This is true because for every $\beta \in (0, \pi)$ there exists some $b > 0$ depending on β so that there is an exterior cone of size $K(\beta, b)$ at every point on ∂D.

If ∂D is C^2, then at every point x on ∂D there is a ball of a fixed size exterior to $\bar{D}$ and tangent to ∂D at x. If v is a positive harmonic function outside a ball and vanishes on the sphere, then the value of v at a point near the sphere is proportional to the distance from that point to the ball. Using this fact instead of Lemma 1 we may replace ϱ by 1 in Theorems 1, 2, and 3. Thus condition (2.1) is weakened but the corresponding exceptional set is enlarged.

Suppose that D is a Liapunov or a Liapunov–Dini region [10]; by an estimate of a harmonic function obtained in [10], we may also replace ϱ by 1 in the above theorems.

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On extending and lifting continuous linear mappings in topological vector spaces

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Abstract. (1) Let $0 < p < 1$. Then there is no non-zero topological vector space which has the extension property for the class of all p-Banach spaces with separating continuous duals.

(2) If $\mathscr{K}$ is the class of all Fréchet spaces (or of all separable Fréchet spaces, or of all nuclear Fréchet spaces, or of all metric vector spaces) and a space P ($P \in \mathscr{K}$) has the lifting property for $\mathscr{K}$, then P is finite-dimensional.

Let $\mathscr{K}$ be any class of topological vector spaces(1) (briefly TVS's), and let E be any TVS. The space E is said to have the *extension property for* $\mathscr{K}$ if for every $X \in \mathscr{K}$ and for every subspace $Y \subset X$, each mapping (= linear continuous mapping) $f\colon Y \to E$ has an extension to a mapping $g\colon X \to E$. Dually, E is said to have the *lifting property for* $\mathscr{K}$ if for every $X \in \mathscr{K}$ and for every closed subspace $N \subset X$, each mapping $f\colon E \to X/N$ has a lifting to a mapping $g\colon E \to X$ (i.e. $f = p \circ g$, where p is the quotient mapping from X onto X/N). If $E \in \mathscr{K}$ and E has the extension property for $\mathscr{K}$ [E has the lifting property for $\mathscr{K}$], then E is called an *injective* [*projective*] *space in* $\mathscr{K}$.

Let $\mathscr{K}$ be the class of all Banach spaces. Then (a) E is an injective space in $\mathscr{K}$ iff E is a P_λ-space for some $\lambda \geqslant 1$; (b) E is a projective space in $\mathscr{K}$ iff E is isomorphic to $l_1(\Gamma)$ for a certain set Γ ([2], [10], [11], [13]). Any product [countable product] of injective Banach spaces is an injective space in the class of all locally convex spaces [of all Fréchet spaces] (see [11]). From an argument of G. Köthe ([10], p. 182; see also S. Rolewicz [12], p. 65) it follows that for each $p \in (0, 1]$ the spaces $l_p(\Gamma)$ are projective in the class of all p-Banach spaces. The author proved in [3] that in the class of all locally convex spaces a space E is projective iff E is a direct sum of one-dimensional spaces. This result is also true for the class of all complete locally convex spaces [4]. Using the method of [3], one can show that in the class of all TVS's a space E is projective iff the topology of E is the finest vector topology for the vector space E.

(1) we include the Hausdorff condition in the definition of TVS.

We show in this paper that if a class $\mathscr{K}$ of TVS's contains all p-Banach spaces with separating continuous duals for some p $(0<p<1)$, then there is no non-zero injective space in $\mathscr{K}$ (Theorem 1.5). The proof of this theorem is a slight modification of N. J. Kalton's idea ([7], proof of Theorem 6.7). In § 2 we show that every projective space in the class of all Fréchet spaces (or all separable Fréchet spaces, or all nuclear Fréchet spaces, or all F-spaces([2])) is finite-dimensional. This answers a question of L. Nachbin ([11], Problem (2)).

