Now a simple calculation shows that $UT_{(w,\omega)}$ is the function $S_{(w,\omega)}$ defined by (2), hence in view of the fact that $U$ is unitary, (17) yields

$$F(w, \omega) = \langle F | S_{(w,\omega)} \rangle_{H^2}.$$

But this is equation (3) in assertion (iv) of the theorem whose proof is now complete.

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# On an integral representation of antisymmetric operations in Hilbert spaces
## I. Bounded operations

by

STANISŁAW GÓŹDŹ (Lublin)

**Abstract.** In this note we give the representation of the bounded and antysymmetric operation $A$ defined and valued in the Hilbert space $H$ (real or complex) in the following form:

$$Ax = \int_{\alpha}^{\beta} \lambda dQ_{\lambda} x,$$

where $Q \in I(H) = \{Q \in L(H): Q^3 = -Q \text{ and } Q^* = -Q\}$.

Moreover, we give the properties of the operation of the class $I(H)$ and some form of the solution of the equation

$$\frac{d}{dt} x(t) = Ax(t), \quad \text{where } A \text{ is antisymmetric}$$

with the initial condition $x(0) = x_0$.

**1. Introduction.** In this paper we give the spectral representation of bounded antisymmetric operations in Hilbert spaces; the case of unbounded operations will be presented in the next paper.

In our theory we formally give an effective solution of the equation

$$\frac{d}{dt} x(t) = Ax(t) \tag{1}$$

with the antisymmetric and bounded operation $A$.

**2. Class of operations** $I(H)$. Let $H$ denote a Hilbert space and let $L(H)$ denote a linear space of all linear and bounded operations in $H$. The class of operations $I(H)$ is defined as follows:

$$Q \in I(H) \equiv Q \in L(H) \quad \text{and} \quad Q^3 = -Q = Q^*, \tag{2}$$

where $Q^*$ denotes the conjugate operation with $Q$.

We shall see later that the class $I(H)$ is non-empty and non-trivial; now we give a simple

EXAMPLE 1.2. Let $H$ denote the three-dimensional real Euclidean space. It is easy to see that the operation $Q$ given by the matrix

$$Q = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \quad \text{where } a^2+b^2+c^2 = 1, \tag{3}$$

belongs to $I(H)$.

Every operation $Q \in I(H)$ has the following properties:

THEOREM 1.2. *If $Q \in I(H)$, then:*

(i) *$-Q^2$ is a projection, i.e. $(-Q^2)^2 = -Q^2$ and $(-Q^2)^* = -Q^2$.*

(ii) $\|Qx\| = \|Q^2x\|$.

(iii) $\|Q\| = 1$.

(iv) *$Q(H) = \overline{Q(H)}$, where $Q(H) = \{Qx\colon x \in H\}$ and $\overline{Q(H)}$ denotes a closure of $Q(H)$.*

Proof. (i) and (ii) are obvious.

From (i) and (ii) we have

$$\|Q\| = \sup\{\|Qx\|\colon \|x\| \leqslant 1\} = \sup\{\|-Q^2x\|\colon \|x\| \leqslant 1\} = 1,$$

which completes the proof of (iii).

It is enough to show that if $x \in \overline{Q(H)}$, then $x \in Q(H)$. In fact, let $x \in \overline{Q(H)}$. Then there exists a sequence $z_n \in H$ such that $Qz_n \to x$. Since $-Q^2(Qz_n) = -Q^3z_n = Qz_n$, so $x = -Q^2x \in Q(H)$.

