Factorization of compact operators and finite representability of Banach spaces with applications to Schwartz spaces

by

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Abstract. Let $X$ and $Y$ be Banach spaces, $X$ with a finite-dimensional decomposition. Necessary and sufficient conditions are given for every compact operator, $X$ to $X$, to factor through a subspace of $Y$. Also derived are sufficient conditions for $X$ uniformly finitely representable in $Y$ to imply that every compact operator, $X$ to $X$, factors through a subspace of $Y$. Examples of spaces $X$ are constructed with $X \times X$ not uniformly finitely representable in $X$. Finally, these results are applied to Schwartz preorders, particularly with respect to the approximation property.

1. Introduction. Let $X$ and $Y$ be Banach spaces. We say that $Y$ has the subspace factorization property [8] (abbreviated s.f.p.) for $X$ if each compact operator $X$ to $X$ factors through a subspace of $Y$. The s.f.p. is related to the approximation property (abbreviated a.p.) of Grothendieck (see [8]). In Theorem 2.3, we obtain conditions equivalent to $Y$ having the s.f.p. for $X$ when $X$ has a finite-dimensional decomposition. These equivalent conditions are the existence of certain kinds of fragmentation, (see Section 2 for definition) in $Y$.

Consider the statements (i) $Y$ has s.f.p. for $X$ and (ii) $X$ is uniformly finitely representable in $Y$. Are (i) and (ii) equivalent? When $X$ has a finite-dimensional decomposition, (i) implies (ii) (Proposition 2.2) and conversely with certain restrictions on $X$ or $Y$ (Theorem 3.1). In particular, if $Y \times Y$ is isomorphic to a subspace of $Y$ and $X$ has a finite-dimensional decomposition, (i) and (ii) are equivalent. Figiel [8] has shown that (i) and (ii) are equivalent if $X = l_1$. (See also remarks at the end of Section 3.)

Another collection of spaces $Y$ for which (ii) implies (i) are the galactic spaces. A Banach space $Y$ is galactic if for separable spaces $X$, $X$ uniformly finitely representable in $Y$ implies that $X$ is isomorphic to a subspace of $Y$. Galactic spaces are considered in Section 4.

Figiel has shown in [7] that there are reflexive Banach spaces $Y$ with $Y \times Y$ not isomorphic to a subspace $Y$. In Section 4, we show that some of these examples are not even locally square, that is, $Y \times Y$ is not uni-

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formally finitely representable in \( Y \). This is done by showing that \( Y \) is a galactic space (Theorem 4.1). The proof of Theorem 4.1 uses non-standard analysis.

The remainder of this paper applies these results to Schwartz pre-varieties of the form \( \mathcal{S}_\mathcal{X} \). For a Banach space \( \mathcal{X} \), \( \mathcal{S}_\mathcal{X} \) is the collection of Schwartz spaces isomorphic to a subspace of \( \mathcal{X}^d \), some power of \( \mathcal{X}^d \). In particular, we consider when \( \mathcal{S}_\mathcal{X} \subset \mathcal{S}_Y \) (Theorem 5.4) and we construct universal generators for some \( \mathcal{S}_\mathcal{X} \) (Corollary 5.6). Lastly, we show that for a Banach space \( \mathcal{X} \), \( \mathcal{X}^* \) has a.p. if the Schwartz space \( \langle X, E \rangle \) has a.p. (Proposition 5.7).

An operator is a continuous linear function. A compact operator between Banach spaces maps bounded sets into relatively compact sets. To say that the operator \( E: \mathcal{X} \rightarrow \mathcal{Y} \) factors through \( Z \), means there are operators \( U: \mathcal{X} \rightarrow \mathcal{Z} \) and \( V: \mathcal{Z} \rightarrow \mathcal{Y} \) with \( V \circ U = E \). The dual of \( \mathcal{X}^* \) is denoted by \( \mathcal{X}^* \) and the transpose of the operator \( \mathcal{X}^* \) is written \( \mathcal{X}^* \). We abbreviate linear span by \( \text{lin span} \) and closed linear span by \( \overline{\text{lin span}} \).

A sequence of finite-dimensional subspaces \( \{E_n\} \) is a finite-dimensional decomposition (abbreviated f.d.d.) for \( X \) if for each \( x \in X \), there is a unique sequence \( \{a_n\} \) with \( a_n \in E_n \) and \( x = \sum a_n \). We reserve the letter \( X \) for a Banach space with at least a f.d.d. \( \{E_n\} \). A f.d.d. is monotone if the projections: \( \sum a_n \rightarrow \sum a_n \) all have norm one. We reserve \( \lambda = (\lambda_n) \) for null sequences of real with \( 1 = \lambda_0 > \lambda_1 > \ldots > 0 \). Also reserved is the letter \( T \), which is always a diagonal operator on a space with a f.d.d. For instance, \( T: \mathcal{X} \rightarrow \mathcal{X} \) is the operator which sends \( x = \sum a_n \) onto \( T\mathcal{X}x = \sum \lambda_n a_n \). The following fact will be needed, a proof is essentially given in [26], p. 40.

**FACT 1.1.** If the Banach space \( X \) has a monotone f.d.d. \( \{E_n\} \) and \( \lambda \) and \( T \) are as above, then \( T \) is a compact operator. Furthermore, if \( m < n \) are integers, letting \( X_m = \text{lin span}(E_{m+1}, \ldots, E_n) \) and \( S_m \), the restriction of \( T \) to \( X_m \), then \( S_m \) is an isomorphism with

\[
\|S_m\| \leq \lambda_{m+1} < \|S_1\| \leq 2\lambda_{m+1},
\]

and

\[
\|S_m^{-1}\| \leq 2\lambda^{-1} < \|S_1^{-1}\| \leq 2\lambda^{-1}.
\]

If \( Y \) and \( Z \) are isomorphic Banach spaces, then the Banach-Mazur distance between \( Y \) and \( Z \) is \( d(Y, Z) = \inf\{|\lambda|: \lambda : Y \rightarrow Z \text{ an isomorphism}\} \). The Banach space \( Y \) is said to be uniformly finitely representable (abbreviated u.f.r.) in the Banach space \( Z \) if there is a constant \( K \) such that, for each finite-dimensional subspace \( Y_0 \) of \( Y \), there is a subspace \( E_0 \subset Z \) of the same dimension and \( d(Y_0, Z_0) \leq K \).