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§ 1. Spaces with the extension property. Let $\mathscr{K}$ be a class of TVS's, and suppose that E is a non-zero TVS which has the extension property for $\mathscr{K}$. For every topological vector space (X,τ), let $\varrho(X)$ be the coarsest vector topology on X which preserves the continuity of all mappings from (X,τ) to E. It is clear that for each base of neighbourhoods of 0, U in E all sets of the form $\bigcap_{i=1}^{n} f_i^{-1}(V)$ (where $V\in U$ and f_i are mappings from (X,τ) to E) form the base of $\varrho(X)$-neighbourhoods of 0.

LEMMA 1.1. *Suppose that $X\in\mathscr{K}$ and let Y be a subspace of X. Then $\varrho(X)$ induces the topology $\varrho(Y)$ on Y.*

Proof. Obvious.

LEMMA 1.2. *Let $(X,\tau)\in\mathscr{K}$. Then τ and $\varrho(X)$ determine the same classes of closed vector subspaces.*

Proof. It is enough to show that each τ-closed subspace $X_0\subset X$ is $\varrho(X)$-closed. Suppose $a\in X\setminus X_0$ and let Y be the linear span of $X_0\cup\{a\}$. Then X_0 is a τ-closed hypersubspace of Y; it follows that there exists a τ-continuous one-dimensional mapping $f\colon Y\to E$ such that $f(a)\neq 0$, $X_0=f^{-1}(0)$. Let g be an extension of f on X. Then $X_0\subset g^{-1}(0)$ and $g(a)\neq 0$. Since g is $\varrho(X)$-continuous, a does not belong to the $\varrho(X)$-closure of X_0. Hence X_0 is $\varrho(X)$-closed.

COROLLARY 1.3. *Let (X,τ) be an F-space belonging to $\mathscr{K}$. Then τ has a base of $\varrho(X)$-closed neighbourhoods.*

Proof. The result follows from [6], Corollary 5.4.

LEMMA 1.4. *Suppose that (H,τ) is a Hilbert space with an orthonormal basis $(e_a)_{a\in A}$ such that $\mathrm{Card}(A)>\mathrm{Card}(E)$. If $A'\subset A$ and $\mathrm{Card}(A')=\mathrm{Card}(A)$, then 0 belongs to the $\varrho(H)$-closure of the set $\{e_a\colon a\in A'\}$.*

([2]) An F-space is a metrizable complete TVS; a *Fréchet space* is a locally convex F-space.

Proof. It is enough to show that, for every mapping $f\colon H\to E$, we have $\mathrm{Card}(\{a\in A'\colon f(e_a)\neq 0\})<\mathrm{Card}(A')$. We have the factorization $f=p\circ f'$ where p is an orthogonal projection from H on some closed subspace $G\subset H$ and f' is an injection from G into E. Let $(g_\beta)_{\beta\in B}$ be an orthonormal basis in G. For each $\beta\in B$ we define $A_\beta=\{a\in A'\colon (e_a|g_\beta)\neq 0\}$. All the sets A_β are countable, and since f' is an injection, we have $\mathrm{Card}(B)\leqslant\mathrm{Card}(E)<\mathrm{Card}(A')$. Hence $\mathrm{Card}(\bigcup_{\beta\in B}A_\beta)<\mathrm{Card}(A')$ and the result follows from the equality

$$A'\setminus\bigcup_{\beta\in B}A_\beta=\{a\in A'\colon f(e_a)=0\}.$$

THEOREM 1.5. *Let $\mathscr{K}$ be any class of TVS's which includes all p-Banach spaces with separating continuous duals for certain p $(0<p<1)$. Then there is no non-zero topological vector space E having the extension property for $\mathscr{K}$.*

