_p^{s}(\boldsymbol{R}^n),\tag{2.4}$$

so that $\mathrm{Tr}\circ\mathrm{Sr}=\mathrm{Id}$.

Proof. The result is known if $p\geqslant 1$ (see Introduction). Thus in proving the theorem we shall, for simplicity only, assume that $0<p\leqslant 1$. We also take $q<\infty$. (The case $q=\infty$ can be handled directly afterwards. Just interpolate.)

In view of Lemma 1.1 (ii) and (iii), it is sufficient to prove (2.2) and (2.3). Furthermore, in view of Lemma 1.1 (i), we may work in $\mathscr{S}_0$ and then extend the operators by continuity.

Instead of (2.2) we shall prove the following stronger statement: if $s > 1/p$, then

$$\mathrm{Tr}\colon \dot{P}_p^s(\boldsymbol{R}^n) \to \dot{\boldsymbol{B}}_p^{s-1/p,p}(\boldsymbol{R}^{n-1}).$$

(That this is indeed sufficient follows by Lemma 1.1 (iv); the analogous modification of (2.1) is also true.) Assume that $f \in \dot{P}_p^s(\boldsymbol{R}^n)$; then

$$f(x', x_n) = I^{-s} g(x', x_n),$$

where $g \in H_p(\boldsymbol{R}^n)$ and

$$\|f\|_{\dot{P}_p^s(\boldsymbol{R}^n)} = \|g\|_{H_p(\boldsymbol{R}^n)}.$$

Let $u(x, t)$, $t > 0$, be the Poisson integral of g. Then as is easily verified

$$f(x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{-1+s} u(x, t)\, dt.$$

(Just take the Fourier transform of both sides.) We consequently have

$$\Delta_{he_i}^m \mathrm{Tr} f(x') = \frac{1}{\Gamma(s)} \int_0^\infty t^{-1+s} \Delta_{he_i}^m u(x', 0, t)\, dt$$

$$= \frac{1}{\Gamma(s)} \sum_{\nu \in \boldsymbol{Z}} \int_{2^\nu h}^{2^{\nu+1}h} t^{-1+s} \Delta_{he_i}^m u(x', 0, t)\, dt, \qquad (i = 1, \ldots, n-1).$$

Hence

$$\text{(2.5)} \qquad \|\Delta_{he_i}^m \mathrm{Tr} f\|_{L_p(\boldsymbol{R}^{n-1})}^p \leqslant C \sum_{\nu \in \boldsymbol{Z}} 2^{\nu s p} h^{sp} \Big\| \sup_{2^\nu h < t < 2^{\nu+1}h} |\Delta_{he_i}^m u(\cdot, 0, t)| \Big\|_{L_p(\boldsymbol{R}^{n-1})}^p,$$

where we have used

$$\text{(2.6)} \qquad (x+y)^p \leqslant x^p + y^p$$

when $0 < p \leqslant 1$ and $x, y \geqslant 0$. Since

$$\sup_{2^\nu h < t < 2^{\nu+1}h} |\Delta_{he_i}^m u(x', 0, t)| \leqslant \sum_{k=0}^m \binom{m}{k} M_1 u(x' + khe_i, 2^\nu h),$$

(2.6) also gives

$$\text{(2.7)} \qquad \Big\| \sup_{2^\nu h < t < 2^{\nu+1}h} |\Delta_{he_i}^m u(\cdot, 0, t)| \Big\|_{L_p(\boldsymbol{R}^{n-1})}^p \leqslant C \|M_1 u(\cdot, 2^\nu h)\|_{L_p(\boldsymbol{R}^{n-1})}^p.$$

On the other hand, using the mean value theorem we see that

$$\sup_{2^\nu h < t < 2^{\nu+1}h} |\Delta_{he_i}^m u(x', 0, t)| \leqslant C h^m \sup_{2^\nu h < t < 2^{\nu+1}h} \sup_{|x'-y'| < hm} |\partial_i^m u(y', 0, t)|$$

$$\leqslant C 2^{-\nu m} \sup_{2^\nu h < t < 2^{\nu+1}h} \sup_{|x'-y'| < tm} |t^m \partial_i^m u(y', 0, t)|$$

$$\leqslant C 2^{-\nu m} M_{m+1}(t^m \partial_i^m u)(x', 2^\nu h)$$

if $\nu \geqslant 0$.