The proof of Theorem 4.1 uses non-standard analysis, in particular, it uses facts about the non-standard hulls of a Banach space found in [3], [10] and [11]. For basic results about non-standard analysis see [17], [16], and [9], the last for non-standard topological vector spaces.

A locally convex space (abbreviated LCS) \( (E, \mathcal{E}) \) is a vector space \( E \) with a Hausdorff locally convex topology \( \mathcal{E} \). Let \( \mathcal{U} \) be a \( \mathcal{E} \)-neighborhood basis of the origin; we will always assume that each \( U \in \mathcal{U} \) is weakly closed and absolutely convex. For \( U \in \mathcal{U} \), the Minkowski functional of \( U \) is \( g_U \) where

\[
g_U(x) = \inf \{z > 0: x \in zU\}.
\]

If \( U \in \mathcal{U} \), \( g_U \) will denote the Banach space formed by taking the completion of the vector space \( E/kU \) with the quotient norm obtained from \( g_U \).

If \( V \subset U \) are elements of \( \mathcal{U} \), there is a natural operator \( E_{x} \rightarrow E_{y} \) induced by the identity on \( E \). A LCS is a Schwartz space if, for each \( U \in \mathcal{U} \), there is a \( V \in \mathcal{U} \), with \( V \subset U \) and the natural operator \( E_{x} \rightarrow E_{y} \) compact.

2. **Factorizations and compact operators.** For this and the next section, let \( \mathcal{X} \) be a Banach space with a finite-dimensional decomposition \( \{E_n\} \). All our results are independent of the choice of norm on \( \mathcal{X} \), so we may and do assume that \( \{E_n\} \) is monotone.

This section is devoted to proving Theorem 2.3, which gives conditions on \( \mathcal{Y} \) equivalent to i.p. for \( \mathcal{X} \). First we give two preliminary results.

**Lemma 2.1.** Let us restrict attention to diagonal operators \( T: \mathcal{X} \rightarrow \mathcal{X} \), where \( \lambda = (\lambda_n) \) is a null sequence with \( 1 = \lambda_0 > \lambda_1 > \ldots > 0 \). That is, every compact operator \( \mathcal{X} \rightarrow \mathcal{X} \) factors through a subspace of \( \mathcal{Y} \) if and only if each \( T: \mathcal{X} \rightarrow \mathcal{X} \) factors through a subspace of \( \mathcal{Y} \). (Lemma 2.1 is similar to Proposition 3.2 of [3], we give a proof of the Lemma for completeness.)

**Lemma 2.1.** If \( U: \mathcal{Z} \rightarrow \mathcal{X} \) is a compact operator between Banach spaces, then there is a null sequence \( \lambda = (\lambda_n) \) with \( 1 = \lambda_0 > \lambda_1 > \ldots > 0 \) and a compact operator \( V: \mathcal{Z} \rightarrow \mathcal{X} \) such that \( U = TV \).

**Proof.** Since \( \{E_n\} \) is a monotone f.d.d. for \( \mathcal{X}, Q_{n} \), the natural projection onto the lin span \( \{E_{n+1}, \ldots, E_{m}\} \), has norm one. Let \( K \) be the closure in \( \mathcal{X} \) of the \( U \)-image of the unit ball of \( Z \). Since \( K \) is compact, for each \( z \geq 0 \), there is an \( m \) so that \( n \geq m \) and \( x \in K \implies \|Q_{n}x - \| < \varepsilon \) ([14], p.12). Let \( n(0) = 0 \) and inductively choose \( n(k+1) > n(k) \) so that \( j \geq n(k+1) \) and \( x \in K \) imply \( \|Q_{j}x - \| < 2^{-k} \). Define \( \lambda = 2^{-k} \) for \( j > n(k-1) \). Define \( V: \mathcal{Z} \rightarrow \mathcal{X} \) by \( Vz = \sum_{n} \lambda_{n} (Q_{n} - Q_{n+1}) Uz + \sum_{n} (Q_{n} - Q_{n}) Uz \).

Suppose \( \|z\| \leq 1 \); then \( \|Q_{n} - Q_{n+1}\| \leq \|U\| \) and, for \( k > 1 \), \( \|Q_{n} - Q_{n+1}\| \leq 2^{-k} \). Hence \( \|V\| = \|U\| + \sum \lambda^{k} 2^{-k} < \infty \), and \( T_{TV} = T_{T} \sum \lambda^{k} (Q_{n} - Q_{n+1}) U = \sum (Q_{n} - Q_{n+1}) U = U \). If \( V \) is not already
compact, let $\mu = (\mu_k)$ be the sequence of positive reals with $\mu_k^2 = \lambda_k$. Now $T_1, V = T_1(T, V)$ and $T_2, V$ is compact. ■

The following proposition is a preliminary version of Theorem 2.3, but it is easier to give direct proof than to deduce it from the theorem. Some partial converses of Proposition 2.2 are the subject of Section 3.

**Proposition 2.2.** If $Y$ has s.f.p. for $X$, then $X$ is u.f.r. in $Y$.

Proof. Let $X_n$ be the linear span of the first $n$ elements of the f.d.d. $(E_n)$. It suffices to show that there are subspaces $X_n \subset Y$ with $d(X_n, X_m)$ uniformly bounded. We complete the proof by showing that we can make $d(X_n, Y)$ tend to infinity as slow as we like.

Let $\lambda = (\lambda_k)$ be any sequence with $\lambda_1 = 1$ and $\lambda_k^2$ monotonically increasing to $\infty$. Since $T_1: X \rightarrow X$ is compact, the hypothesis implies that there is a subspace $W$ of $Y$ and operators $U: X \rightarrow W$ and $V: W \rightarrow X$ with $T_1 = UV$. Let $X_n = U(X_n) = V^{-1}(X_n)$. Thus restricting to $X_n$, or $Y$, we have $U^{-1} = T_1^{-1}$ restricted to $X_n \subset \{0\} \parallel V \parallel \{0\}$ by Fact 1.1.