Proof. Suppose $E\neq\{0\}$ has the extension property for $\mathscr{K}$. Let (H,τ) be any Hilbert space of Lemma 1.4. It is clear that $(H,\tau)\in\mathscr{K}$. Let $A'=A\setminus\{\gamma\}$, where γ is a fixed element of A. Denote by C the τ-closure of the absolutely p-convex hull of the set $\{e_a+e_\gamma\colon a\in A'\}$. Let $|\cdot|$ be the Hilbert norm on H and denote by V the unit ball in $(H,|\cdot|)$. For each $n\geqslant 1$, let $\|\cdot\|_n$ be the p-norm on H whose unit ball is the τ-closure of $C+n^{-1}V$. Since C is τ-bounded, the p-norms $\|\cdot\|_n$ define the topology τ on H. Let $(X,\|\cdot\|)$ be the p-Banach space of all sequences $(x_n)_{n\geqslant 1}$, where $x_n\in H$, such that $\|(x_n)\|=\sum_n\|x_n\|_n<+\infty$; it is clear that the dual space X' is point-separating. By Corollary 1.3, there exists a $\delta>0$ such that the $\varrho(X)$-closure of the unit ball $\{x\in X\colon \|x\|\leqslant 1\}$ is contained in $\{x\in X\colon \|\delta x\|\leqslant 1\}$. Let $X_k=\{(x_n)\in X\colon x_n=0,\ n\neq k\}$; then X_k is isomorphic to $(H,\|\cdot\|_k)$. By Lemma 1.4, the vector e_γ is in the $\varrho(H)$-closure of C. Hence, by Lemma 1.1, we have $\|\delta a_m\|\leqslant 1$ for the elements $a_m=(\delta_{mn}e_\gamma)_{n\geqslant 1}$. Thus $\|\delta e_\gamma\|_m\leqslant 1$, and hence $\delta e_\gamma\in C+\frac{2}{m}V$ for every $m\geqslant 1$. Since C is τ-closed, we have $\delta e_\gamma\in C$. Thus, for every $\varepsilon>0$, there exist indices $a_1,\ldots,a_n\in A$ and scalars $t_1,\ldots,t_n$ such that

$$\sum_{i=1}^{n}|t_i|^p\leqslant 1 \quad\text{and}\quad \Big|\delta e_\gamma-\sum_{i=1}^{n}t_i(e_{a_i}+e_\gamma)\Big|\leqslant\varepsilon.$$

Taking the inner products of $(\delta e_\gamma-\sum t_i(e_{a_i}+e_\gamma))$ and $e_\gamma, e_{a_1},\ldots,e_{a_n}$ successively, we obtain

$$\Big|\delta-\sum_i t_i\Big|\leqslant\varepsilon \quad\text{and}\quad |t_i|\leqslant\varepsilon \quad (i=1,\ldots,n).$$

Now

$$\Big|\sum_i t_i\Big| \leqslant \sum_i |t_i| \leqslant \max_i |t_i|^{1-p} \sum_i |t_i|^p \leqslant \varepsilon^{1-p};$$

thus $\delta \leqslant \varepsilon+\varepsilon^{1-p}$ and hence $\delta \leqslant 0$, which contradicts $\delta > 0$. This completes the proof.

§ 2. Projective spaces in certain classes of F-spaces. Let X be a TVS; by the *density character* of X (in symbols $\mathrm{Dens}(X)$) we mean the minimal cardinal number $\mathfrak{m}$ such that X has a dense subspace $Y \subset X$ with $\dim(Y) = \mathfrak{m}$ ($\dim(Y)$ denotes the algebraic dimension of Y).