THEOREM 2.2. *Let $Q_1, Q_2 \in I(H)$ and $Q_1Q_2 = Q_2Q_1$; then $Q_1+Q_2 \in I(H)$ if and only if $Q_1Q_2$ is a projection.*

Proof. *Necessity.* Let $Q_1, Q_2 \in I(H)$, $Q_1Q_2 = Q_2Q_1$, $Q_1+Q_2 \in I(H)$. We have $(Q_1+Q_2)^3 = -(Q_1+Q_2)$; hence

$$Q_1Q_2^2+Q_1^2Q_2 = 0. \tag{a}$$

Multiplying both sides of equation (a) by $Q_2$, we obtain

$$Q_1^2Q_2^2 = Q_1Q_2.$$

Thus

$$(Q_1^1Q_2)^2 = Q_1^2Q_2^2 = Q_1Q_2, \quad (Q_1Q_2)^* = Q_2Q_1 = Q_1Q_2,$$

i.e. $Q_1Q_2$ is a projection.



*Sufficiency.* By using our assumptions and a definition of the class $I(H)$, it is easy to see that

$$(Q_1+Q_2)^3 = -(Q_1+Q_2).$$

As a simple conclusion of Theorem 2.2 we get

THEOREM 3.2. *Let $Q_1, Q_2 \in I(H)$ and $Q_1Q_2 = Q_2Q_1$; then $Q_2-Q_1 \in I(H)$ if and only if $Q_1^2Q_2 = Q_1Q_2^2$.*

COROLLARY 1.2. *If $Q_1, Q_2 \in I(H)$ and $Q_1Q_2 = Q_1^2$, then $Q_2-Q_1 \in I(H)$*

The proof results from Theorem 3.2.

DEFINITION 1.2. An operation $Q_1 \in I(H)$ is said to *precede* an operation $Q_2 \in I(H)$ when $Q_1Q_2 = Q_1^2$, and we shall write

$$Q_1 \leqslant Q_2 \overset{\text{def}}{\equiv} Q_1Q_2 = Q_1^2.$$

THEOREM 4.2. *The class $I(H)$ with the relation $\leqslant$ is a partly ordered set.*

Proof. It is easy to see that $Q \leqslant Q$ for each $Q \in I(H)$, and $Q_1 \leqslant Q_2$ and $Q_2 \leqslant Q_3$ implies $Q_1 \leqslant Q_3$.

We shall show that $Q_1 \leqslant Q_2$ and $Q_2 \leqslant Q_1$ implies $Q_1 = Q_2$.

If $Q_1 \leqslant Q_2$ and $Q_2 \leqslant Q_1$, then $Q_1^2 = Q_2^2$ and $(Q_2-Q_1)^2 = Q_1^2-Q_2^2 = 0$, and so we have

$$\begin{aligned} 0 &= \big((Q_2^2-Q_1^2)x, x\big) = \big((Q_2-Q_1)^2x, x\big) \\ &= \big((Q_2-Q_1)x, -(Q_2-Q_1)x\big) = -\|(Q_2-Q_1)x\|^2 \end{aligned}$$

for each $x \in H$, which means that $Q_2x = Q_1x$.

THEOREM 5.2. *$I(H)$ is a closed subset of $L(H)$ in the sense of strong convergency, i.e. if $Q_nx \to Qx$ for each $x \in H$ and $Q_n \in I(H)$, then $Q \in I(H)$.*

Proof. In fact, for any sequence $\{Q_n\}$ of operations such that $Q_n \in I(H)$ and $Q_nx \to Qx$ we have

$$\|Q^3x+Qx\| \leqslant \|Q^3x-Q_n^2Qx\|+\|Q_n^2Qx-Q_n^3x\|+\|Q_n^3x+Qx\| \to 0 \quad \text{if} \quad n \to \infty.$$

Evidently, $Q^* = -Q$.

