[2], for instance. The proof of Theorem (a) can now be followed word for word, except that formula (1) requires an alteration. If \( g_M \), \( g' \) are the Riemannian structures on \( M \) and \( N \), respectively (both \( C^\infty \) and complete; \( g_N \) is bounded), and if \( f \) is a \( C^\infty \) function on \( N \) which is everywhere strictly positive and has infimum zero, define for \( i = 1, 2, \xi_i \in T_M \), \( \eta_i \in T_N \),

\[
g(\xi, \eta) = (f(\xi))^2 \xi_2 \eta(\xi_1) + \xi_2 g'(\eta, \eta),
\]

The previous proof now shows that \( g \) is a complete bounded \( C^\infty \) Riemannian structure on \( M \times N \), which may be transferred by the given diffeomorphism to \( M \).

References


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Interpolation of weighted \( L_p \)-spaces

by

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Abstract. We characterize the interpolation spaces with respect to couples of weighted \( L_p \)-spaces. This is done in terms of the \( K \)-functional of Peetre. The main tool is a generalization of the theorem of Hardy, Littlewood, and Polya on doubly stochastic matrices.

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0. Introduction. We are concerned with the following problem:

Let \( \overline{A} = \{A, A\} = \{L_{p_{1}}, L_{p_{2}}\}, \overline{B} = \{B, B\} = \{L_{p_{1}}, L_{p_{2}}\}, 0 < p_{1}, p_{2}, p_{t} < \infty \), be two couples of weighted \( L_p \)-spaces assigned to some, not necessarily the same, measure spaces. Then characterize all interpolation spaces with respect to \( \overline{A}, \overline{B} \), i.e. all spaces \( A, B \) obeying

\[
T: A \rightarrow B \quad (\mu = 0, 1) \quad \text{implies} \quad T: A \rightarrow B,
\]

\[
\|T\|_{\omega_p(\mu, 1)} \leq c \max_{p_{1}, p_{2}} \|T\|_{\omega_p(\mu, \nu)},
\]

(0.1)

(operator norms). A both necessary and sufficient condition is found to be

\[
K(t, \alpha, \overline{A}, \overline{B}) \leq K(t, f; \overline{A}), \quad t > 0, \quad \text{implies} \quad \|g\|_{\omega_p} \leq C\|f\|_{\omega_p},
\]

("\( K \)-monotonicity"), where \( K \) is the Peetre functional

\[
K(t, f; \overline{A}) = \inf_{\|\phi\|_{\omega_p} + \|f\|_{\omega_p}}\left\{\int_{\overline{A}} |f(t, \alpha)|^2 \phi(\alpha) \right\}.
\]

(0.2)
Careing also about the constants $c$ and $C$, we find that (0.2) implies (0.1) with $c = O$ ("exactness"), while in the other direction (0.1) only implies (0.2) with $C \leq c \rho_{p_3, p_1}, 1 \leq \rho_{p_3, p_1} \leq 2$. Using instead a modified $K$, we obtain the equality $O = c$ in the latter direction (but in general not in the former). If $0 < p_3, p_1 \leq 1$, however, we get a complete characterization even in the exact sense.

Our results were announced in [30]. They generalize those of Mitjavide [19], Calderón [6], and Cotlar [7] for $(L_1, L_\infty)$; Lorentz–Shimogaki [17] for $(L_p, L_\infty)$ and $(L_\infty, L_p)$, $1 \leq p < \infty$; Sedaev–Semenov [35], and Sedaev [36] for $(L_{p_3}, L_{p_1}), 1 \leq p \leq \infty$. Recently, and independently of us, Oikawa [8] has treated the general Banach case $(L_{p_3}, L_{p_1}), 1 \leq p_3, p_1 \leq \infty$. His results are about the same as ours, although obtained by different methods.

The plan of the paper is as follows. In Section 1 we give some preliminaries on interpolation theory. In Section 2, also for background purposes, we briefly recapitulate the classical results and methods for $(L_1, L_\infty)$.

In Section 3 we introduce and study the modified $K$-functionals, mentioned above. In Section 4 we establish our main results, having as a consequence the equivalence between (0.1) and (0.2). In Section 5 we investigate the character of the constants $c$ and $C$. In Section 6 we illustrate how our methods apply to the interpolation of Lorentz spaces. In the Appendix, finally, we derive a matrix lemma which plays a major role throughout the paper. This lemma generalizes a classical result of Hardy, Littlewood, and Polya on doubly stochastic matrices.

1. General background. As a general source for the theory of interpolation we refer to [4]. Here we recapitulate a few notions only.

Let $A_1$ and $A_2$ be two (quasi-) Banach spaces, both continuously imbedded in some Hausdorff topological vector space $A$. They then constitute what is called a (quasi-) Banach couple, denoted by $A = (A_1, A_2)$. (For a quasi-Banach space only holds a quasi-triangle inequality: $\|f + g\| \leq c(\|f\| + \|g\|)$, $c > 1$.) To $A$ are associated the (quasi-) Banach spaces $\Sigma(A)$ and $\delta(A)$, defined by the (quasi-) norms

$$\|f\|_{\Sigma(A)} = \inf_{r > 1} \left\{ \|f\|_{L_r} + \|f\|_{L_\infty} \right\},$$

$$\|f\|_{\delta(A)} = \max(\|f\|_{L_1}, \|f\|_{L_\infty}).$$

Given two spaces $A$ and $B$, we denote by $\mathcal{L}^2(A; B)$ the set of continuous linear operators $T: A \rightarrow B$, provided with the operator (quasi-) norm

$$\|T\|_{\mathcal{L}^2(A; B)} = \sup\{\|Tf\|_B : \|f\|_A \leq 1\}.$$

Analogously, if $\tilde{A}$ and $\tilde{B}$ are two (quasi-) Banach couples we denote by $\mathcal{L}^2(\tilde{A}; \tilde{B})$ the set of operators $T: \Sigma(\tilde{A}) \rightarrow \Sigma(\tilde{B})$ such that $T \in \mathcal{L}^2(A_1, B_1)$.

$$\mu = 0, 1. \text{ Here and in the sequel, by abuse of notation, we do not distinguish between an operator and its restrictions to various subspaces.}$$

A (quasi-) norm on $\mathcal{L}^2(\tilde{A}; \tilde{B})$ is defined by

$$\|T\|_{\mathcal{L}^2(\tilde{A}; \tilde{B})} = \max_{r > 1} \|T\|_{\mathcal{L}^2(A_r; B_r)}.$$

Denote by $\mathcal{L}^2(A; B)$ and $\mathcal{L}^2(\tilde{A}; \tilde{B})$ the respective balls of radius $\epsilon$, i.e.,

$$T \in \mathcal{L}^2(A; B) \iff \|T\|_{\mathcal{L}^2(A; B)} < \epsilon,$$

$$T \in \mathcal{L}^2(\tilde{A}; \tilde{B}) \iff \|T\|_{\mathcal{L}^2(\tilde{A}; \tilde{B})} < \epsilon.$$

If $A = B$, $\tilde{A} = \tilde{B}$, we simply write $\mathcal{L}^2(A)$, $\mathcal{L}^2(\tilde{A})$, $\mathcal{L}^2(A; \tilde{A})$.