**Definition of fragmentation.** Suppose that $(Y_k)$ is a sequence of finite-dimensional subspaces of $Y$ with $Y_k \cap Y_{k+1} = \{0\}$, if $j \neq k$. Define $P_j$ to be the projection from $\operatorname{lin}span(Y_k)_{\cap} Y_k$ given by $P_j(\sum a_k) = a_j$ if $a_k \in Y_k$. Suppose further that, for each $j$, $P_j \parallel < \infty$. Then each $P_j$ has a norm preserving extension to $W = \operatorname{lin}span(Y_k)_{\cap} Y_k$ which we will also call $P_j$. A triple $(Y_k, P_j, W)$ satisfying all the above conditions is a fragmentation of $Y$.

Let us make the following notations. Let $\mathcal{A}$ be the set of all sequences $(n(k))$ of integers with $0 \leq n(0) < n(1) < \ldots$. Let $B$ be the set of all sequences of positive reals $(\mu_k)$ increasing to infinity with $\mu_1 = 1$. Since $X$ has a monotone f.d.d. $(E_n)$, for each $(n(k)) \in \mathcal{A}$ we define $Z_k = \operatorname{lin}span(E_{n(k-1)}, \ldots, E_{n(k)})$. And, finally, for $(\mu_k) \in B$, we represent the following statement by:

- For each $(n(k)) \in \mathcal{A}$, there is a fragmentation $((Y_k), (P_j), W)$ of $Y$ with $\max d(Z_k, Y_k, \{P_j\}) = 0 (\mu_k)$.

**Theorem 2.3.** For $X$ and $Y$ Banach spaces, $X$ with a f.d.d., the following are equivalent:

(i) $Y$ has s.f.p. for $X$.

(ii) For each $(\mu_k) \in B$, $(\mu_k)$ is true.

(iii) For some $(\mu_k) \in B$, $(\mu_k)$ is true.

Proof. Suppose first that (i) is true; then for each $(\mu_k) \in B$ and $(n(k)) \in A$, define $\lambda = (\lambda_k)$ by $\lambda_1 = \mu_1^2$ for $n(k-1) < n(k)$. By the hypothesis, the compact operator $T_1: X \rightarrow X$ factors through $W$, a subspace of $Y$, by operators $U: X \rightarrow W$ and $V: W \rightarrow X$. We may and do assume that $W = \operatorname{cl}U(Y)$. Now let $Y_k = U(Z_k) = V^{-1}(Z_k)$. Thus, restricting to $Z_k$, we have $U^{-1} = T_1^{-1}$ restricted to $Z_k \subset \{0\} \parallel V \parallel \{0\}$, by Fact 1.1.

Again, let $Q_k$ be the natural projection onto $\operatorname{lin}span(E_{n(k)}, \ldots, E_{n(k)})$.

Since $P_k = U T_1^{-1} (Q_{n(k)} - Q_{n(k)-1}) V$ are the projections needed to make $((Y_k), (P_j), W)$ a fragmentation and $\|P_k\| \leq 4 \|U\| \|V\| \lambda_k^{-1} \leq 4 \|U\| \|V\| \lambda_k < \infty$, by Fact 1.1. We have shown (ii).

Now suppose that (iii) is true for the sequence $(\mu_k) \in B$. Let $(\mu_k)$ be a null sequence with $1 = \mu_1 \geq \mu_2 \geq \ldots > 0$ and $\sum \mu_k \mu_{k+1} < \infty$. To show (i), by Lemma 2.1, it suffices to show that each diagonal compact operator $T_1: X \rightarrow X$ factors through a subspace of $Y$. Let $\lambda = (\lambda_k)$ be a null sequence with $1 = \lambda_1 \geq \lambda_2 \geq \ldots > 0$ and let $\mu = (\mu_k)$ be the sequence such that $\mu_0 > 0$ and $\mu_k^2 = \lambda_k$. Let $n(0) = 0$ and inductively choose $n(k+1) > n(k)$ so that $\lambda_j > n(k+1)$ implies $\mu_j < \mu_{k+1}$.

By hypothesis, since $(n(k)) \in \mathcal{A}$, there is a constant $M$ and a fragmentation $((Y_k), (P_j), W)$ with $d(Z_k, Y_k) < M \lambda_k$ and $\|P_j\| < M \lambda_k$. Let $j_1: Z_k \rightarrow Y_k$ be an isomorphism with $\|j_1\| < M \lambda_k$ and $\|j_1\| < 1$. Let $j_2: Y_k \rightarrow Z_k$ be the restriction of $P_j$ to $Z_k$.

Define $U: X \rightarrow W$ and $V: W \rightarrow X$ by $U = \sum T_k T_j (Q_{n(k)} - Q_{n(k)-1})$ and $V = \sum T_k T_j T_k^{-1} P_j P_k$. Now

- $\|U\| \leq \sum \|j_1\| \|T_k\| \|Q_{n(k)} - Q_{n(k)-1}\| 
\leq \sum (M \lambda_k) (2 \mu_{k+1-1}) \|Q_{n(k)} - Q_{n(k)-1}\| < 4 \sum \mu_k \mu_{k+1} < \infty$;

- $\|V\| \leq \sum \|T_k\| \|T_k^{-1}\| \|P_k\| < \sum (2 \mu_{k+1-1}) (M \lambda_k) \|P_k\| < 2 \sum \mu_k \mu_{k+1} < \infty$.

And finally

- $V U = \sum T_k T_j T_k^{-1} P_k \|j_1 T_k (Q_{n(k)} - Q_{n(k)-1}) \|
\leq \sum T_k T_j T_k^{-1} j_1 T_k (Q_{n(k)} - Q_{n(k)-1}) - \sum T_k^2 (Q_{n(k)} - Q_{n(k)-1}) = T_1$. ■

3. Uniform finite representability and factoring compact operators. Theorem 3.1 is a partial converse to Proposition 2.2. First we need some definitions we remind the reader that $X$ has a monotone f.d.d. $(E_n)$.

Let $Y$ and $Z$ be Banach spaces.
We say $Y$ is finitely stable if there is a constant $K$ with $K$-u.f.r. in each subspace $Y_c = Y$ with $\dim(Y/Y_c) < \infty$. We say $Z$ is stably finitely representable in $Y$ if there is a constant $K$ with $Z$-u.f.r. in each subspace $Y_c = Y$ with $\dim(Y/Y_c) < \infty$. We say $Y$ is locally square if $X \times Y$ is u.f.r. in $Y$.