LEMMA 1.2. *For $Q_1, Q_2 \in I(H)$ and $Q_1 \leqslant Q_2$ we have $\|Q_1x\| \leqslant \|Q_2x\|$ for each $x \in H$.*

Proof. From Theorem 1.2 we have

$$\|Q_1x\| = \|Q_1^2x\| = \|Q_1Q_2x\| \leqslant \|Q_1\|\,\|Q_2x\| = \|Q_2x\|.$$

THEOREM 6.2. *If $Q_1 \leqslant Q_2 \leqslant \ldots \leqslant Q_n \leqslant \ldots$ and $Q_n \in I(H)$ for $n = 1, 2, \ldots$, then there exists $\lim\limits_{n\to\infty} Q_nx = Qx$ for each $x \in H$ and $Q \in I(H)$.*

*Moreover, if $\{Q_n\}$ is an arbitrary sequence of operations from $I(H)$, where $\lim_{n\to\infty} Q_n x = Qx$ exists for any $x \in H$ and $Q_n \leqslant Q'$ for all $n$, then $Q \leqslant Q'$.*

Proof. Let $x \in H$; from Lemma 1.2 and Theorem 1.2 (iii) we have

$$\|Q_n x\| \leqslant \|Q_{n+1} x\| \leqslant \|x\|,$$

and so a sequence $\{\|Q_n x\|^2\}$ is convergent for an arbitrary $x \in H$. Hence we obtain

$$\|Q_{n+k}x - Q_n x\|^2 = \|Q_{n+k}x\|^2 - \|Q_n x\|^2 \to 0 \quad \text{if} \quad n, k \to \infty.$$

Then we have

$$\lim_{n\to\infty} Q_n x = Qx \quad \text{and} \quad Q \in I(H).$$

Likewise, from the Lemma 1.2 if $Q_n \leqslant Q'$, then

$$\lim_{n\to\infty} Q_n \leqslant Q'.$$

**3. Antisymmetric spectral family.**

Definition 1.3. Let us establish an abstract function $\boldsymbol{R} \ni \lambda \to Q_\lambda \in I(H)$ (where $\boldsymbol{R}$ denotes the set of all real numbers) satisfying the following conditions:

(i) If $\lambda_1 \leqslant \lambda_2$ then $Q_{\lambda_1} \leqslant Q_{\lambda_2}$;

(ii) for each $x \in H$ and $\lambda \in \boldsymbol{R}$ we have

$$\lim_{0<\varepsilon\to 0} Q_{\lambda-\varepsilon} x = Q_{\lambda-0} x \quad \text{and} \quad Q_{\lambda-0} \leqslant Q_\lambda.$$

The function thus defined we shall call the *antisymmetric spectral family*.

Theorem 1.3. *If $\{Q_\lambda\}_{\lambda\in\boldsymbol{R}}$ is an antisymmetric spectral family, then we have*

$$\lim_{\lambda\to-\infty} Q_\lambda x = Q_{-\infty} x \quad \text{and} \quad \lim_{\lambda\to+\infty} Q_\lambda x = Q_{+\infty} x.$$

Proof. It is clear that a function $\varphi_x(\lambda) = \|Q_\lambda x\|^2$ is non-decreasing and bounded, and so we have

$$\lim_{\lambda\to+\infty} \|Q_\lambda x\|^2 \quad \text{for an arbitrary } x \in H.$$

Hence we have

$$\|Q_{\lambda_2} x - Q_{\lambda_1} x\|^2 = \|Q_{\lambda_2} x\|^2 - \|Q_{\lambda_1} x\|^2 \to 0 \quad \text{if} \quad \lambda_1, \lambda_2 \to \infty$$

for an arbitrary $\lambda_1 < \lambda_2$.

Similarly, for $\lambda \to -\infty$.



**4. Antisymmetric spectral integral.** Let $\{Q_\lambda\}_{\lambda\in\boldsymbol{R}}$ be an antisymmetric spectral family and let $\{f(\lambda)\}_{\lambda\in\langle\alpha,\beta\rangle}$ be a real continuous function defined on an interval $\langle\alpha, \beta\rangle$.