THEOREM 2.1. *Suppose that* $\mathfrak{m}$ *is a cardinal number and let* $\mathscr{K}_i$ ($i = 1, \ldots, 6$) *be the class*: (1) *of all Fréchet spaces*; (2) *of all Fréchet spaces* X *with* $\mathrm{Dens}(X) \leqslant \mathfrak{m}$; (3) *of all Fréchet–Montel spaces*; (4) *of all nuclear Fréchet spaces*; (5) *of all F-spaces*; (6) *of all F-spaces* X *such that* $\mathrm{Dens}(X) \leqslant \mathfrak{m}$, *respectively. Then for each* $i = 1, \ldots, 6$ *every projective space* $P \in \mathscr{K}_i$ *is finite-dimensional.*

Remark. Let $\mathscr{K}$ be any class of TVS's, let $\hat{\mathscr{K}}$ be the class of all spaces which are the completions of spaces of $\mathscr{K}$; supose that $\hat{\mathscr{K}} \subset \mathscr{K}$. It is easy to show that if P is a projective space in $\mathscr{K}$, then the completion of P is a projective space in $\hat{\mathscr{K}}$. Hence, for instance, every projective space in the class of all metrizable TVS's is finite-dimensional.

(I) We first describe the general construction to be applied in the proof of Theorem 2.1.

Let $A = (a_j^{(n)})_{j,n\geqslant 1}$ be any infinite numerical matrix such that (a) $\forall j, n$: $0 \leqslant a_j^{(n)} \leqslant a_j^{(n+1)}$; (b) $\forall j \exists n$: $a_j^{(n)} > 0$. We will denote by $L(A)$ the Köthe space of all sequences $x = (\xi_j)_{j\geqslant 1}$ such that $\forall n$

$$p_n(x) = \sum_j a_j^{(n)} |\xi_j| < +\infty.$$

The space $L(A)$ equipped with the sequence of the seminorms (p_n) is a Fréchet space ([9], p. 422). We will denote by e_n the sequence $(\delta_{mn})_{m\geqslant 1}$; it follows that $x = \sum_j \xi_j e_j$ in $L(A)$. In the sequel we will use the following results (we assume $0/0 = 0$):

(M) $L(A)$ *is Montel iff for every infinite subset* $K \subset N$ *and every* $n \geqslant 1$ *there exists an* $m \geqslant n$ *such that*

$$\inf\{a_j^{(n)}/a_j^{(m)}\colon j \in K\} = 0 \qquad \text{(G. Köthe [9], p. 424).}$$

(N) $L(A)$ *is nuclear iff for every* $n \geqslant 1$ *there exists an* $m \geqslant n$ *such that* $\sum_j a_j^{(n)}/a_j^{(m)} < +\infty$ (A. Grothendieck [5], Chap. II, Prop. 8).

Suppose we are given the matrix A and two matrices $(\beta_j^{(n)})_{j,n\geqslant 1}$ and $(\varkappa_j^{(n)})_{n\geqslant 1, i\geqslant n}$ such that

(i) $\forall j, n$: $0 < \beta_j^{(n)} \leqslant \beta_j^{(n+1)}$;

(ii) $\forall m, n$: $\lim\limits_{j\to\infty} a_j^{(m)}/\beta_j^{(n)} = 0$;

(iii) $\forall n \geqslant 1, i \geqslant n$: $1 \leqslant \varkappa_i^{(n)} \leqslant \varkappa_{i+1}^{(n)}$;

(iv) $\forall n \geqslant 1, i \geqslant n+1$: $\varkappa_i^{(n)} \leqslant \varkappa_i^{(n+1)}$.

We put for $i, j, n \geqslant 1$

$$\gamma_{ij}^{(n)} = \begin{cases} \beta_j^{(n)} & \text{if } i < n, \\ \varkappa_i^{(n)} a_j^{(i+n-1)} & \text{if } i \geqslant n. \end{cases}$$