**Theorem 3.1.** If $X$ and $Y$ are Banach spaces, $X$ with a f.d.d. and $X$ is u.f.r. in $Y$, then each of the following is sufficient to imply $Y$ has a f.d.d. for $X$.

(i) $(X \times X \times \ldots \times X)_p$ is isomorphic to a subspace of $X$, for some $p$, $1 \leq p \leq \infty$. (By $p = \infty$ we mean the $\ell^\infty$-sum.)

(ii) $X$ is stably finitely representable in $Y$.

(iii) $(X \times X \times \ldots \times X)_p$ is isomorphic to a subspace of $X$, for some $p$, $1 \leq p \leq \infty$.

(iv) $Y$ is finitely stable.

(v) $X \times Y$ is isomorphic to a subspace of $Y$.

**Proof.** We will show (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (v), (iv) $\Rightarrow$ (ii) and complete the proof by showing that each of the conditions (ii) and (v) imply condition (iii) of Theorem 2.3.

(iv) $\Rightarrow$ (ii): If $X$ is $K$-u.f.r. in $Y$ and $X$ is $K$-u.f.r. in $Y$, then $X$ is $K$-u.f.r. in $Y$.

(iii) $\Rightarrow$ (v): Obvious.

(i) $\Rightarrow$ (ii): Let $Y$ be a finite-codimensional subspace of $Y$. By hypothesis, we may assume $(X \times X \times \ldots \times X)_p$ is $K$-u.f.r. in $Y$. Let $E$ be a finite rank projection with $E_Y = \ker E$. Let $F$ be an $r$-dimensional subspace of $X$. Let $m = \dim(Y/Y_c)$ and let $F'$ be the $l_p$-sum of $(m + 1)^r$ copies of $E$. Since $F = (X \times X \times \ldots \times X)_p$, there is a subspace $Z \subset X$ with $\dim(Z) < \infty$. Let $P$ be any projection from $X$ onto $Y$. By Lemma 4.1 of [8], there is a projection $Q$ in $Y$ with $Q\ker E = \ker E = X$ and $\dim(Q(Y), Y) = K$. Now $\ker E = X$ implies $Q(Y) = \ker E = Y$; thus $X$ is u.f.r. in $Y$. Therefore, (iii) is true. We observe that this proof also shows (iii) $\Rightarrow$ (iv), and that (ii) is implied by the following weaker condition:

(i') The sequence $(X, (X \times X \times \ldots \times X \times \ldots)_p)$ of Banach spaces are each $K$-u.f.r. in $X$, for some $K$ and some $p$, $1 \leq p \leq \infty$.

(ii) $\Rightarrow$ Theorem 2.3, condition (ii): We actually show more, namely that $\gamma_k$ can be a bounded sequence. Let $(\gamma_k)$ be a and $(\gamma_k)$ as in Theorem 2.3, and suppose $X$ is $K$-u.f.r. in $X$. Inductively choose $(\gamma_k)$ and $(\gamma_k)$ so that

1. $d(X, \gamma_k) \leq K$,
2. $Y_{k+1} \subset W_k$,
3. $Y = W_1 \subset W_2 \subset \ldots$,
4. $\dim(Y/Y_k) < \infty$,
5. The projection from $Y_1 + \ldots + Y_{k+1}$ onto $Y_1 + \ldots + Y_k$ with kernel $W_k$ has norm $\leq 1 + \delta_k$.

We can choose $(\gamma_k)$ satisfying (1) and (5) by hypothesis, where as choosing $W_{k+1}$ can be modelled on the standard construction of a basic sequence in any Banach space as in [14] (see p. 10).

(v) $\Rightarrow$ Theorem 2.3, condition (iii): The idea of this proof is fairly easy. We construct a sequence $(W_k)$ of subspaces of $X$, with each $W_k$ isomorphic to $Y$. By picking $W_k = W_k$ we will obtain a fragmentation of $X$ with the desired properties. We obtain the $(W_k)$ by induction, starting with $Y \times Y \subset Y$ into the second factor of $Y \times Y \subset Y$. We pass to the details.

Let $J_t: Y \times Y \to Y$ be an isomorphism of $Y \times Y$ onto a subspace of $Y$. Let $Q_t: Y \times Y \to Y \times Y$ be the projection onto the $i$th factor, $i = 1, 2$. We may assume $|Q_0| = 1$. Let $J_t: Y \times Y \to Y$ be the injection of $Y$ onto the $i$th factor, $i = 1, 2$. Let $M = ||J||J^{-1}||$.

We inductively define:

1. $J_1: Y \to Y$ an isomorphism into a subspace of $Y$, for $i = 1, 2, \ldots$ and $k = 1, 2, \ldots$.
2. $W_k = J_1(Y)$, for $i = 1, 2$ and $k = 1, 2, \ldots$.
3. $J_t = J_1(Y)$, for $i = 1, 2$ and $k = 1, 2$.

Fix a projection from $W_t_{i}$ onto $W_t_{i+1}$, for $i = 1, 2$.

First let $J_t = J_t$, and $J_t = J_tJ_t^{-1}$; we note that $d(W_t, Y) \leq M$ and $|J_t| \leq M$, for $i = 2, 2$.

Define $J_tJ_t = J_tJ_tJ_t$ and $P_tJ_t = J_tJ_t^{-1}J_tJ_t$.

Now let $W_t = \text{lin span}(W_t)$ and let $E_t$ be the natural projection from $W_t$ onto $W_t$. Now $E_t = P_tP_t\ldots P_tP_t$ on $W_t$ and so $|E_t| \leq M^{k+1} \ldots 1 = M^{k+1}$.

Now let $(\gamma_k)$ be a sequence in $d$, and $Z_k$ as in Theorem 2.3. Since $X$ is $K$-u.f.r. in $Y$, there is a subspace $Y_{k+1}$ in $W_1$ with $d(Z_k, \gamma_k) \leq M^{k+1}$. Furthermore, the projections $E_t$ on $Y_{k+1}$ onto $Y_{k+1}$ are the just the restriction of $E_t$ to $Z_k$ on $Z_k$; hence $|E_t| \leq M^{k+1}$. Letting $W = \text{lin span}(Z_k)$, the fragmentation $(X, (X_k), (Y_k), W)$ satisfies $(\ast)$ for $(\gamma_k)$ in $d$, where $s_k = 1$ and $\delta_k = M^{k+1}$ for $k > 1$.