Thus

$$\text{(2.8)} \qquad \Big\| \sup_{2^\nu h < t < 2^{\nu+1}h} |\Delta_{he_i}^m u(\cdot, 0, t)| \Big\|_{L_p(\boldsymbol{R}^{n-1})}^p \leqslant C 2^{-\nu m p} \|M_{m+1}(t^m \partial_i^m u)(\cdot, 2^\nu h)\|_{L_p(\boldsymbol{R}^{n-1})}^p \quad \text{if } \nu \geqslant 0.$$

Combining (2.5), (2.7) and (2.8), we find that

$$\int_0^\infty h^{-sp} \|\Delta_{he_i}^m \mathrm{Tr} f\|_{L_p(\boldsymbol{R}^{n-1})}^p dh \leqslant C \Big\{ \sum_{-\infty}^{-1} 2^{\nu s p} \int_0^\infty \|M_1 u(\cdot, 2^\nu h)\|_{L_p(\boldsymbol{R}^{n-1})}^p dh +$$

$$+ \sum_0^\infty 2^{\nu p(s-m)} \int_0^\infty \|M_{m+1}(t^m \partial_i^m u)(\cdot, 2^\nu h)\|_{L_p(\boldsymbol{R}^{n-1})}^p dh \Big\}.$$

If we now change variable ($2^\nu h = x_n$) and use Lemma 1.2, we easily get

$$\|\mathrm{Tr} f\|_{\dot{\boldsymbol{B}}_p^{s-1/p,p}(\boldsymbol{R}^{n-1})}^p \leqslant C \Big\{ \sum_{-\infty}^{-1} 2^{\nu(s-1/p)p} + \sum_0^\infty 2^{\nu(s-1/p-m)p} \Big\} \|g\|_{H_p(\boldsymbol{R}^n)}^p \leqslant C \|g\|_{H_p(\boldsymbol{R}^n)}^p$$

since $0 < s - 1/p < m$. This concludes the proof of (2.2).

To prove (2.3) we assume that $f \in \dot{\boldsymbol{B}}_p^{s-1/p,q}(\boldsymbol{R}^{n-1})$. Let $\{\varphi_\nu'\}$ and $\{\psi_\nu\}$ be sequences of testfunctions on $\boldsymbol{R}^{n-1}$ and $\boldsymbol{R}^1$, respectively, satisfying in addition to the analogues of (1.1) also

$$\sum_\nu \varphi_\nu' = \delta; \qquad \psi_\nu(x_n) = 2^\nu \psi_0(2^\nu x_n), \qquad \psi_0(0) = 1.$$

(δ is Dirac's δ-function.) Put

$$\mathrm{Sr} f(x', x_n) = \sum_\mu 2^{-\mu} \varphi_\mu' * f(x') \otimes \psi_\mu(x_n) = \sum_\mu F_\mu.$$

We obviously have

$$\mathrm{Sr} f(x', 0) = f(x')$$

(i.e. $\mathrm{Tr} \circ \mathrm{Sr} = \mathrm{Id}$). To avoid some trivial technicalities we assume that on $\operatorname{supp} \hat{F}_\mu$, the testfunctions $\hat{\varphi}_\nu$ on $\boldsymbol{R}^n$ are of the form $\hat{\varphi}_\nu' \otimes \hat{\psi}_\nu$. Observe that $\operatorname{supp} \hat{F}_\mu$ and $\operatorname{supp} \hat{\varphi}_\nu$ overlap only if $|\mu - \nu| \leqslant 2$. To estimate $\varphi_\nu * \mathrm{Sr} f$ we have therefore to consider terms

$$a = 2^{-\nu} \varphi_\nu' * \varphi_\mu' * f \otimes \psi_\nu * \psi_\mu$$

with $|\mu - \nu| \leqslant 2$. Since

$$\|\psi_\nu * \psi_\mu\|_{L_p(\boldsymbol{R}^1)} \leqslant C 2^{\nu(1-1/p)},$$

we get

$$\|a\|_{L_p(\boldsymbol{R}^n)} \leqslant C 2^{-\nu/p} \|\varphi_\nu' * f\|_{L_p(\boldsymbol{R}^{n-1})}.$$

Inserting this estimate in the definition of the quasi-norm in $\dot B_p^{sq}$ gives

$$\|\mathrm{Sr} f\|_{\dot B_p^{sq}(\mathbf{R}^n)} \leqslant C\|f\|_{\dot B_p^{s-1/p,q}(\mathbf{R}^{n-1})}.$$

This completes the proof.