Now consider spaces $A$, $B$ such that, with continuous imbeddings,

$$\Sigma(\tilde{A}) \subseteq \mathcal{L}^2(\tilde{A}; B) \subseteq \mathcal{L}^2(A; B).$$

We say that $A$, $B$ are interpolation spaces with respect to $\tilde{A}$, $\tilde{B}$ if

$$T \in \mathcal{L}^2(\tilde{A}; B)$$

implies $T \in \mathcal{L}^2(A; B)$, $\|T\|_{\mathcal{L}^2(A; B)} \leq c\|T\|_{\mathcal{L}^2(\tilde{A}; B)}$, for some constant $c$, independent of $T$. By homogeneity this is equivalent to

$$\mathcal{L}^2(\tilde{A}; B) \subseteq \mathcal{L}^2(A; B).$$

We use the symbol $(\mathcal{I})$ for the condition above. Then $A \subseteq \mathcal{L}^2(\tilde{A}; B)$ implies $\|T\|_{\mathcal{L}^2(A; B)} \leq \|T\|_{\mathcal{L}^2(A; B)}$ for some constant $c$, independent of $T$. By homogeneity this is equivalent to

$$\mathcal{L}^2(\tilde{A}; B) \subseteq \mathcal{L}^2(A; B).$$

For some $c$. (In fact, by the closed graph theorem, $(\mathcal{I})$ is equivalent to the seemingly less restrictive condition $\mathcal{L}^2(\tilde{A}; B) \subseteq \mathcal{L}^2(A; B).$) When on places we want to emphasize a particular value of $c$, this is done by referring to the above inclusion as $(\epsilon \mathcal{I})$. The case $\epsilon = 1$ is important in many applications. $A$, $B$ are then called exact interpolation spaces with respect to $\tilde{A}$, $\tilde{B}$, obeying thus

$$\mathcal{L}^2(\tilde{A}; B) = \mathcal{L}^2(A; B).$$

If, in particular, $\tilde{A} = B$, $\tilde{B} = A$, we simply say that $A$ is an (exact) interpolation space with respect to $\tilde{A}$, this case will be referred to as the diagonal case.

Given $\tilde{A}$, $\tilde{B}$, the problem of characterizing the interpolation spaces $A$, $B$ thus can be treated on two different levels, a non-exact and an exact one. In the former case one does not distinguish between spaces having equivalent norms. However, it can be shown ([13], p. 74) that to $A$, $B$ obeying $(\mathcal{I})$ there exist equivalent spaces $A'$, $B'$ obeying $(\epsilon \mathcal{I})$. Hence, whichever level one works on, it suffices to consider the condition $(\epsilon \mathcal{I})$.

As was announced in the Introduction, the solution of the problem can be expressed in terms of the $K$-functional (0.3). To this end we define a quasi-order (relative to $\tilde{A}$, $\tilde{B}$)

$$g \leq f [k] \iff K(t, g, \tilde{B}) \leq K(t, f, \tilde{A}), t > 0.$$
Two spaces $A, B$ obeying (1.1) then are said to be $K$-monotonic with respect to $A, B$ if

\[ f \in A, \ g \leq f(K) \implies g \in B, \ \lVert g \rVert_A \leq C \lVert f \rVert_B \]

for some $C$, independent of $f, g$. Wanting to stress a particular value of $C$, they are said to be $(C; K)$-monotonic. If $C = 1$, i.e.

\[ f \in A, \ g \leq f(K) \implies g \in B, \ \lVert g \rVert_B \leq \lVert f \rVert_A, \]

we speak about exact $K$-monotonicity.

The applicability of these notions depends on

**Lemma 1.1.** If $g = Tf$ with $T \in \mathcal{L}(A; B)$, then $g \leq f(K)$.

**Proof.** The assumption $T \in \mathcal{L}(A; B)$ yields

\[ K(t, g; B) = K(t, Tf; B) \leq \inf_{f \in F} \lVert f \rVert_{\mathcal{L}(A; B)} = K(t, f; A), \]

i.e. $g \leq f(K)$. \[ \Box \]

We now immediately get

**Theorem 1.1.** (Riesz Representation) is a consequence of exact $K$-monotonicity.

**Proof.** Let $T \in \mathcal{L}(A; B)$. Then, by Lemma 1.1, $Tf \leq f(K)$ for every $f \in \mathcal{L}(A)$. Hence, if $A, B$ are exactly $K$-monotonic, $\lVert T \rVert_B \leq \lVert f \rVert_A$, i.e. $\lVert T \rVert_{\mathcal{L}(A; B)} \leq 1$. \[ \Box \]

A natural question is what extent and for which purposes the converse of this theorem holds true. To describe that situation, we say that $A, B$ are $K$-adequate if the conditions (Int) and $K$-monotonicity are equivalent. In particular, the norm $K(t, f; A)$ is called the $K$-adequate norm. (In [8] the term “Calderon couple” was used in the same sense.)

Not every couple is $K$-adequate. Counterexamples were given in [25], (even a finite-dimensional one) and in [8] $(L_1, W^1_1)$, where $W^1_1$ is a Sobolev space. What we intend to prove is that every two couples of weighted $L^p$-spaces are $K$-adequate, provided they refer to the same $p$'s.

Using this fact, a quite general class of $K$-adequate couples was discovered by Cwikel ([9]). In fact, for arbitrary $A = (A_0, A_1)$, every couple $(A_{0,1}, A_{1,2})$,

\[ 0 < a_1, b_1 < 1, \ 1 < b_1 \leq \infty, \]

is $K$-adequate. Here, as usual, $A_{a,b}$ stands for the space defined by

\[ \lVert f \rVert_{A_{a,b}} = \left( \int |t^{-a}K(t, f)|^b \, dt \right)^{1/b}. \]

This is an exact interpolation space, by virtue of Theorem 1.1. For special examples of $K$-adequate couples, see Remark 6.1. Below.

**2. Review of the case $[L_1, L_{\infty}]$.** Let us briefly recapitulate the results and methods in the case $[L_1, L_{\infty}]$. We restrict ourselves to the diagonal case and to functions defined on $R$. The exact interpolation spaces then can be characterized in the following two, seemingly different, ways:

**Theorem M** (Mitjagin [19]). Let $\mathcal{B}$ be the set of operators of the form

\[ Tf(x) = e(\gamma(x))g(x), \]

where $\gamma$ is a measure preserving bijection on $R$ and $|e(\gamma)| = 1$. Then $A$ is an exact interpolation space if and only if

\[ \mathcal{B} = \mathcal{L}(A). \]

**Theorem C** (Calderon [6], Cotlar [7]). $A$ is an exact interpolation space if and only if

\[ f \in A, \ \int \varphi^*(s)ds \leq \int \varphi^*(t)ds \text{ for } t > 0 \implies g \in A, \ \lVert g \rVert_A \leq \lVert f \rVert_A. \]

(Here $\varphi$ stands for the non-increasing rearrangement of $|f|$, i.e. $\varphi^*: R_+ \to R_+$ is non-increasing and $\text{mes} \{ x : \varphi^*(x) > y \} = \text{mes} \{ x : |f(x)| > y \}$, $t > 0$.)

It is well known that

\[ K(t, f; (L_1, L_{\infty})) = \int \varphi^*(s)ds. \]

Hence Theorem C may be restated as

**Corollary.** $[L_1, L_{\infty}]$ is exactly $K$-adequate.

The search for a link between these theorems, by approximation with simple functions one reduces to the $n$-dimensional case $\mathbb{R}^n = [0, 1]^n$. We note that a matrix $T = (t_{ij})$ belongs to $\mathcal{L}(A)$ if and only if

\[ [T]_{ij} \leq \lVert f \rVert_A \leq [T]_{ij} \]

(2.1)

As an important subset we recognize the doubly stochastic matrices $\mathcal{D}$, defined by

\[ t_{ij} \geq 0, \ \sum_{j} t_{ij} = 1 = \sum_{i} t_{ij} \]

$\mathcal{D}$ in turn contains as a subset the permutation matrices $\mathcal{P}$. There exists an extensive literature about doubly stochastic matrices, cf. the survey [18]. The two major theorems are

**Theorem B** (Birkhoff). $\mathcal{D} = \text{the convex hull of } \mathcal{P}$.

**Theorem HLP** (Hardy, Littlewood, Pölya). Let $f = (f_1, \ldots, f_n)$,
is covered. In fact, since \( a_0 \) and \( a_1 \) both are absolutely continuous with respect to \( \alpha \) and \( \alpha_1 \) by the Radon-Nikodym theorem there exist \( \alpha \) and \( \alpha_1 \) such that \( d\alpha = a_0^* d\alpha \), \( d\alpha_1 = a_1^* d\alpha \).