**Remarks.** 1. Since $\delta_k$ satisfies condition (i), we obtain as a corollary Figiel's Theorem 7.7 [8] that $X$ has the s.f.d. if $\delta_k$ if and only if $\delta_k$ is u.f.r. in $Y$.

2. We also obtain Figiel's result [8], Theorem 6.1, that every compact operator factors through a subspace of $X$ and is only if $\delta_k$ is u.f.r. in $Y$ (i.e. $Y$ is universal for finite-dimensional spaces). Since such an $Y$ satisfies condition (iv).

3. It seems reasonable that Theorem 3.1 could be extended to cases where
(vi) $X$ is finitely stable and
(vii) $X$ is locally square.

I do not know if this can be done, but methods used above, in particular Lemma 4.1 of [8], require too much dependence on the dimensions of $Z$. 4. It is easy to see that each of conditions (i) through (vii) imply that $X \times X$ is u.f.r. in $Y$. Figiel in [7] constructs examples of Banach spaces $X$, for which $X \times X$ is not isomorphic to a subspace of $X$. Actually, he shows that $X$ is not locally square nor finitely stable. We give another proof in the next section.

5. A similar development shows that Theorem 3.1 remains true if we drop the condition that $X$ has a f.d.d. and weaken the conclusion to every compact operator $T: X \to X$, which is the uniform limit of finite rank operators, factors through a subspace of $Y$. Hence Theorem 3.1 is true if $X$ has the a.p.

6. Condition (ii) implies that there is a subspace $Y_0$ of $Y$ such that each compact operator $T: X \to X$ factors through $Y_0$. This subspace can be constructed in a fashion similar to the spaces $G'$; (see [8]). The sequence $(G', E_k)$ (p. 194) are chosen to be dense in the finite-dimensional subspaces of $X$, rather than in all Banach spaces as in [8]. The construction proceeds as in the proof of part (ii) of the theorem.

1. If $Z \oplus Z$ is isomorphic to a complemented subspace of $Z$, and $Y$ has the a.p. for $Z$, then there is a constant $K$ such that for each compact $T: X \to Z$, the factorization operators $\bar{U}$ and $V$ can be chosen to satisfy $|TV||W| \leq K|Z|$. The proof combines the techniques of Theorem 3.1 (v) and (i) $\Rightarrow$ (ii) of Theorem 2.2 of [8]. Hence if such a $Z$ has the b.a.p. ([14], p. 123), then $Z$ is u.f.r. in each $Y$ having the a.p. for $Z$.

4. **Galactic Banach spaces.** A Banach space $Z$ will be called galactic if for each separable Banach space $Y$, then $Y$ u.f.r. in $Z$ implies $Y$ is isomorphic to a subspace of $Z$. Roughly speaking, a galactic space is a subspace universal for separable Banach spaces made out of $Z$'s universe of finite-dimensional subspaces.

An obvious example of a galactic space is $l_2$ space and every galactic space must contain a subspace isomorphic to $l_2$ again. Other examples include $l_1$ and $C[0, 1]$ as well as $L_p[0, 1]$ (see remark after Lemma 4.2 below). In this section, we will construct other examples of galactic spaces which have unconditional basis and some of which are neither finitely stable nor locally square. For $Y$ a galactic space and $X$ with a f.d.d., clearly $Y$ has the a.p. if and only if $X$ is u.f.r. in $Y$.

Let $(E_n)$ be a sequence of finite-dimensional Banach spaces with the property, for each integer $m$ and $\varepsilon > 0$, there is $X$ so that for each $m$-dimensional subspace $Y$ of some $E_n$ with $n \geq N$, we have $d(Y', Y) < 1 + \varepsilon$. Examples of such sequences $(E_n)$, include $(G', E_k)$ for any sequence of integers $(d(n))$ and any sequence of reals $(p(n))$ with limit two (Corollary 2.2 of [15]).

Let $X_0 = (E_0 \times E_1 \times \cdots )_0$ and let $X = X_1$. $X$ satisfies:

- for each integer $m$ and $\varepsilon > 0$, there is an $X$, so that for each $m$-dimensional subspace $Y$ of $X$, we have $d(Y', Y) < 1 + \varepsilon$.

We observe that $X$ has a complemented subspace isomorphic to $l_1$, hence by Pelczynski's decomposition method (see [14], p. 30) $X$ is isomorphic to $l_1$. 4.1. $X$ is a galactic space.

Remark. Figiel has shown in [7] that if $E = l_2 \oplus l_2$ with $p(n)$ strictly decreasing with limit two, then for some choices of $d(n)$, $X \times X$ is not isomorphic to a subspace of $X$. By the theorem (or by inspecting Figiel's proof [7]), $X$ is not locally square nor finitely stable. For other results about Figiel's space see [4].

We need some simple facts about non-standard norm spaces. First we may quote results, particularly of [10] and [11], we make the technical assumption that our non-standard model is an enlargement which is at least $\aleph_1$-saturated. If $Y$ is a Banach space, we will write $Y$ for the non-standard Banach space. We note that $Y$ is a subset of $Y$ and $|Y|$ is an extension to $|Y|$ of the norm on $Y$.

Define $\text{fin} = \{x \in Y: |x| < \text{finite} (i.e. |x| = 0 \text{ is finitely close to a standard real number, } |x| < |y| \} \text{ and let } \mu = \{x \in Y: |x| = \text{infinitesimal or equivalently } |x| = 0 \}. \text{ The quotient vector space } \text{fin} / \mu = \tilde{Y} \text{ is a norm. In fact, } (\tilde{Y}, \|\|) \text{ is a standard Banach space called the non-standard hull of } (Y, \|\|). \text{ We now restate Theorem 2.3 of [10] as Lemma 4.2.}

**Lemma 4.2.** If $(Y, \|\|)$ is a Banach space, then its non-standard hull $(\tilde{Y}, \|\|)$ is galactic.

**Remark.** We can now give an easy proof that $L_2[0, 1]$ is galactic. Let $Y = L_2$; then $Y$ is isomorphic to an $L_2(a)$-space by Corollary 2.5 of [11]. Furthermore any separable subspace of $L_2(a)$ is isomorphic to a subspace of $L_2[0, 1]$ ([14], p. 124). Of course, this result is well known, for example, see [14], p. 122, where a "similar" proof is given.