Let $\pi$: $\alpha = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \beta$ be a division of the $\langle\alpha, \beta\rangle$; denote by $s(\pi)x$ the sum

$$s(\pi)x = \sum_{i=1}^{n} f(\lambda_{i_1})(Q_{\lambda_i} - Q_{\lambda_{i-1}})x$$

for arbitrary $x \in H$ and write

$$\Delta(\pi) = \max\{|\lambda_i - \lambda_{i-1}|:\ i = 1, 2, \ldots, n\}.$$

Theorem 1.4. *For an arbitrary antisymmetric spectral family $\{Q_\lambda\}_{\lambda\in\boldsymbol{R}}$ and for an arbitrary real continuous function $\{f(\lambda)\}_{\lambda\in\boldsymbol{R}}$ and for an arbitrary $x \in H$ there exists an integral*

$$\int_\alpha^\beta f(\lambda)\, dQ_\lambda x$$

*understood as the bound of a sequence of sums $s(\pi_n)x$, where $\{\pi_n\}$ is a sequence of divisions such that $\Delta(\pi_n) \to 0$ and denotes a bounded and antisymmetric linear operation.*

Outline of the proof. From Theorem 4.4. V [1], to show the existence of $\int_\alpha^\beta f(\lambda)\, dQ_\lambda x$ it is sufficient to prove that a variation of a function $x_\lambda \overset{\text{def}}{=} Q_\lambda x$ is bounded.

In fact,

$$\operatorname{var}_\alpha^\beta(x_\lambda) = \sup\left\{\left\|\sum_{i=1}^{n} \varepsilon_i (Q_{\lambda_i} - Q_{\lambda_{i-1}})x\right\|:\ \pi:\ \alpha = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \beta\right\},$$

where $|\varepsilon_i| = 1$,

$$\left\|\sum_{i=1}^{n} \varepsilon_i (Q_{\lambda_i} - Q_{\lambda_{i-1}})x\right\|^2 = \sum_{i=1}^{n} \|(Q_{\lambda_i} - Q_{\lambda_{i-1}})x\|^2$$
$$= \left\|\sum_{i=1}^{n} (Q_{\lambda_i} - Q_{\lambda_{i-1}})x\right\|^2 = \|(Q_\beta - Q_\alpha)x\|^2 \leqslant \|x\|^2,$$

where the last estimation follows from Theorem 1.2 (iii). Hence

$$\operatorname{var}_\alpha^\beta(x_\lambda) \leqslant \|x\|$$

and $\int_\alpha^\beta f(\lambda)\, dQ_\lambda x$ is an antisymmetric linear operation as a bound of a sequence of antisymmetric operations.

Let

$$Ax = \int_\alpha^\beta f(\lambda)\,dQ_\lambda x;$$

then

$$\|Ax\|^2 = \|\lim_{\Delta(\pi)\to 0} \sum_{i=1}^n f(\lambda_{i-1})(Q_{\lambda_i} - Q_{\lambda_{i-1}})x\|^2$$

$$= \lim_{\Delta(\pi)\to 0} \sum_{i=1}^n f^2(\lambda_{i_1})\|(Q_{\lambda_i} - Q_{\lambda_{i-1}})x\|^2 \leqslant M^2\|x\|^2,$$

where $M = \max\{|f(\lambda)|: \lambda \in \langle\alpha, \beta\rangle\}$ so $A$ is a bounded operation.

Now we shall see that for an arbitrary bounded antisymmetric and linear operation there exists an antisymmetric spectral family.

**5. Representation of antisymmetric operations.** We shall prove the following

THEOREM 1.5. *For each antisymmetric bounded operation $A$ there exist two bounded operations $V$, $B$ such that*

(i) $V \in I(H)$,

(ii) $B = B^*$,

(iii) $A = VB = BV$, $AB = BA$.

Proof. Let us consider two cases: there exists an $A^{-1}$.

From Theorem 2.3 IV [2] for $-A^2$ there exists a symmetric and non-decreasing operation $B$ such that $-A^2 = B^2$ and there also exists a $B^{-1}$.

If $B^{-1}$ is bounded, then we put

$$V \stackrel{\text{def}}{=} AB^{-1}.$$

It is obvious that $A = VB$ and that $V \in I(H)$ (since $V^2 = -I$).