To make certain that \( X \) is a (quasi-)Banach couple, we claim that \( a_0 > 0 \) (\( \mu = 0,1 \)), or, equivalently, that the measures \( \alpha_0 \) and \( \alpha_1 \) are absolutely continuous with respect to each other. However, after minor modifications also the semi-normed case \( \alpha_0 > 0 \) (\( \mu = 0,1 \)) can be included in our treatment.

For couples like \( X \) it is advantageous besides \( K \) to work with

\[
K_{x}^{\pm}(t, f) = K_{x}^{\pm}(t, f; X) = \inf \left\{ \int f_{\alpha} d\alpha \mid f = f_{\alpha} \right\}
\]

cf. [22] where this functional was denoted by \( L \). Its usefulness depends on the formula

\[
K_{x}^{\pm}(t, f) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{|x-y|} \left( \int f_{\alpha}(\beta) d\beta \right) \left( \int f_{\alpha}(\gamma) d\gamma \right) d\alpha d\beta d\gamma
d\alpha d\beta d\gamma
\]

We shall also use another functional, similar to \( K_{x}^{\pm} \),

\[
X_{x}^{\pm}(t, f) = X_{x}^{\pm}(t, f; X) = \inf \left\{ \int f_{\alpha}(\beta) d\beta \mid f_{\alpha}(\beta) \in L_{x}(\beta, X) \right\}
\]

**LEMMA 3.1.** \( K_{x}^{\pm} \) and \( X_{x}^{\pm} \) are equivalent. More precisely,

\[
x_{\alpha}X_{x}^{\pm}(t, f) < K_{x}^{\pm}(t, f) < x_{\alpha}X_{x}^{\pm}(t, f),
\]

where \( x_{\alpha} = \inf \left( x_{\alpha} + y_{\alpha} \right) \). If \( 0 < p_{x} < 1 \), then \( K_{x} = X_{x}^{\pm} \).

**Proof.** It suffices to consider pointwise the scalar-valued integrands in (3.1) and (3.2). The right inequality then is obvious. The left one follows from

\[
f_{\alpha}^{p_{x}} + y_{\alpha}^{p_{x}} = \left( \frac{1}{x_{\alpha}} \right) f_{\alpha}^{p_{x}} + \left( \frac{1}{y_{\alpha}} \right) y_{\alpha}^{p_{x}} > \left( \frac{1}{x_{\alpha}} \right) f_{\alpha}^{p_{x}} + \left( \frac{1}{y_{\alpha}} \right) y_{\alpha}^{p_{x}},
\]

i.e.

\[
\inf_{t=\alpha_1} \left( f_{\alpha}^{p_{x}} + y_{\alpha}^{p_{x}} \right) > \inf_{t=\alpha_1} \left( f_{\alpha}^{p_{x}} + y_{\alpha}^{p_{x}} \right).
\]

Since \( x_{\alpha} = 1 \) if \( 0 < p_{x} < 1 \), \( K_{x} \) and \( X_{x}^{\pm} \) coincide in that case. Now let

\[
B = \left( L_{p_{x}}(\beta, X), L_{p_{x}}(\beta, X) \right) = \left( L_{p_{x}}(\beta, X), L_{p_{x}}(\beta, X) \right)
\]

be another (quasi-) Banach couple, assigned to the same \( B \). In analogy with (1.3), using \( K_{x}^{\pm} \) and \( X_{x}^{\pm} \) instead of \( K_{x} \) and \( X_{x}^{\pm} \), respectively, we obtain the definitions of \( K_{x}^{\pm} \) and \( X_{x}^{\pm} \).
monotonicity, with the different prefixe. In analogy with the $E$-case we also define the notions of (exact) $K_\mathcal{E}$ and $\mathcal{E}$-monotonicity.

**Lemma 3.2.** Generally holds:

(i) The concepts of $K$, $K_\mathcal{E}$ and $\mathcal{E}$-monotonicity coincide.

In particular cases moreover hold:

(ii) If $1 < p \leq p_1 < \infty$, then exact $K$ and exact $K_\mathcal{E}$-monotonicity coincide.

(iii) If $0 < p < p_1 < 1$, then exact $K_\mathcal{E}$ and exact $\mathcal{E}$-monotonicity coincide.

This lemma is an immediate consequence of **Lemma 3.3.** Generally equivalent:

(i) The following statements are equivalent:

\[
(3.3) \quad g \in \mathcal{E}[K] \quad \text{for some } c > 0,
\]

\[
(3.4) \quad g \in \mathcal{E}[K_\mathcal{E}] \quad \text{for some } c > 0,
\]

\[
(3.5) \quad g \in \mathcal{E}[\mathcal{E}] \quad \text{for some } c > 0.
\]

In particular cases moreover hold:

(ii) If $1 < p \leq p_1 < \infty$, then $g \in \mathcal{E}[K]$ iff $g \in \mathcal{E}[K_\mathcal{E}]$.

(iii) If $0 < p < p_1 < 1$, then $g \in \mathcal{E}[K_\mathcal{E}]$ iff $g \in \mathcal{E}[\mathcal{E}]$.

**Proof.** Part (iii) and the equivalence between (3.4) and (3.5) are immediate consequences of Lemma 3.1. What remains are the statements about $K$ and $K_\mathcal{E}$. To verify them, we use the functional

\[
E(t) = E(t, f) = \inf_{s \in \mathbb{R}} \int f \leq \lambda p_2.
\]

There is a close connection between $K$ and $E$, cf. [24], notably Proposition 4.4. In fact,

\[
(3.6) \quad K(t) = \mathcal{E}[K_\mathcal{E}, f] = \inf_s [s + tE(s)],
\]

\[
(3.7) \quad K_\mathcal{E}(t) = K(t, f) = \inf_s [s^{\mathcal{E}} + tE^\mathcal{E}(s)] = \inf_s [s + tE^\mathcal{E}(s^{\mathcal{E}})].
\]

Put

\[
E_\mathcal{E}(s) = E^\mathcal{E}(s^{\mathcal{E}}).
\]

The formulas (3.6) and (3.7) then state that $K$ and $K_\mathcal{E}$ are Legendre transforms of $E$ and $E_\mathcal{E}$, respectively.