It is clear that for every $n \geqslant 1$ we have $\gamma_{ij}^{(n)} \leqslant \gamma_{ij}^{(n+1)}$ for all but finitely many pairs (i, j). Let $L(\Gamma)$ (where $\Gamma = (\gamma_{ij}^{n})$) be the Fréchet–Köthe space of all double sequences $z = (\zeta_{ij})$ such that

$$q_n(z) = \sum_{i,j} \gamma_{ij}^{(n)} |\zeta_{ij}| < +\infty,$$

equipped with the sequence of seminorms (q_n). For each $z = (\zeta_{ij}) \in L(\Gamma)$ and for each $j \geqslant 1$, we have $\sum_{i\geqslant 1} |\zeta_{ij}| < +\infty$. In fact, let us consider an integer r such that $a_j^{(r)} > 0$; then

$$\sum_{i\geqslant r} |\zeta_{ij}| \leqslant [a_j^{(r)}]^{-1} \sum_{i\geqslant r} a_j^{(r)} \varkappa_i^{(1)} |\zeta_{ij}| \leqslant [a_j^{(r)}]^{-1} \sum_{i\geqslant 1} \gamma_{ij}^{(1)} |\zeta_{ij}| < +\infty.$$

LEMMA 2.2. *The correspondence* $\pi\colon (\zeta_{ij})_{i,j\geqslant 1} \mapsto \big(\sum_{i\geqslant 1} \zeta_{ij}\big)_{j\geqslant 1}$ *is a mapping from* $L(\Gamma)$ *onto* $L(A)$. *Hence* $L(A) \cong L(\Gamma)/\pi^{-1}(0)$.

Proof. Let $z = (\zeta_{ij}) \in L(\Gamma)$ and $n \geqslant 1$. By (ii) there exists a j_n such that if $j \geqslant j_n$ then $a_j^{(n)} \leqslant \beta_j^{(n)}$. Now

$$\begin{aligned} p_n(\pi(z)) &= \sum_{j\geqslant 1} a_j^{(n)} \Big|\sum_{i\geqslant 1} \zeta_{ij}\Big| \\ &\leqslant \sum_{j<j_n} \sum_{i<n} a_j^{(n)} |\zeta_{ij}| + \sum_{j\geqslant j_n} \sum_{i<n} a_j^{(n)} |\zeta_{ij}| + \sum_{j\geqslant 1} \sum_{i\geqslant n} a_j^{(n)} |\zeta_{ij}| \\ &\leqslant \sum_{j<j_n} \sum_{i<n} a_j^{(n)} |\zeta_{ij}| + \sum_{j\geqslant 1} \sum_{i<n} \beta_j^{(n)} |\zeta_{ij}| + \sum_{j\geqslant 1} \sum_{i\geqslant n} \varkappa_i^{(n)} a_j^{(i+n-1)} |\zeta_{ij}| \\ &= q_n(z) + \sum_{j<j_n} \sum_{i<n} a_j^{(n)} |\zeta_{ij}|. \end{aligned}$$

Hence $\pi(z) \in L(A)$ and $\pi\colon L(\Gamma) \to L(A)$ is continuous. It remains to show that the conjugate mapping $\pi'\colon L(A)' \to L(\Gamma)'$ is an injection and $\pi'(L(A)')$ is weakly closed in $L(\Gamma)'$ ([1], p. 106). It is known (see [9], p. 422) that $L(A)'$ is the space of all sequences (η_j) such that $|\eta_j| \leqslant c a_j^{(n)}$ for some $c \geqslant 0$ and some $a_j^{(n)}$ (analogously, $L(\Gamma)'$). It is clear that π' is injective and it is easy to show that $\pi'(L(A)')$ consists of all double sequences (η_{ij}) such that for each i, j we have $\eta_{ij} = \eta_{1j}$. Thus $\pi'(L(A)')$ is weakly closed.

COROLLARY 2.3. *Every separable Fréchet space is a quotient of a Montel–Köthe space.*

Proof. Let X be a separable Fréchet space, let (p_n) be an increasing sequence of seminorms which defines the topology in X, and let (x_j) be a dense sequence in $X\setminus\{0\}$. Put $a_j^{(n)} = p_n(x_j)$ and define $\sigma\colon L(A)\to X$ by

$$\sigma\Big(\sum_{j\geqslant 1}\xi_j e_j\Big) = \sum_{j\geqslant 1}\xi_j x_j.$$

Then σ is a mapping from $L(A)$ onto X. Let $\beta_j^{(n)} = j^n(a_j^{(n)}+1)$, $\varkappa_i^{(n)} = i^n$; then theorem (M) implies that $L(\Gamma)$ is a Montel space. By Lemma 2.2, X is isomorphic to a quotient of $L(\Gamma)$.