**Proof of Theorem 4.1.** Let $X$ be as given. Since $X$ is reflexive, by [1], Theorem 3.1 and 2, Theorem 4.1 the non-standard hull $\tilde{X}$ is isomorphic to $X \oplus H$, where $H$ is defined below. Let $\varphi$ be the quotient operator $X \to \tilde{X}$, and let $P = \{x \in X: |\varphi(x)| \text{ is finitely close to any standard } Y' \in \tilde{X}'\}$. Then $H = \varphi(X)$. By the remark before Theorem 4.1, it suffices to show that $H$ is isomorphic to Hilbert space. This will be
done by showing each finite-dimensional subspace of $H$ is close to Hilbert space of the same dimension.

Let $Y$ be a finite-dimensional subspace of $H$. Let $Y_1$ be a subspace of $P = \mathcal{X}$ of the same dimension as $Y$ and with $\psi(Y_1) = Y$. Let $w = \dim Y = \dim Y_1$ and let $\varepsilon > 0$; by (*), there is an $N$ so that each $m$-dimensional subspace of $X_N$ is $1+\varepsilon$ close to $Y$. The same statement is true for $m$-dimensional subspaces of $\mathcal{X}$. Now $P_N = \bigoplus_{1 \leq i \leq m} \mathcal{X}_i$, and $X_N = \ker(I - P_N)$. Now $\mathcal{X}_N = \bigoplus_{1 \leq i \leq m} \mathcal{X}_i = \ker(I - P_N)$. Letting $Y_1 = (I - P_N)X_1$, and since $Y_1 \subseteq Y$, we have $d(Y_1, X_N) = \inf_{Y_1 \subseteq Y} d(Y_1, X_N) \leq 1 + \varepsilon$, since $X_N = \ker(I - P_N)$.

**Remark.** A modification of the proof of Theorem 4.1 yields that the space $\mathcal{X} \otimes L_2$ is galactic, when $\mathcal{X} = \langle \mathcal{E}_1 \times \mathcal{E}_2 \times \ldots \mathcal{E}_q \rangle$, $\mathcal{E}_m = \mathcal{E}_m(0)$, and $d(\mathcal{E}_m, \mathcal{E}_n) = \sum \delta_{mn}$. (See (Theorem A) of [13]).

**5. Schwartz spaces.** In this section we explore interconnections between Banach spaces and Schwartz spaces, particularly with respect to the approximation property. We find it convenient to use the notions of prevarieties. A collection of LCS's is a prevariety [1] if it is closed with respect to subspaces, products and isomorphic images. The collection $\mathcal{S}$ of all Schwartz spaces is a prevariety (see [12], p. 275).

For a Banach space $\mathcal{X}$, let $\mathcal{S}_{\mathcal{X}}$ be the set of Schwartz spaces isomorphic to a subspace of a power of $\mathcal{X}$ (i.e. $\mathcal{X}^n$ for some index set $I$). It is easy to see that $\mathcal{S}_{\mathcal{X}}$ is a prevariety; in fact, it is the intersection of $\mathcal{S}$ with $\mathcal{S}(\mathcal{X})$, the smallest prevariety containing $\mathcal{X}$. If $\mathcal{X} = \mathcal{E}_1^I$, we will write $\mathcal{S}_{\mathcal{E}_1^I}$.

Our first order of business is to construct some examples of spaces in $\mathcal{S}_{\mathcal{X}}$, when $\mathcal{X}$ has a finite-dimensional decomposition $\langle \mathcal{E}_m \rangle$.

**Example 5.1.** Let $\lambda = (\lambda_n)$ be a null sequence with $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq 0$. Let $T_k$ be the diagonal operator from $\mathcal{X}$ to $\mathcal{X}$, corresponding to $\mu = (\mu_n)$, where $\mu_n > 0$ and $\mu_{k+1} = \lambda_k$ for $k = 1, 2, \ldots$. Let $X_k$ be the projective limit of the sequence:

$$\cdots \rightarrow X_{k+1} \rightarrow X_k \rightarrow X_k \rightarrow X_{k+1} \rightarrow \cdots$$

where $X_k = \mathcal{X}$. If we let $S_k: X \rightarrow X_k$ be $T_k$, then $T_k: X \rightarrow X_k$ is the composition $T_k T_{k+1} \cdots T_2 T_1$. Hence there is a natural continuous operator $T_0: \mathcal{X} \rightarrow X_1$, whose image in $X_1$ is the same as $T_1: \mathcal{X} \rightarrow X_1$. It is easy to see that $X_1$ is a Schwartz–Fréchet space ([12], Proposition 9, p. 283) in $\mathcal{S}_{\mathcal{X}}$.

Before the next example, some definitions are in order. Let $\mathcal{X}$ have a f.d.d. $\langle \mathcal{E}_m \rangle$ and let $\mathcal{P}_m$ be the projection $\sum \mathcal{E}_m \rightarrow \mathcal{E}_m$. If $E'_n = \mathcal{P}_n(X')$, then $\langle E'_n \rangle$ is said to be shrinking if $\langle E'_n \rangle$ is a f.d.d. for $X$. Let $\mathcal{X}$ be a Banach space with norm topology $\tau$. Then $\xi_\mathcal{E}$, the topology of uniform convergence on $\mathcal{E}$-norm null sequences, is the strongest Schwartz topology on $\mathcal{X}$ weaker than $\tau$ ([1]). If $(Z, \eta)$ is a norm subspace of $(\mathcal{X}, \xi)$, then $\eta_\mathcal{E}$ is the restriction of $\xi_\mathcal{E}$ to $Z$. If $A \subseteq \mathcal{X}$, then $A^\perp$ (the absolute polar) $= \{y \in \mathcal{X}: \langle y, a \rangle \leq 1 \text{ for all } a \in A\}$.