Beginning with (ii), we assume that $1 < p \leq p_1 < \infty$. Then, as is readily verified, $E$ is convex and decreasing. This is in fact the case with $E_\mathcal{E}$ too. To verify that, we use in order the concavity of $x \mapsto x^{p_1}$ and the decreasingness of $E$, the convexity of $E$, and the convexity of $x \mapsto x^{p_1}$, thus obtaining, with $\lambda_1 + \lambda_2 = 1$,

\[
E_\mathcal{E}(\lambda_1 s_1 + \lambda_2 s_2) = E^\mathcal{E}(\lambda_1 s_1^{p_1} + \lambda_2 s_2^{p_1}) \leq \lambda_1 E^\mathcal{E}(s_1^{p_1}) + \lambda_2 E^\mathcal{E}(s_2^{p_1})
\]

\[
= \lambda_1 (E_\mathcal{E}(s_1)) + \lambda_2 (E_\mathcal{E}(s_2)).
\]

This proves the convexity. Under these circumstances, making an inverse Legendre transformation in (3.8) and (3.7), we get

\[
E(t) = \sup_t \left( \frac{E(t)}{t} - \frac{s}{t} \right),
\]

\[
E_\mathcal{E}(t) = \sup_t \left( \frac{E_\mathcal{E}(t)}{t} - \frac{s}{t} \right).
\]

Now by virtue of (3.6) (3.9) we have the chain of equivalences

\[
K(t, \frac{s}{t}) \leq K^\mathcal{E}(t, \frac{s}{t}), \quad t > 0 \Rightarrow E(t, \frac{s}{t}) \leq E^\mathcal{E}(t, \frac{s}{t}), \quad s > 0
\]

\[
\Rightarrow E_\mathcal{E}(t, \frac{s}{t}) \leq K_\mathcal{E}(t, \frac{s}{t}), \quad t > 0 \Rightarrow E_\mathcal{E}(t, \frac{s}{t}) \leq K_\mathcal{E}(t, \frac{s}{t}), \quad t > 0.
\]

This proves (ii).

Turning to the general case $0 < p < p_1 < \infty$, $E$ and $E_\mathcal{E}$ need not be convex. Consequently, the inverse transformation applied to (3.8) and (3.7) does not, in general, give back $E$ and $E_\mathcal{E}$. Instead we get their greatest convex minorants $E^\mathcal{E}$ and $E^\mathcal{E}_\mathcal{E}$. However, taking into account that

\[
E^\mathcal{E}(t) \leq E(t) \leq 2E^\mathcal{E}(s),
\]

\[
E^\mathcal{E}_\mathcal{E}(t) \leq E_\mathcal{E}(t) \leq 2E^\mathcal{E}_\mathcal{E}(s),
\]

by an argument similar to that above the equivalence between (3.3) and (3.4) can be proved. We omit the details. ■

**Remark 3.1.** The definitions of $K$, $K_\mathcal{E}$ and $E_\mathcal{E}$ have significance for general quasi-Banach couples. With no changes the proof above applies to this general situation. Part (ii) of the lemma then generalizes a result of [34] for the case $p_1 = p_2$.

The next section will motivate a closer study of the relationship between $K_\mathcal{E}$ and $\mathcal{E}$-monotonicity. Such investigations are then carried out in Section 5. They are based on the following integral representation of $K_\mathcal{E}$.

**Lemma 3.4.** Let $1 < p_1 < \infty$. Put

\[
\Delta(t) = \frac{p_1}{p_0} \left( 1 - \frac{p_1}{p_0} \right)^{p_1-1}
\]
Then
\[
\inf_{a \geq 0} \{a^p + t P_1 \} = \frac{1}{p} \int_{a}^{\infty} \min \{ \frac{P_1 a^{p-1}}{p}, s^{p-1} \} \, ds \\
= \frac{1}{p} \int_{a}^{\infty} \min \{ a^{p}, t A_2(\sigma \, a^{p}) \} \, ds,
\]
(3.12)
\[
K_2(t, f) = \frac{1}{t} \int \mathcal{X}_\sigma(t A_2(\sigma, f)) \, d\sigma.
\]

Proof. Here (3.12) is a consequence of (3.11) and (3.1). In fact,
\[
K_2(t, f) = \frac{1}{t} \int \mathcal{X}_\sigma(\int (f a^{p} + t f a^{p})) \, d\sigma = \frac{1}{t} \int \mathcal{X}_\sigma(\int (f a^{p} + t f a^{p})) \, d\sigma \, ds = \frac{1}{t} \int \mathcal{X}_\sigma(\int (f a^{p} + t f a^{p})) \, d\sigma = \frac{1}{t} \int \mathcal{X}_\sigma(\int (f a^{p} + t f a^{p})) \, d\sigma.
\]

In verifying (3.11), without restrictions we may assume \(a = 1\). Put
\[
k(s) = \inf_{s^{p} + t P_1} = \inf_{s^{p} + t (1 - \sigma)^{p_1}},
\]
(3.13)

Being an infimum of posite concave functions on \(R_+\), vanishing at the origin, \(k\) itself has these properties. Moreover \(k(s) < 1\), \(s \to \infty\). By partial integrations one then verifies the formula
\[
k(t) = - \frac{1}{t} \min \{ s, t \} \, dk(s),
\]
(3.14)

cf. formula (A.5) of the Appendix and the reference given there.

We now restrict ourselves to the case \(p_1, p_2 > 1\). The remaining cases need some minor modifications, omitted here. To get an expression for \(k'\) in (3.14), first consider \(k\). A derivation yields that the infimum in (3.13) is attained for \(\sigma\) obeying
\[
p_1 \sigma^{p-1} = P_1 (1 - \sigma)^{p_1 - 1},
\]
(3.15)

This formula defines a bijection between \(0 < s < \infty\) and \(0 < \sigma < 1\). Expressing \(k\) and \(k'\) in terms of \(\sigma\), we get
\[
k(s) = s^{p} + s (1 - \sigma)^{p_1},
\]
\[
k'(s) = \frac{d k}{d s} = \frac{P_1 \sigma^{p-1} - \sigma (1 - \sigma)^{p_1 - 1} - \frac{d s}{ds} (1 - \sigma)^{p_1}}{d s} = (1 - \sigma)^{p_1}.
\]

Thus, making the change of variables (3.15) in (3.14), we obtain
\[
k(t) = - \frac{1}{t} \int \min \{ t P_1 (1 - \sigma)^{p_1 - 1}, t \} \, d(1 - \sigma)^{p_1} = \frac{1}{t} \int \min \{ t P_1 (1 - \sigma)^{p_1 - 1}, t \} \, d(1 - \sigma)^{p_1},
\]
i.e. the first equality in (3.11). Making another change of variables, we get the second one
\[
k(t) = - \frac{1}{t} \int \min \{ 1, t P_1 (1 - \sigma)^{p_1 - 1}, t \} \, d(1 - \sigma)^{p_1} = \frac{1}{t} \int \min \{ t P_1 (1 - \sigma)^{p_1 - 1}, t \} \, d(1 - \sigma)^{p_1}.
\]

4. The general case \((L_{p_1}, L_{p_2})_\beta, 0 < p_1, p_2 < \infty\). As in the preceding section, when not otherwise stated, let
\[
\begin{align*}
\tilde{A} = (L_{p_1}, \beta, X, L_{p_2}, (a, \beta, X)) &= (L_{p_1}, L_{p_2}), \\
\tilde{B} = (L_{p_1}, \beta, X, L_{p_2}, (\beta, \beta, X)) &= (L_{p_1}, L_{p_2}).
\end{align*}
\]

A sufficient condition for \(A, B\) to be interpolation spaces with respect to these couples was given in Theorem 1.1 above. However, for comparison with the necessary condition to be derived below, it is convenient to restate it by means of \(K_2\) instead of \(K\).

**Theorem 4.1.** \((Ex Int)\) is a consequence of exact \(K_2\)-monotonicity.

The proof is analogous to that of Theorem 1.1, depending on Lemma 4.1. If \(g = T_\beta \in X\), then \(g \in F K_2\).

Below a necessary condition is derived in two (overlapping) cases:
(i) \(1 < p_1, p_2 < \infty\) with arbitrary measure spaces,
(ii) \(0 < p_1, p_2 < \infty\) with the following restrictions imposed on the measures:

- (M1) \((\beta, \beta, X)\) is non-atomic,
- (M2) For every \(D \in \beta, E \in \beta\) with \(a(D) + \beta(E) < \infty\), the normalized measure spaces \(a(a(D), \beta(D), D)\) and \(\beta(\beta(E), \beta(E), E)\) are isomorphic.