We conclude with the following remarks:

Remark 2.1. Our results for the Besov spaces can be summarized in the following diagram:

$$\begin{array}{ccc} \dot B_p^{s-1/p,q}(\mathbf{R}^{n-1}) & \xrightarrow[s\in\mathbf{R}^1]{\mathrm{Sr}} & \dot B_p^{s,q}(\mathbf{R}^n) \\ \Big\downarrow \mathrm{Id} & & \Big\downarrow \mathrm{Tr}\ \ s>1/p \\ \dot B_p^{s-1/p,q}(\mathbf{R}^{n-1}) & \xleftarrow[s>1/p+\max(0,(n-1)(1/p-1))]{\mathrm{Id}} & \dot{\boldsymbol{B}}_1^{s-1/p,q}(\mathbf{R}^{n-1}) \end{array}$$

We have an analogous diagram in the case of potential spaces.

Remark 2.2. Peetre [4] has shown that a necessary condition for the trace X of $\dot B_p^{s,q}(\mathbf{R}^n)$ to exist as a distributions pace (i.e. $\mathcal{S} \subset X \subset \mathcal{S}'$) is that $s \geqslant 1/p+\max\big(0, (n-1)(1/p-1)\big)$. Theorem 2.1 establishes that $s > 1/p+\max\big(0, (n-1)(1/p-1)\big)$ is sufficient $(0<q\leqslant\infty)$. Incidentally, we have proved that the trace also exists if $1/p < s \leqslant 1/p+(n-1)(1/p-1)$, $0<p<1$, $0<q\leqslant\infty$, but as a space of measurable functions:

$$0 \neq X \subset \dot{\boldsymbol{B}}_p^{s-1/p,q}(\mathbf{R}^{n-1}).$$

(Note that it also follows that Lemma 1.1 (iv) is nearly best possible.) This is also true if $s = 1/p$, $0<p\leqslant 1$ and $0<q\leqslant p$; it is then possible to prove

$$0 \neq X \subset L_p(\mathbf{R}^{n-1}).$$

Remark 2.3. If we are only interested in the result for Besov spaces (viz. (2.1) and (2.3)), a direct proof of (2.1) runs as follows:

Take sequences of testfunctions $\{\varphi_\nu\}$ and $\{\varphi'_\nu\}$ on $\mathbf{R}^n$ and $\mathbf{R}^{n-1}$ respectively with the properties (1.1) and furthermore $\sum_\nu \varphi_\nu = \delta$. Then

$$\mathrm{Tr} f*\varphi'_\mu = \sum_{\nu\geqslant\mu-1} \mathrm{Tr}(f*\varphi_\nu)*\varphi'_\mu.$$

Now using the following two facts (see [3]) it is easy to complete the proof:

(i) If $\hat f$ has support in $\{|\xi|\leqslant r\}$ and $0<p<\infty$, then

$$\|\sup_{x_n} f(\cdot, x_n)\|_{L_p(\mathbf{R}^{n-1})} \leqslant C r^{1/p}\|f\|_{L_p(\mathbf{R}^n)}.$$

(ii) If $\hat f, \hat g$ have support in $\{|\xi|\leqslant r\}$ and $0<p<1$, then

$$\|f*g\|_{L_p(\mathbf{R}^n)} \leqslant C r^{n(1/p-1)}\|f\|_{L_p(\mathbf{R}^n)}\|g\|_{L_p(\mathbf{R}^n)}.$$

((ii) also appears implicitly in the proof of (2.3) in Theorem 2.1.)

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Received February 5, 1976 (1120)