**Example 5.2.** Suppose that $\mathcal{X}$ has a shrinking f.d.d. $\langle \mathcal{E}_m \rangle$ then $\langle (\mathcal{X}, \xi_\mathcal{E}) \rangle^{\mathcal{S}_{\mathcal{X}}}$ is a norm subspace of $\mathcal{X}$. Since $\langle \mathcal{E}_m \rangle$ is shrinking, Lemma 3.1 applies to $\mathcal{X}$. Thus there is a null sequence $\lambda = (\lambda_n)$ with $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq 0$, and if $U$ is the unit ball of $\mathcal{X}$, then $T_1: \mathcal{X} \rightarrow \mathcal{X}'$ maps some multiple of $U$ onto a subset of $K$. We may assume $T_1(U) = K$. Since $T_1$ is the transpose of $T_1: X \rightarrow X$, and by Lemma 6.39 of [18], $T_1(U)^* = T_1^{-1}(U)^*$, where $U$ is the unit ball of $\mathcal{X}$. But $K \subseteq T_1(U)$, so that $K^* = T_1^{-1}(U)$. We have shown that for each $\xi_\mathcal{E}$-neighborhood $V = K^*$, of the origin, there is a $\xi_\mathcal{E}$-neighborhood $W = T_1^{-1}(U)$, with $W \subseteq V$ and $W$ isomorphic to $V$ (By Fact 5.3 below). Hence $\langle \mathcal{X}, \xi_\mathcal{E} \rangle \in \mathcal{S}_{\mathcal{X}}$.

Applying our earlier results to the prevarieties of the form $\mathcal{S}_{\mathcal{X}}$, we obtain Theorem 5.4. We need the following fact whose proof is easy and will not be given.

**FACT 5.3.** Let $(E, \xi)$ be a LCS, $\mathcal{X}$ a Banach space with unit ball $U$, and $S: E \rightarrow \mathcal{X}$ be an operator. Then $W = S^{-1}(U)$ is an $\xi$-neighborhood with $W$ isomorphic to a subspace of $\mathcal{X}$.

**Theorem 5.4.** If $\mathcal{X}$ has f.d.d. and if $\mathcal{Y} \subseteq \mathcal{X}$ is isomorphic to a subspace of $\mathcal{X}$. Then $\mathcal{S}_{\mathcal{Y}} \subset \mathcal{S}_{\mathcal{X}}$, if and only if $\mathcal{X}$ is u.f.r. in $\mathcal{Y}$.

**Proof.** Suppose $\mathcal{S}_{\mathcal{X}} \subset \mathcal{S}_{\mathcal{Y}}$; by the proof of Proposition 2.2, $\mathcal{X}$ will be u.f.r. in $\mathcal{Y}$ if each $T_k: X \rightarrow X$ factors through a subspace of $\mathcal{Y}$. By Example 5.1, $X \in \mathcal{S}_{\mathcal{X}} \subset \mathcal{S}_{\mathcal{Y}}$, hence $X_1$ is isomorphic to a subspace of a power of $\mathcal{Y}$. In the notation of Example 5.1, let $U_0$ be the unit ball of $X_1$. Then there is an $\xi_\mathcal{X}$-neighborhood $V$ of $U_0$, with $U_0 \subseteq V \subset U_0$ and $(X_1)_V$ isomorphic to a subspace of $\mathcal{Y}$, a product of a finite number of copies of $\mathcal{Y}$. Thus, by Example 5.1 and by hypothesis, $T_1: X \rightarrow X$ factors through a subspace of $\mathcal{Y}$. Therefore, Proposition 2.3 implies $X$ is u.f.r. in $\mathcal{Y}$.

Conversely, suppose that $\mathcal{X}$ is u.f.r. in $\mathcal{Y}$. Since $\mathcal{Y} \subseteq \mathcal{X}$ is isomorphic to a subspace of $\mathcal{X}$, $\mathcal{X}$ is u.f.r. in $\mathcal{Y}$. Now $\mathcal{X}$ has a f.d.d., so Theorem 3.1 implies that $\mathcal{X}$ has the s.l.p. for $\mathcal{X}$.

Now let $\mathcal{E}$ be a LCS in $\mathcal{S}_{\mathcal{X}}$. Thus for every neighborhood $V$ of the origin in $\mathcal{E}$, there are neighborhoods $V$ and $W$ with $V \subset W \subset U$, $E_V \rightarrow E_W$.
compact and $E_W$ isomorphic to a subspace of $X'$. By considering $E_Y \to E_W$ as a compact operator $E_Y \to X'$ and applying Lemma 2.4, we have a factorization $E_Y \to X' \to X''$. Now the latter operator factors through a subspace of $Y$. Fact 5.3 yields a neighborhood $G$ with $V = U \subset W$ and $E_0$ isomorphic to a subspace of $Y$. Hence $E \in \mathcal{P}_Y$.

**Remark.** We can now give a complete description of the inclusion relations among the prevarieties $\mathcal{P}_P$, for $1 \leq p \leq \infty$:

(i) $\mathcal{P}_p \subset \mathcal{P}_q$, for $1 \leq p < q < \infty$.

(ii) $\mathcal{P}_1 = \mathcal{P}_\infty$.

(iii) $\mathcal{P}_p \subset \mathcal{P}_q$, for $1 \leq q \leq p \leq \infty$.

(iv) $\mathcal{P}_p \subset \mathcal{P}_q$, otherwise. These inclusions are familiar, since $\mathcal{P}_1 = \mathcal{P}_\infty$ if and only if $L_1$ is isomorphic to a subspace of $L_\infty$. This follows since $L_1$ and $L_\infty$ generate the same Schwartz prevariety $\mathcal{P}_1$, $\mathcal{P}_\infty$ is a galactic space, and by Theorem 5.4. (See [14] for which $L_p$ are subspaces of $L_1$.)

A universal generator for the prevariety $\mathcal{P}_X$, is a $\mathcal{P}_X \in \mathcal{P}_X$, such that each $F \in \mathcal{P}_X$ is isomorphic to a subspace of $E$. The following proposition has been proved for some special cases [13, 16, 17], and [18].

**Proposition 5.5.** If $X$ is a Banach space with $X \times X$ isomorphic to a subspace $Y$, then each $E \in \mathcal{P}_X$ is isomorphic to a subspace of a power of $(X, \xi_X)$. Thus if $(X, \xi_X) \in \mathcal{P}_X$, then $(X, \xi_X)$ is a universal generator for $\mathcal{P}_X$.