Here the latter condition means that there exists a measurable bijection \(\pi: D \to E\) such that for every measurable set \(D' \subset D,
\[
\frac{a(D')}{a(D)} = \frac{\beta(D')}{\beta(E)},
\]

\[
\frac{a(D')}{a(D)} = \frac{\beta(D')}{\beta(E)}.
\]
We note that if $X = Y = R, \alpha = \beta =$ Lebesgue measure, (M.1) and (M.2) are both obeyed. Consequently the way they are whenever $X$ and $Y$ are isomorphic to $R$ (or some subinterval of $R$). This in turn is guaranteed by their being separable, non-atomic and $\sigma$-finite (cf. e.g. [12], p. 170). Also note that if $X$ is discrete, then (M.2) is automatically fulfilled.

The case $p = \infty$ is commented upon in Remark 5.1 below.

We are now prepared to state the main result:

**Theorem 4.2.** Under the hypotheses of (i) or (ii), (ExInt) implies exact $\mathcal{X}_\varphi$-monotonicity.

Theorem 4.1 and 4.2 now yield, by virtue of Lemma 3.2.

**Corollary 4.1.** $A, B$ are $K$-adequate.

**Corollary 4.2.** If $0 < p_0, p_1 < 1$ and (M.1), (M.2) are obeyed, then $A, B$ are exactly $\mathcal{X}_\varphi$-adequate.

Theorem 4.2 is a consequence of the following lemma. When combined with Lemma 4.1 we get the generalization of Theorem HLP inquired for at the end of Section 2, cf. also Remark 4.1 below.

**Lemma 4.2.** Under the hypotheses of (i) or (ii), if $g < f[X_\varphi]$ then to every $q > 1$ there exists $T \in \mathcal{L}(A, B)$ such that $T \varphi = g$.

**Proof of Theorem 4.2.** Let $A, B$ obey (ExInt) and $g < f[X_\varphi]$. For $q > 1$, let $T_q$ be given by the lemma. Then

$$\|g\|_q = \|T_q f\|_q < \|f\|_q.$$  

Since this holds for any $q > 1$, $\|g\|_q < \|f\|_q$. Hence $A, B$ are exactly $\mathcal{X}_\varphi$-monotonic.

Before proving Lemma 4.2 we introduce some more notations. Let $\mathcal{X}_0$ denote the characteristic function of the set $D$. For arbitrary functions $f$ we write $f_D = f_{\mathcal{X}_0}. We consider functions $f = \sum f x_i x_i, i \in \mathcal{X},$ with the property that the values $(f_i)$ can be renumbered in increasing order. Such functions are called *elementary*, or if $I$ is finite, *simple*.

**Proof of Lemma 4.1.** The disposition is as follows: In Step 1 we interpret the condition $g < f[X_\varphi]$ for certain elementary functions $f, g, \varphi, h, (\mu = 0, 1).$ In Steps 2(i) and 2(ii), for such functions we construct the in the respective cases the operator $T.$ Here we even get $T \in \mathcal{L}(A, B).$ In Step 3 is performed the passage from elementary to arbitrary functions. Steps 1 and 3 are common for (i) and (ii).

**Step 1.** Suppose that $f, g, \varphi, h, (\mu = 0, 1)$ are are constant. Hence

$$f = \sum f_i x_i = \sum f_i, g = \sum g_i x_i = \sum h_i.$$  

Without restrictions we may assume that $\alpha, \alpha, b_i, h_i$ are constant on $F_i (i \in I)$ and $G_j (j \in J)$ respectively. For the $\mathcal{X}_\varphi$-functional we then have

$$\mathcal{X}_\varphi(t, f_1, \varphi_1, \varphi_2) = \int \min \{\|f_0\|^p_\mu, t\|f_0\|^p_\mu\} \, dt$$

$$= \sum \int \min \{\|f_0\|^p_\mu, t\|f_0\|^p_\mu\} \, dt$$

$$= \sum \int \min \{\|f_0\|^p_\mu, t\} \, dt + \int \|f_0\|^p_\mu \, dt$$

$$= \sum \min \{\|f_0\|^p_\mu, t\} \|f_0\|^p_\mu \, dt,$$

analogously for $\mathcal{X}_\varphi(t, g_1, \varphi_1, \varphi_2).$ The condition $\varphi < f[X_\varphi]$ thus means

$$\sum \min \{\|g_0\|^p_\mu, t\} \|g_0\|^p_\mu \, dt < \sum \min \{\|f_0\|^p_\mu, t\} \|f_0\|^p_\mu \, dt, \quad t > 0.$$  

Now suppose that each set of plane vectors $(\|x_0\|_\mu, \|y_0\|_\mu), (i \in I),$ and $(\|z_0\|_\mu, \|y_0\|_\mu, \|y_0\|_\mu), j \in J,$ can be arranged in non-decreasing order with respect to the order relation $\in$ of the Appendix. Then Lemma A.2 applies. It provides us with numbers $\theta_0 \geq 0$ such that

$$(4.1) \quad \sum \theta_0 \varphi_1 = 1 \quad (j \in J).$$

$$(4.2) \quad \sum \theta_0 \varphi \|g_0\|^p_\mu \leq \|g_0\|^p_\mu \|z_0\|^p_\mu \quad \mu = 0, 1, (i \in I).$$

These relations shall now be used in constructing the operators.

**Step 2(i).** Under the hypothesis (i), for $\varphi \in \mathcal{X}(A, B)$ we define

$$T \varphi = \sum \theta_0 \left(\frac{1}{a(F_0)} \int_{F_0} \varphi \, df \right) g_0.$$  

Then obviously, $T \varphi = g$. We prove that $T \in \mathcal{L}(A, B).$ To begin with, (4.1) and the convexity of $a \mapsto a^2$ ($\mu = 0, 1$) yield

$$(4.4) \quad \|T \varphi\|_\mu = \sum \theta_0 \left(\frac{1}{a(F_0)} \int_{F_0} \varphi \, df \right) \|g_0\|^p_\mu \|z_0\|^p_\mu 

\leq \sum \theta_0 \left(\frac{1}{a(F_0)} \int_{F_0} \varphi \, df \right) \|z_0\|^p_\mu \|z_0\|^p_\mu 

\mu = 0, 1.$$  

But here, again by the convexity and the fact that $\varphi$ and $\varphi_0 (\mu = 0, 1)$ are constant on each $F_0 (i \in I),$

$$\left(\frac{1}{a(F_0)} \int_{F_0} \varphi \, df \right) \|g_0\|^p_\mu \leq \frac{1}{a(F_0)} \int_{F_0} \varphi \, df \|g_0\|^p_\mu 

= \frac{1}{a(F_0)} \int_{F_0} \varphi \, df \|g_0\|^p_\mu = \frac{1}{a(F_0)} \int_{F_0} \varphi \, df \|g_0\|^p_\mu 

= \|g_0\|^p_\mu \|z_0\|^p_\mu \mu = 0, 1.$$  

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Inserting this in (4.3), we get, by virtue of (4.2),
\[ \|T\psi\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} \leq \sum_{\xi} \|\psi_{\xi}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} \sum_{\tau} q_{\xi} \|q_{\tau}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} \|f_{\tau}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}}, \]
\[ \leq \sum_{\xi} \|\psi_{\xi}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} = \|\psi\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}}, \quad (\mu = 0, 1). \]

Hence \( T \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}; \mathcal{B}) \).