(II) Now let $L(A)$ be a Montel space; we may assume that the sets $\{x\in L(A)\colon p_n(x)\leqslant 1\}$ form a base of zero neighbourhoods. If $L(A)$ is nuclear, we shall assume also that $\sum_{j\geqslant 1} a_j^{(n)}/a_j^{(n+1)}\leqslant 1$ for each $n\geqslant 1$. Let P be an *infinite-dimensional* subspace of $L(A)$ such that the restriction of p_1 on P is a norm on P. We define by induction a sequence of elements $x_k = (\xi_j^{(k)})\in P$ $(k\geqslant 1)$ and a sequence of natural numbers t_k $(k\geqslant 0)$ such that

(i) $1 = t_0 < t_1 < \ldots < t_k < \ldots$;

(ii) $p_1(x_k) = 1$ $(k\geqslant 1)$;

(iii) x_k has the form $\sum_{j\geqslant t_{k-1}}\xi_j^{(k)}e_j$;

(iv) $p_k\big(\sum_{j\geqslant t_k}\xi_j^{(k)}e_j\big)\leqslant 1/2$.

We write $y_k = \sum_{t_{k-1}\leqslant j<t_k}\xi_j^{(k)}e_j$ (in general, $y_k\notin P$). For each $j\geqslant 1$, we denote by $k(j)$ the natural number k such that $t_{k-1}\leqslant j<t_k$.

Now we put

$$\sigma_j' = \begin{cases} j\left(a_j^{(j)}+1+\dfrac{p_j(x_{k(j)})}{|\xi_j^{(k(j))}|}\right) & \text{if}\quad \xi_j^{(k(j))}\neq 0,\\ j(a_j^{(j)}+1) & \text{if}\quad \xi_j^{(k(j))} = 0;\end{cases}$$

$$\sigma_j = \max\{\sigma_k'\colon k\leqslant j\}.$$

Let $\beta_j^{(n)} = \sigma_j^{2n}$, $\varkappa_i^{(n)} = i^{2n}$ $(j, n\geqslant 1,\ i\geqslant n)$. Then we have

(v) if $j\geqslant n\geqslant 1$, $m\geqslant 1$ then $\beta_j^{(m)}/a_j^{(n)}\geqslant j^{2m}$;

(vi) if $j\geqslant n\geqslant 1$, $m\geqslant 1$ and $\xi_j^{(k(j))}\neq 0$, then

$$|\xi_j^{(k(j))}|\beta_j^{(m)}\geqslant j^{2m}p_n(x_{k(j)}).$$

By the definition of $\beta_j^{(n)}$, $\varkappa_i^{(n)}$ and by (v), the conditions (i)–(iv) of (I) are satisfied. Let Γ be the matrix from (I); then it is easy to show that

$$\begin{aligned}\gamma_{ij}^{(n)}/\gamma_{ij}^{(n+1)} &\leqslant j^{-2} && \text{if}\quad i<n,\\ &= i^{-2}(a_j^{(i+n-1)}/a_j^{(i+n)}) && \text{if}\quad i>n,\\ &= n^n j^{-2(n+1)} && \text{if}\quad i=n,\ j\geqslant 2n-1.\end{aligned}$$

Thus, using theorems (M) and (N), we show that $L(\Gamma)$ is a Montel space, and if $L(A)$ is a nuclear space, then $L(\Gamma)$ is also nuclear. Let $\pi\colon L(\Gamma)\to L(A)$ be the mapping from (I).