If $B^{-1}$ is not bounded, we see that

$$\overline{R(B)} = \overline{\{Bx: x \in H\}} = H;$$

this follows directly from a decomposition $H = N(A) \oplus \overline{R(A^*)}$, where $N(A) = \{x \in H: Ax = 0\}$.

We put

$$V'x = AB^{-1}x \quad \text{for} \quad x \in R(B)$$

($B^{-1}$ is defined on a dense domain). We have

$$\|V'x\|^2 = (V'x, V'x) = (AB^{-1}x, AB^{-1}x) = \|x\|^2,$$

and so $V'$ is an isometry.

Let us extend by continuity the operation $V'$ on the space $H$ and let us denote it by $V$. Evidently, $V \in I(H)$ and $A = VB$.



Now we shall prove that $V'$ is an antisymmetric operation. Let $x, y \in R(B)$; then we have

$$(V'x, y) + (x, V'y) = (AB^{-1}x, y) + (x, AB^{-1}y)$$

$$= (AB^{-1}Bu, y) + (Bu, AB^{-1}y) = (Au, y) + (Bu, AB^{-1}y)$$

$$= (Au, y) + (u, Ay) = (u, -Ay) + (u, Ay) = 0,$$

where $x = Bu$. Since $V'$ is an antisymmetric operation, it is also true for $V$. It is easy to see that $V \in I(H)$, because $V^2 = -I$.

It remains to show that $A = VB$. Let $x \in H$, $x_n \to x$ and $x_n \in R(B)$. Then we have

$$\|Ax - VBx\| \leqslant \|Ax - Ax_n\| + \|Ax_n - VBx_n\| + \|VBx_n - VBx\| \to 0$$

if $n \to \infty$.

Proof of Theorem 1.5. In view of Lemma 1.5 it is enough to consider the case where $A$ is singular. Since $A = -A^*$, we have $H = N(A) \oplus \overline{R(A)}$. Obviously, $A(\overline{R(A)}) \subset \overline{R(A)}$; let us put

$$Cx = Ax \quad \text{for} \quad x \in \overline{R(A)}.$$

We note that the operation $C$ satisfies the assumptions of Lemma 1.5. There exist $V'$, $B'$ such that $Cx = V'B'x$ for $x \in \overline{R(A)}$ and $V' \in I(\overline{R(A)})$. We put

$$Vx = V'x_2 \quad \text{where} \quad x = x_1 + x_2;\ x_1 \in N(A),\ x_2 \in \overline{R(A)}$$

and

$$Bx = B'x_2.$$

It is easy to see that $V \in I(H)$, $B = B^*$ and $A = VB$.

**6. Existence of an antisymmetric spectral family.** Now we shall prove

THEOREM 1.6. *For each antisymmetric linear and bounded operation $A$ there exists an antisymmetric spectral family $\{Q_\lambda\}_{\lambda\in\mathbf{R}}$ such that*

$$Ax = \int_\alpha^\beta \lambda\,dQ_\lambda x.$$

Proof. By Theorem 1.5, for an arbitrary operation A satisfying the assumptions of Theorem 1.6 there exist operations $V$ and $B$ such that $A = VB$.

By Theorem 9.4 [1], for an operation $B$ we have a representation

$$B = \int_\alpha^\beta \lambda\,dE_\lambda,$$

where $E_\lambda$ is a spectral family of $B$. Putting

$$Q_\lambda \stackrel{\text{def}}{=} VE_\lambda,$$

we see that

$$A = \int_\alpha^\beta \lambda dQ_\lambda.$$

**7. Equation** $\frac{d}{dt}x(t) = Ax(t)$. One of the applications of an antisymmetric spectral family is the equation (1′):

$$\text{(1′)} \qquad \frac{d}{dt}x(t) = Ax(t),$$

with the initial condition

$$x(0) = x_0 \in H,$$

where the operation $A$ is antisymmetric and bounded.