Step 2(ii). We make use of the following fact: Let \( T \) be a mapping defined for all characteristic functions \( \chi_{\mathcal{D}} \), \( D \in \mathcal{A} \), with values in \( \sum_{\mathcal{B}} \), such that
\[ T_{\chi_{\mathcal{D}}} \circ \chi_{\mathcal{E}} = T_{\chi_{\mathcal{D}}} + T_{\chi_{\mathcal{E}}} \quad \text{if} \quad D \cap E = \emptyset, \]
\[ \supp T_{\chi_{\mathcal{D}}} \cap \supp T_{\chi_{\mathcal{E}}} = \emptyset \quad \text{if} \quad D \cap E = \emptyset, \]
\[ \|T_{\chi_{\mathcal{D}}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} = \|\chi_{\mathcal{D}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} \quad (\mu = 0, 1). \]

Then, extending \( T \) by linearity to simple functions and thereafter by continuity to \( \sum_{\mathcal{B}} \), we obtain an operator \( T \in \mathcal{L}_{\mathcal{B}}(\mathcal{F}; \mathcal{B}) \). Thus, to verify the lemma it suffices to construct \( T \) obeying (4.4)–(4.6) and, in addition, \( T f = g \). (Also note that \( T \) constructed this way gets the property \( \supp T \cap \supp \psi = \emptyset \) if \( \supp \psi \cap \supp \sigma = \emptyset \).)

We begin by cutting the sets \( G_{j} \) (\( j \in J \)) into pieces. By virtue of (M.1) and (4.1) there exist sets \( G_{j} \) such that, disjointly,
\[ G_{j} = \bigcup_{i} G_{j_{i}} \text{ with } \beta(G_{j_{i}}) = \beta_{j_{i}}(j_{i}) \quad (i \in I, j \in J) \]
(cf. [13], p. 174). Since \( g \) and \( \beta_{j} \) (\( \mu = 0, 1 \)) are constant on \( G_{j} \), it follows that
\[ \theta_{j} \int_{G_{j_{i}}} |g| d\beta = \int_{G_{j_{i}}} |g|| \beta d\beta, \]
i.e.
\[ \theta_{j} \|g_{j_{i}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} = \|g_{j_{i}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}}, \quad (\mu = 0, 1, i \in I, j \in J). \]
Hence, by (4.3),
\[ \sum_{j} \|g_{j_{i}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} \leq \|g_{j_{i}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} \quad (\mu = 0, 1, i \in I). \]
Now put
\[ G_{j} = \bigcup_{i} G_{j_{i}} \quad (i \in I). \]
These sets are pairwise disjoint. Thus, defining \( T \) so that
\[ \supp T_{\chi_{\mathcal{D}}} = \bigcup_{i} G_{j_{i}} \quad \text{if} \quad D \subseteq F_{i}, \]
it suffices to verify (4.4)–(4.6) for \( D \subseteq F_{i} \) with \( i \in I \) fixed. Since \( f \) is constant on \( F_{i} \), we may use \( f_{D} \) instead of \( T_{\chi_{\mathcal{D}}} \).

To this end, for fixed \( i \), let \( n_{D} : F_{i} \to G_{j} \) be chosen in accordance with (M.2). For \( D \subseteq F_{i} \), put
\[ n_{D} = \frac{\alpha(D)}{\alpha(F_{i})} \frac{\beta_{j}(n_{D})}{\beta(n_{D})} \quad (j \in J). \]
Then
\[ \|T n_{D}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} = \|n_{D}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}}, \quad \|g_{j_{i}} n_{D}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} = \|g_{j_{i}} n_{D}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}}. \]
Now define (cf. Fig. 4.1)
\[ T f_{D} = \sum_{j} n_{D}. \]
Then (4.4) and (4.5) are satisfied. To verify (4.6), by virtue of (4.7) and (4.8) we have
\[ \|T f_{D}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} = \sum_{j} \|g_{j_{i}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}} \leq \|g_{j_{i}}\|_{\mathcal{L}^{p}_{\mathcal{B}}_{\mathcal{P}}}, \quad (\mu = 0, 1). \]
Finally, since \( T f_{D} = g_{F_{i}} \), we also get \( T f = g \). This proves the assertion.

\[ \text{Fig. 4.1} \]

Step 3. Now consider the general case, dropping thus the assumption the functions to be elementary. In order to reduce to that case, for \( q > 1 \) and positive functions \( \varphi \) we define the discretizing operator \( \varphi : \varphi^{k} \) by
\[ \varphi^{k}(x) = \varphi^{k} \quad \text{if} \quad \varphi^{k} \leq \varphi(x) < \varphi^{k+1} \quad (k \in \mathbb{Z}). \]
Then
\[ \varphi^{k} \leq \varphi \leq \varphi^{k+1}. \]
Let \( A \) be a linear functional on \( L_\infty \), of norm one, such that \( A(\varepsilon_n) = \lim a_n \) if this limit exists. Let \( \lambda_n \) be the operator obtained from Lemma 4.2 if \( \varepsilon = 1 + 1/n \). For \( \varphi \in \tilde{X}(\tilde{A}; \tilde{B}) \) one verifies that

\[
\lambda_n(\tilde{B}) = A \left( \frac{1}{1+1/n} \int E \tilde{A} \varphi \tilde{B} \right)
\]

defines a measure on \( Y \), absolutely continuous with respect to \( \beta \). Let \( T_\varphi = d\lambda_n/d\beta \), the Radon-Nikodym derivative. Then

\[
\int \tilde{E} T_\varphi \varphi \tilde{B} = A \left( \frac{1}{1+1/n} \int \tilde{E} \tilde{A} \varphi \tilde{B} \right).
\]

In particular, with \( \varphi = f \),

\[
\int \tilde{E} T_\varphi f \tilde{B} = A \left( \frac{1}{1+1/n} \int \tilde{E} \tilde{A} f \tilde{B} \right) = A \left( \frac{1}{1+1/n} \int \tilde{E} g \tilde{B} \right) = \int \tilde{E} g \tilde{B}.
\]

Hence \( T_\varphi = g \). To prove that \( T \in \tilde{L}_2(\tilde{A}; \tilde{B}) \), consider simple functions \( \varphi \in \tilde{L}_2(\tilde{A}; \tilde{B}) \), where \( 1/p_2 + 1/p_\mu = 1 \) \((\mu = 0, 1)\). Then by virtue of (4.10)

\[
\int \tilde{E} T_\varphi \varphi \tilde{B} = A \left( \frac{1}{1+1/n} \int \tilde{E} \tilde{A} \varphi \tilde{B} \right).
\]

Hence

\[
\int \tilde{E} T_\varphi \varphi \tilde{B} \leq \sup_n \left( \frac{1}{1+1/n} \int \tilde{E} \tilde{A} \varphi \tilde{B} \right) \leq \sup_n \left( \frac{1}{1+1/n} \int \tilde{E} \tilde{A} \varphi \tilde{B} \right)
\]

It then follows that \( T \in \tilde{L}_2(\tilde{A}; \tilde{B}) \), which concludes the proof.

We do not know whether Lemma 4.3 admits a generalization to \( \varepsilon = 1 \) also in the quasi-Banach case (ii).