LEMMA 2.4. *There is no mapping $\varrho\colon P\to L(\Gamma)$ such that $\pi(\varrho(x)) = x$ for each $x\in P$.*

Proof. Suppose that such a mapping exists. Then there exists an increasing sequence $(m_n)_{n\geqslant 1}$ of natural numbers such that for each $x\in P$ we have

$$(*)\qquad 5q_n(\varrho(x))\leqslant p_{m_n}(x).$$

Let $\varrho(x_k) = (\zeta_{ij}^{(k)})$; then $\sum_{i\geqslant 1}\zeta_{ij}^{(k)} = \xi_j^{(k)}$, and therefore

$$\sum_{i\geqslant 1}|\zeta_{ij}^{(k)}|\geqslant|\xi_j^{(k)}|.$$

Now we put

$$r(j) = \begin{cases}\max\Big\{r\in\mathbf{N}\colon \sum_{1\leqslant i\leqslant r}|\zeta_{ij}^{(k(j))}|\leqslant\tfrac12|\xi_j^{(k(j))}|\Big\} & \text{if}\quad \xi_j^{(k(j))}\neq 0,\\ +\infty & \text{if}\quad \xi_j^{(k(j))} = 0.\end{cases}$$

Then

$$\sum_{i\leqslant r(j)+1}|\zeta_{ij}^{(k(j))}|\geqslant\tfrac12|\xi_j^{(k(j))}|$$

and

$$\sum_{i\geqslant r(j)+1}|\zeta_{ij}^{(k(j))}| = \sum_{i\geqslant 1}|\zeta_{ij}^{(k(j))}| - \sum_{i\leqslant r(j)}|\zeta_{ij}^{(k(j))}|\geqslant\tfrac12|\xi_j^{(k(j))}|.$$

(Clearly these inequalities make sense if $r(j) = +\infty$, too.)

Now let us consider two cases.

(a) There exists a subsequence $j_1<j_2<\ldots<j_s<\ldots$ such that $\sup_s r(j_s)<n_0<+\infty$. Then

$$\begin{aligned} q_{n_0+1}\varrho(x_{k(j_s)}) &= \sum_{i,j\geqslant 1}\gamma_{ij}^{(n_0+1)}|\zeta_{ij}^{(k(j_s))}|\\ &\geqslant\sum_{i\leqslant n_0}\sum_{j\geqslant 1}\gamma_{ij}^{(n_0+1)}|\zeta_{ij}^{(k(j_s))}|\geqslant\sum_{i\leqslant n_0}\gamma_{ij_s}^{(n_0+1)}|\zeta_{ij_s}^{(k(j_s))}|\\ &= \beta_{j_s}^{(n_0+1)}\sum_{i\leqslant n_0}|\zeta_{ij_s}^{(k(j_s))}|\geqslant\tfrac12\beta_{j_s}^{(n_0+1)}|\xi_{j_s}^{(k(j_s))}|.\end{aligned}$$

If s is sufficiently large, then $j_s \geqslant m_{n_0+1}$. Since $r(j_s) < +\infty$, we have $\xi_{j_s}^{(k(j_s))} \neq 0$. Hence by (vi) we have

$$\tfrac{1}{2}\beta_{j_s}^{(n_0+1)} |\xi_{j_s}^{(k(j_s))}| \geqslant \tfrac{1}{2} j_s^2 p_{m_{n_0+1}}(x_{k(j_s)}) \geqslant 2 p_{m_{n_0+1}}(x_{k(j_s)}).$$

This inequality contradicts (*) since $p_{m_{n_0+1}}(x_k) \geqslant p_1(x_k) > 0$.