**Proof.** By hypothesis, each $E \in \mathcal{P}_X$ has a neighborhood basis $\mathcal{W}$ such that $U \in \mathcal{W}$ implies $E_U$ is isomorphic to a subspace of $Y$. Therefore there is a natural isomorphism $E$ into $Y^X$:

$$E \to \bigcap_{E_U \in \mathcal{W}} E_U \to Y^X \to (X, \xi_X)^X.$$ Since the latter operator is continuous, the proof will be complete if the composition $E \to (X, \xi_X)^X$ is open.

Let $U \in \mathcal{W}$ and let $V \in \mathcal{W}$ so that $V \subset U$ and $E_V \to E_U$ is compact. The transposes $(E_V)\to (E_U)\to Y^X$ is also compact. Let $K = (E_U)^X$ be the image of $U^X$ by the transpose. There is a norm null sequence $\{a_n^U\}$ in $(E_U)^X$ with $K$ contained in the closed absolutely convex hull of $\{a_n^U\}$ ([21], p. 111). Considering $E_U$ as a subspace of $Y$, use the Hahn-Banach Theorem to extend $a_n^U$ to $a_n^Y$ a null sequence in $Y^X$. Now $W = (E_U)^X$ is a $\xi_X$-neighborhood of $Y$. We have $E_U = (E_U)^Y \subset K = S^{-1}(U)$, where $S: E_U \to E_U$ is the natural operator (the last equality is Lemma 6, p. 30 [18]). Thus $B \in (X, \xi_X)^X$ is an isomorphism.

**Corollary 5.6.** $(X, \xi_X)$ is a universal generator for $\mathcal{P}_X$ if $X$ has a shrinking f.d.d. and $X \times X$ is isomorphic to a subspace of $X$.

**Remark.** Since $l_p$, $1 \leq p < \infty$, and $l_\infty$ satisfy these conditions, $(l_p, \xi_\infty)$ is a universal generator for $\mathcal{P}_p$ and $(l_\infty, \xi_p)$ is a universal generator for $\mathcal{P}_p$.

The LCS $E$ has the approximation property if the identity operator $E \to E$ is in the closure of the set of finite rank operators when the space of operators: $E \to E$ is given the precompact-open topology [21]. It is known that if there is an $E \in \mathcal{P}_X$ and the a.p., then there is a subspace of $Y$ without the a.p. ([21], p. 109) Proposition 5.7 provides a converse to this result if $Y$ has a shrinking f.d.d. (Apply Example 5.2 and [21], p. 113). As a corollary, each $E \in \mathcal{P}_p$, $1 \leq p < \infty$, has a LCS without the a.p. in light of [6] and [7].

**Proposition 5.7.** Let $Y$ be a Banach space with dual $X'$. Then $(Y, \xi_Y)$ has the a.p., implies that $Y'$ has the a.p., and conversely if $Y$ is reflexive.

**Proof.** Now $(Y, \xi_Y)$ has the a.p. if and only if, for each precompact set $K$ and open set $U$, there is a finite rank operator $F$ with $(I - F)(K) \subset U$. Since $\xi_Y$ is a Schwartz topology of the dual pair $(Y, X')$ and bounded sets are precompact in a Schwartz space, we may assume that $K$ runs over the scalar multiples of the unit ball of $Y$. By the definition of $\xi_Y$, $U$ may be assumed to be the polar of a norm null sequence of $Y'$. By Lemma 6, p. 30 of [18],

$$[(I - F)(K)]^* = [(I - F')^*]^*(K)^*.$$ Thus $(I - F)(K) \subset U$ if and only if $(I - (F')^*)(U) \subset K$. Now as $U$ runs over the open sets of $\xi_Y$, $U^*$ is running over the compact sets of $Y'$, and as $K$ runs over the precompact sets of $\xi_Y$, $K^*$ is running over the open sets of $Y$. Thus $(Y, \xi_Y)$ has the a.p. implies $Y'$ has the a.p.

For the converse implication, observe that the above proof is reversible if in $Y'$ we can choose the finite rank operator to be the transpose of a finite rank operator on $Y$. That this is the case is an easy consequence of reflexivity.

**References**


* Added in proof: It is possible to modify ([21], p. 109) so as to deduce the existence of a Fréchet space in $\mathcal{P}_p$, $2 < p < \infty$, without the a.p.
Boundary limits of Green’s potentials along curves II
Lipschitz domains

by

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Abstract. On a Lipschitz domain D in space, let μ be a mass distribution and w the Green potential of μ. Conditions on μ are given so that w ≠ +∞; under the same condition we show that the boundary limits of w along curves with certain differentiability properties are zero almost everywhere.

Green’s potential occurs in the study of subharmonic and superharmonic functions via Riesz decomposition theorem ([5], p. 116). Let D be an open subset of R^n having a Green’s function G; Green’s potential w given by a mass distribution μ is defined to be

\[ w(x) = \int_D G(x,y) \, d\mu(y) \]  

for every \( x \in D \). When D is the unit disk in the plane, the necessary and sufficient condition for \( w \neq +\infty \) is

\[ \int_D (1 - |y|) \, d\mu(y) < +\infty; \]

under this condition w has radial limit zero at almost every point on the unit circle, see Littlewood [6]. Later in 1938, Privalov [7] proved the similar result for Green’s potentials on the unit ball in \( \mathbb{R}^n \). The non-tangential limit of Green’s potential need not exist at any point on the boundary, as pointed out by Zygmund, [9], pp. 644-645.

The purpose of this paper is to study the boundary limits of Green’s potentials in a Lipschitz domain D in \( \mathbb{R}^n \), \( n \geq 3 \) along curves with certain differentiability properties. The problem for \( n = 2 \) was studied in [11], where, with the aid of conformal mapping, we need only to study the limit of Green’s potentials on \( |\cdot| < 1 \) along curves with the same differentiability properties. When \( n \geq 3 \) the conformal mapping technique does not apply and it is not even obvious for which \( \mu \) the Green’s potential of \( \mu \) is not identically +∞. Our main tool is an estimate on a certain harmonic function in a cone derived from a series representation of that

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