As demonstrated in [3], for an arbitrary and bounded linear operation $A$ there exists a semigroup

$$I_t x = \exp(tA)x$$

such that $x(t) = I_t x_0$ for any $x_0 \in H$ and $x(t)$ satisfies the equation (1′) with the initial condition $x(0) = x_0$.

We know that if $A$ is an antisymmetric operation, then $I_t$ is an isometry for any $t$.

Preserving the notations of Section 6, we shall prove

THEOREM 1.7. *If $A$ is an antisymmetric and bounded linear operation, then an analytic semigroup $I_t$ for which the operation $A$ is an infinitesimal generator is expressed by the formula*

$$\text{(2′)} \qquad I_t x = \Big[I + V^2 - V^2 \int_\alpha^\beta \cos\lambda t\, dE_\lambda + V\int_\alpha^\beta \sin\lambda t\, dE_\lambda\Big]x.$$

*It determines the solution $x(t) = I_t x_0$ of equation* (1) *satisfying the initial condition $x(0) = x_0 \in H$, where $A = VB$ and $B = \int_\alpha^\beta \lambda dE_\lambda$.*

Proof. We shall prove that the solution of equation (1′) is given by the integral

$$\text{(3′)} \qquad I_t x = \int_\alpha^\beta \exp(t\lambda V)dE_\lambda x_0$$

which is the limit of sums

$$\text{(4′)} \qquad S_t(\pi)x = \sum_{i=1}^n \exp(t\lambda_{i-1}V)(E_{\lambda_i} - E_{\lambda_{i-1}})x$$



of divisions $\pi$: $\alpha = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \beta$ of the interval $\langle\alpha, \beta\rangle$, where $\Delta(\pi) \to 0$ with respect to a norm.

Let us note that

$$\begin{aligned}\exp(t\lambda V) &= \sum_{n=0}^\infty \frac{1}{n!}(t\lambda V)^n \\ &= I + t\lambda V + \frac{1}{2!}t^2\lambda^2V^2 - \frac{1}{3!}t^3\lambda^3V - \frac{1}{4!}t^4\lambda^4V^2 + \ldots \\ &= I - \Big(-1 + 1 - \frac{1}{2!}\lambda^2t^2 + \frac{1}{4!}\lambda^4t^4 - \ldots\Big)V^2 + \Big(\lambda t - \frac{1}{3!}\lambda^3t^3 + \ldots\Big)V \\ &= I - (-1 + \cos\lambda t)V^2 + (\sin\lambda t)V.\end{aligned}$$

Hence we obtain

$$\text{(5′)} \qquad \exp(t\lambda V) = I + V^2 - \cos\lambda t\, V^2 + \sin\lambda t\, V.$$

Putting (5′) into (4′), we get

$$\text{(6′)} \qquad \begin{aligned} S_t(\pi)x = x + V^2x - V^2 \sum_{i=1}^n (\cos\lambda_{i-1}t)(E_{\lambda_i} - E_{\lambda_{i-1}})x + \\ + V\sum_{i=1}^n (\sin\lambda_{i-1}t)(E_{\lambda_i} - E_{\lambda_{i-1}})x.\end{aligned}$$

Operations

$$\text{(7′)} \qquad \begin{aligned} B_t(\pi)x &= \sum_{i=1}^n (\cos\lambda_{i-1}t)(E_{\lambda_i} - E_{\lambda_{i-1}})x, \\ C_t(\pi)x &= \sum_{i=1}^n (\sin\lambda_{i-1}t)(E_{\lambda_i} - E_{\lambda_{i-1}})x \end{aligned}$$

are partial sums of integrals

$$\text{(8′)} \qquad B_t x = \int_\alpha^\beta \sin\lambda t\, dE_\lambda x, \qquad C_t x = \int_\alpha^\beta \cos\lambda t\, dE_\lambda x;$$

their existence follows from 9.7 IX of [1].