**Example 4.1.** Let \( \mathbb{R}^n \rightarrow \mathbb{R} \) be homogeneous, i.e. \( H(\lambda t^a, \lambda b^a) = \lambda H(t^a, t b^a), \lambda > 0 \). We say that \( H \) is an exact \( \beta \)-interpolation function if, for arbitrary \( A = (L_{p_\beta}, L_{q_\beta}), B = (L_{p_\beta}, L_{q_\beta}) \),

\[
\varphi = T_\varepsilon \quad \text{with} \quad T \in \tilde{L}_2(\tilde{A}; \tilde{B}) \quad \text{implies} \quad \int \tilde{E} (gb^\varepsilon \varphi, gb^\varepsilon \varphi) \tilde{B} \leq \int \tilde{E} (f^\varepsilon \varphi, f^\varepsilon \varphi) \tilde{B}.
\]

We try to characterize such functions \( H \). There are also the analogous non-exact problems, with a constant \( c \) inserted in (4.11). In that case the interpolation functions are characterized by being equivalent to the
exact ones. Since the functional
\[(4.12)\]
f \mapsto \int H (|f_{0} + f_{1}|^{p_{0}} + |f_{0} + f_{1}|^{p_{1}}) \, dx
\]
only accidentially defines a quasi-norm, in general in (4.11) we do not
deal with interpolation spaces. However, if \( p_{0} = p_{1} = p \) (the most
interesting case), (4.12) reads
\[f \mapsto \int |f|^{p} H (a_{0}^{p}, a_{1}^{p}) \, dx,
\]
i.e. it is the \( p \)-th power of a weighted \( L_{p} \)-norm. This case has been
extensively studied, cf. notably [19], [24], [25], and [27]. In the case \( p_{0} \neq p_{1}, \)
claiming on \( H \) some kind of convexity property, by means of (4.12) it is possible
to define a quasi-normed space \( O_{H} \) (a “weighted Orlicz space”) by
\[\inf \left\{ \epsilon \int H \left( \frac{|f_{0}|}{\epsilon}, \frac{|f_{0} + f_{1}|}{\epsilon} \right) \, dx : \epsilon \leq 1 \right\}.
\]
In this way we thus obtain exact interpolation spaces. For the case \( a_{0}, b_{0} = 1
\) (\( \mu = 0, 1 \), see [22]).

Returning to (4.11), by the homogeneity of \( H \) we may write \( H (a^{p}, b^{p}) \)
\[= a^{p} H (a^{p}, b^{p})\] with \( h (a) = H (a, 1) \). Hence there is a one-to-one correspon-
dence between the exact \( \beta \)-interpolation functions and a class, denoted by \( \mathcal{F}_{\beta} \), of non-negative functions \( h \) on \( R_{+} \). As in the Appendix, let \( \mathcal{P}_{+} \)
be the set of non-negative concave functions on \( R_{+} \). For \( 1 \leq p_{0}, p_{1} < \infty \),
we define a class \( \mathcal{F}_{\beta}^{\infty} \) consisting of functions of the form
\[h (a) = \int A_{\beta} (a) \varphi (|a|^{\beta} (a)) \, dx,
\]
where \( \varphi \in \mathcal{P}_{+}, \varphi (a) = o \max (1, t) \) as \( t \to 0 \) or \( \infty \), and \( A_{\beta} \) is defined by
\[(3.10). \] Note that here, by (3.5),
\[\varphi (a) = \min \{ a, 1 \} \, d\omega (t), \]
with a positive measure \( \omega \).

We are now able to prove
\[\mathcal{F}_{\beta}^{\infty} \subset \mathcal{F}_{\beta} \subset \mathcal{P}_{+} \,.
\]
In the cases \( p_{0} = p_{1} \) and \( p_{0} \neq p_{1}, a_{0}, b_{0} = 1 \) (\( \mu = 0, 1 \)), these inclusions
are well known, cf. the references given above. (One verifies that \( p_{0} = p_{1} \),
our class \( \mathcal{F}_{\beta}^{\infty} \) coincides with that of, e.g., (12).) It will be apparent from
the proof below, using also Lemma 3.2(iii), that if \( 0 < p_{0}, p_{1} \leq 1 \), then \( \mathcal{F}_{\beta} \)
\(= \mathcal{P}_{+} \). If \( p_{0} = p_{1} \), we thus obtain a result of [21]. By means of R-
example 5.1 below one easily constructs an example showing that if \( 1 \leq p_{0}, p_{1} \)< \( \infty \) the right inclusion in (4.14) is strict. Concerning the left one, in [11]
Then
\[ X_p(t, \sigma_p) = \min \{ f \in L^p \cap \text{Int}, \| f \|_p \leq \| f \|_p \} \]
\[ = \min \{ f \in L^p \cap \text{Int}, \| f \|_p \leq \| f \|_p \} \]
\[ = \min \{ f \in L^p \cap \text{Int}, \| f \|_p \leq \| f \|_p \} \]

If follows that
\[ \sigma_p \leq \max \{ \sigma^n_p, \sigma^p_1 \} \frac{\| f \|_p}{\| f \|_p} \]
\[ \sigma_p \leq \max \{ \sigma^n_p, \sigma^p_1 \} \frac{\| f \|_p}{\| f \|_p} \]

Hence, by virtue of Theorem 4.2, for any \( \varepsilon \)-interpolation space \( A \) holds
\[ (4.16) \quad \| \sigma_p \|_A \leq \varepsilon \max \{ \sigma^n_p, \sigma^p_1 \} \frac{\| f \|_p}{\| f \|_p} \]

This should be compared with the results for interpolation of weak type operators, i.e., operators \( T : L^p \to L^{p, \infty} \), where \( \| f \|_{p, \infty} = (\int |f|^p)^{1/p} \). Following Boyd [3], we define
\[ u(s) = \sup f \| f \|_p \| f \|_p, \quad \sigma_p = \lim log u(s)/log \varepsilon, \]
\[ \beta_p = \lim log u(s)/log s \]

Then by (4.16) a necessary condition for \( A \) to be a (strong) interpolation space is
\[ 1/p_1 < \beta_p < \sigma_p < 1/p_1, \quad \text{where} \quad 0 < p_0 < p_1 < \infty. \]

(Here the information about \( \varepsilon \) is lost.) In the Banach case \( 1 < p_0 < p_1 < \infty \), under additional, rather restrictive, assumptions on \( A \), (4.17) (or (4.16)) is also sufficient for \( A \) to be an interpolation space, cf. [27] with the addendum made in [15]. For weak interpolation, on the other hand, by [3] a both necessary and sufficient condition is, generally,
\[ 1/p_1 < \beta_p < \sigma_p < 1/p_1 \quad \text{if} \quad 1 < p_0 < p_1 < \infty. \]

5. On the gap between the necessary and the sufficient conditions. In Section 4 was proved that \( K_p \)-monotonicity is sufficient and exact \( X_p \)-monotonicity necessary for the condition (Ex, Int) to be satisfied. We now investigate quantitatively the relationship between the two kinds of monotonicity. When not otherwise stated, let \( \bar{A} \) and \( \bar{B} \) denote the same couples as in Section 4. However, we now restrict ourselves to \( 1 \leq p_0 < p_1 < \infty \). Since \( \bar{p} = (1, 1) \) was fully covered by Corollary 4.5, that case is excluded too.

It is natural first to compare the quasi-orders \( g \leq f(K_p) \) and \( g \leq f(X_p) \).

One result in this direction can be derived from Lemma 3.1, with the deviation expressed in terms of \( \alpha_p \). However, this yields too rough an estimate. In fact, since \( \alpha_p (p_0, p_1) \to 0 \), \( p_1 \to \infty \), measured this way the gap between the necessary and sufficient conditions would increase unboundedly as \( p \to \infty \). In the limit case \( \bar{p} = (1, \infty) \), however, we have reason to expect Theorem C of Section 2, at least to within norm equivalences.

The following lemma settles these objections. In order to obtain there the best constant, not just an estimate, we suppose that the measure spaces are not purely atomic with finitely many atoms and that if \( p_0 = p_1 \), the weight functions are non-equivalent.