(b) Let $\lim\limits_{j\to\infty} r(j) = +\infty$. Let j' and k be so chosen that $k \geqslant m_1$, $t_{k-1} \geqslant j'$ and, for each $j \geqslant j'$, $r(j) \geqslant m_1$. Then

$$q_1(\varrho(x_k)) = \sum_{i,j\geqslant 1} \gamma_{ij}^{(1)} |\zeta_{ij}^{(k)}| \geqslant \sum_{i\geqslant m_1} \sum_{t_{k-1}\leqslant j<t_k} \gamma_{ij}^{(1)} |\zeta_{ij}^{(k)}|$$
$$\geqslant \sum_{t_{k-1}\leqslant j<t_k} \sum_{i\geqslant r(j)+1} a_j^{(m_1)} |\zeta_{ij}^{(k)}| \geqslant \tfrac{1}{2} \sum_{t_{k-1}\leqslant j<t_k} a_j^{(m_1)} |\xi_j^{(k)}| = \tfrac{1}{2} p_{m_1}(y_k).$$

But

$$p_{m_1}(y_k) \geqslant p_{m_1}(x_k) - p_{m_1}(x_k - y_k) \geqslant p_{m_1}(x_k) - p_k(x_k - y_k) \geqslant p_{m_1}(x_k) - \tfrac{1}{2}$$

(see (iv)). Further, by (ii), $\frac{1}{2} \leqslant \frac{1}{2} p_{m_1}(x_k)$. Hence $p_{m_1}(y_k) \geqslant \frac{1}{2} p_{m_1}(x_k)$, and therefore

$$q_1(\varrho(x_k)) \geqslant \tfrac{1}{4} p_{m_1}(x_k) > \tfrac{1}{5} p_{m_1}(x_k).$$

This inequality contradicts (*). The lemma is proved.

(III) LEMMA 2.5. *Let P be a projective space in the class $\mathscr{K}_i$ ($i = 1, 2, 3, 4$) Then P is a Montel space which admits a continuous norm.*

Proof. See [4].

(IV). Proof of Theorem 2.1.

(a) Case $i = 1, 2, 3$. In this case P is a Montel space which admits a continuous norm (Lemma 2.5). Thus there exist a Montel–Köthe space $L(A)$ and a mapping φ from $L(A)$ onto P (Corollary 2.3). Since P is a projective space, there exists a mapping ψ: $P \to L(A)$ such that $\varphi \circ \psi = 1_P$. Hence P is isomorphic to some subspace of $L(A)$. By Lemma 2.4 P cannot be infinite-dimensional.

(b) Case $i = 4$. By a theorem of T. Komura and Y. Komura [8], P is isomorphic to some subspace of the nuclear Köthe space $(s)^N$. Further, apply Lemmas 2.4 and 2.5.

(c) Case $i = 5, 6$. Let p be any F-norm which defines the topology of P ([12], p. 14). Let $(x_a)_{a\in A}$ be a dense family of elements of $P \setminus \{0\}$ such that $\mathrm{Card}(A) = \mathrm{Dens}(P)$. For each $a \in A$ we define the function f_a: $R_+ \to R_+$ by the equality $f_a(t) = p(tx_a)$. Let $M(A)$ be the F-space of all families of numbers $\xi = (\xi_a)_{a\in A}$ such that $q(\xi) = \sum\limits_{a\in A} f_a(\xi_a) < +\infty$, equipped with the F-norm q. Then the mapping π: $M(A) \to P$ such that $\pi(\xi) = \sum\limits_a \xi_a x_a$ is onto ([14], 0.3.11). Since $M(A) \in \mathscr{K}_i$, there exists an isomorphic embedding $P \to M(A)$. It is clear that the dual space $M(A)'$ is point-separating, and hence P' is point-separating. Therefore, the topology in P, generated by the convex hulls of zero neighbourhoods in P, is Hausdorff. Let P_0 be the vector space P equipped with this locally convex topology. Then the completion of P_0, say $\hat{P}$, belongs to $\mathscr{K}_{i-4}$ and is a projective space in $\mathscr{K}_{i-4}$. Thus, by (a), $\hat{P}$ is finite-dimensional. The proof is complete.

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