From (6′) we obtain

$$\text{(9′)} \qquad \lim_{\Delta(\pi)\to 0} S_t(\pi)x = x + V^2x - V^2\int_\alpha^\beta \cos\lambda t\, dE_\lambda x + V\int_\alpha^\beta \sin\lambda t\, dE_\lambda x;$$

thus integral (3′) exists and

(10′)

$$I_t x = \int_\alpha^\beta \exp(t\lambda V)dE_\lambda x = \Big[I + V^2 - V^2\int_\alpha^\beta \cos\lambda t\, dE_\lambda + V\int_\alpha^\beta \sin\lambda t\, dE_\lambda\Big]x.$$

Moreover, the operation $I_t$ is an isometry for any $t$; it is easy to see that $\|S_t(\pi)x\|^2 = \|x\|^2$. Equation (10′) with respect to Theorem 9.7 IX [1] takes the form

$$I_t x = [I + V^2 - V^2 \cos tB + V \sin tB]x, \tag{11'}$$

where $A = VB$.

If $H$ is a finitely-dimensional space with even dimension, then (11′) takes the form

$$I_t x = \cos tBx + V \sin tBx. \tag{12'}$$

Now, we shall see that the function $x(t) = I_t x_0$ is the solution of equation (1′).

Let us consider a differential quotient

$$\frac{1}{\varepsilon}[x(t+\varepsilon) - x(t)] = \frac{1}{\varepsilon}[I_{t+\varepsilon}x_0 - I_t x_0] = I_t \frac{1}{\varepsilon}[I_\varepsilon x_0 - x_0].$$

It is enough to show that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[I_\varepsilon x_0 - x_0] = Ax_0 \quad \text{for any } x_0 \in H.$$

From (10) we have

$$\frac{1}{\varepsilon}[I_\varepsilon x_0 - x_0] = V^2 \int_\alpha^\beta \frac{1}{\varepsilon}[1 - \cos\lambda t]\, dE_\lambda x_0 + V \int_\alpha^\beta \frac{1}{\varepsilon} \sin\lambda\varepsilon\, dE_\lambda x_0 \tag{13'}$$

and at the same time uniform convergence of functions takes place:

$$\varphi_\varepsilon(\lambda) = \frac{1}{\varepsilon}[1 - \cos\lambda t] \rightrightarrows 0 \quad \text{and} \quad \psi_\varepsilon(\lambda) = \frac{1}{\varepsilon}\sin\lambda\varepsilon \rightrightarrows \lambda.$$

From 9.2 IX [1] and (13′) we obtain

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[I_\varepsilon x_0 - x_0] = V^2 \int_\alpha^\beta 0\, dE_\lambda x_0 + V \int_\alpha^\beta \lambda\, dE_\lambda x_0,$$

which gives

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[I_\varepsilon x_0 - x_0] = Ax_0 \quad \text{for any } x_0 \in H.$$

Thus from the theorem about the equivalence of solutions of equation (1′), $x(t) = I_t x_0$ is a solution of equation (1′) with the initial condition $x(0) = x_0$.

Let us consider the case where the operation $-A^2$ has a discrete spectrum; then there exist an orthonormal and complete basis $(e_i)_{i=1,2,\ldots}$



of its own vectors and a sequence $\{\lambda_i\}_{i=1,2,\ldots}$ of its eigenvalues such that

$$-A^2 x = \sum_{i=1}^{\infty} \lambda_i (x, e_i) e_i.$$

We put

$$\lambda_i x \stackrel{\text{def}}{=} (x, e_i) e_i;$$

then formula (10′) has the following form:

$$I_t x = \Big[I + V^2 - V^2 \sum_{i=1}^{\infty} \cos t\sqrt{\lambda_i} P_i + V \sum_{i=1}^{\infty} \sin t\sqrt{\lambda_i} P_i\Big] x.$$

These formulas are also true for an inifinetely-dimensional space.

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