**Lemma 5.1.** Under the above assumptions we have
\[ (5.1) \quad g \leq f(X_p) \quad \text{implies} \quad g \leq f(K_p), \]
\[ (5.2) \quad g \leq f(K_p) \quad \text{implies} \quad g \leq f(X_p), \]

where \( Y_p \), the smallest constant possible, is determined by
\[ (5.3) \quad \inf_{\varphi \in \varphi} (\alpha_{p_0} \varphi + \beta_{p_1} \varphi) = 1. \]

Here, generally speaking,
\[ (5.4) \quad 1 < Y_p < 2. \]

**Proof.** The main tool is Lemma 3.4, expressing \( K_p \) as a certain mean value of \( X_p \)-functions. As a first consequence, we immediately obtain (5.1).

Less trivial is (5.2). There the best constant \( Y_p \) can be expressed as
\[ (5.5) \quad Y_p = \sup_{\varphi \in \varphi} (\varphi_0 + \varphi_1), \quad \text{where} \quad \varphi_0 = \inf \{ \varepsilon \mid g \leq f(X_p) \}. \]

After approximation it suffices to consider simple functions \( f, g, a_0, b_1 \), \( (\mu = 0, 1) \), \( X_p(t, f) \) and \( X_p(t, g) \) are then piecewise linear. By the definition of \( \alpha_p \), for some \( r > 0 \) we have
\[ (5.6) \quad X_p(t, g) = X_p(t, \alpha_p g). \]

Here
\[ (5.7) \quad X_p(t, \alpha_p) = \alpha_p X_p(t, \alpha_p g, f). \]

Let \( p_0 > p_1 \). Consider the points
\[ P : \{ (t, \alpha_p g), f(t, \alpha_p g) \}, \quad \text{Q : (t, X_p(t, g))}. \]

Defining the multiplication \( \# : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[ c \# (a, b) = (a/c^p_b, c^{p_0} b), \]

(5.6) and (5.7) express that
\[ c_p \triangleq Q = P. \]

\( Q \) is, or may be chosen as, a corner of the \( X_p^2(t; f) \)-polygone. If \( c_p \gg 1 \), which we may assume, the relative localization of \( P \) and \( Q \) is indicated by Figure 5.1.

![Figure 5.1](image)

In fact, comparing (5.5) and (5.8), by the considerations above every \( c = c_p \) in (5.5) also appears in (5.8). Hence \( \gamma_p \triangleq \sup_p c \), where \( c \) obeys the conditions in (5.8). (Actually this estimate is all we need in the sequel.) On the other hand, since \( \varphi(t) = X_p^2(t; f) \) for some \( f \) and \( \varphi(t) \) may be approximated by \( X_p^2(t; f) \)-functions, there is an inequality in the reverse direction too. (It is here the additional assumptions we made on the measures and the weight functions are needed.) We omit the details on this point.

We have to make the conditions on \( c \) in (5.8) more explicit. Without restrictions, let \( \varphi = (1, 1) \), i.e. \( \varphi(t) = \min(1, t) \) and then \( P = (e^{\psi_p}, e^{\psi_p}) \). Writing \( \varphi(t) = \xi^p + ty^p \), \( P \) is graph of \( \varphi \) means that \( \xi^p + ty^p = \sigma^p \), i.e.
\[(e^{\psi_p})^p + (e^p)^p = 1.\]

Turning to the integral inequality in (5.8), we handle the two members separately. Since \( \int \varphi(t\Delta(\sigma))d\sigma = 1 \), for \( \varphi \) holds
\[ \int \varphi(t\Delta(\sigma))d\sigma = \int \left[ e^{\psi_p} + t\Delta(\sigma) \psi^p \right] d\sigma = e^{\psi_p} + ty^p. \]

For \( \varphi \) we get, using the decreasingness of \( \Delta \) and defining \( x = x(t) \) by \( t\Delta(x) = 1 \),
\[ \int \varphi(t\Delta(\sigma))d\sigma = \int \min(1, t\Delta(\sigma))d\sigma = x + \frac{1}{2} \Delta(\sigma)d\sigma. \]

If \( 1 - p < p \), this formula needs in fact a minor modification, omitted here, originating from the fact that \( \Delta(\sigma) \) then attains no values smaller than \( 1/p \). The inequality in (5.8) now can be rewritten as
\[ x + \frac{1}{2} \Delta(\sigma)d\sigma \leqslant e^{\psi_p} + ty^p, \]
or, since \( t = 1/\Delta(x) \),
\[(5.10) \quad e\Delta(\sigma) + \frac{1}{2} \Delta(\sigma)d\sigma \leqslant e^{\psi_p} e\Delta(\sigma) + \eta^p. \]

In (5.8), for the extremal choice of \( \varphi \), for some \( t \) there holds equality between the integrals. Hence, for some \( x \) this must be the case in (5.10) too.

A derivation yields \( x = e^{\psi_p} \). Inserting this in (5.10), we get
\[ \int \Delta(\sigma)d\sigma = \psi^p. \]
But here
\[
\int_{\mathcal{A}} \delta(\alpha) \, d\sigma = \int_{\mathcal{A}} p_1 \left( \frac{1 - \sigma^{1/p_1} \rho^{1-1}}{\sigma^{1/p_1} \rho^{1-1}} \right) \, d\sigma = \int_{\mathcal{A}} p_1 \left( 1 - \sigma^{1/p_1} \rho^{1-1} \right) \, d\sigma = \int_{\mathcal{A}} p_1 (1 - \sigma)^{p_1-1} \, d\sigma = (1 - \xi)^{p_1}.
\]

Thus, for the extremal choice of $\varphi$ we have
\[
1 - \xi = \eta.
\]

In conclusion, the conditions on $\varphi$ in (5.8) are equivalent to (5.9) and (5.11). Hence
\[
\gamma_2 = \sup \{ \| \varphi \|_{L^p} + \| \varphi \|_{L^q} : \varphi \in C_0^\infty(\mathbb{R}) \} \quad \text{with} \quad \xi + \eta = 1.
\]

But this is the same as
\[
\inf_{\varphi \in C_0^\infty(\mathbb{R})} (\gamma_2 \xi + (\gamma_2 \eta)^p) = 1,
\]

which in turn is the same as (5.3).

What remains is (5.4). Obviously, $\gamma_2 > 1$ (remember the assumption $p \neq (1, 1)$). On the other hand, in view of (3.11) we have
\[
1 = \int_{\mathcal{A}} \min \left\{ p_1 \gamma_2^p \rho^{p_1-1}, p_1 \gamma_2^p (1 - \sigma^{p_1-1}) \right\} \, d\sigma
\]
\[
= \int_{\mathcal{A}} p_1 \gamma_2^p \rho^{p_1-1} \, d\sigma + \int_{\mathcal{A}} p_1 \gamma_2^p (1 - \sigma^{p_1-1}) \, d\sigma
\]
for some $\sigma$, $0 < \sigma < 1$. If $p > 1/\beta$ we get, cancelling the second and estimating the first integral,
\[
1 > \int_{\mathcal{A}} p_1 \gamma_2^p \rho^{p_1-1} \, d\sigma > \int_{\mathcal{A}} p_1 \gamma_2^p \rho^{p_1-1} \, d\sigma = (\gamma_2 \beta)^p;
\]
i.e. $\gamma_2 < \beta$. Treating in the same way the case $a < 1/\beta$, we end up with (5.4).

It is now possible to formulate partial converses of the theorems and lemmas of Section 4. Thus, combining Lemma 5.1 with Lemma 4.2 and Lemma 4.1 respectively, we get

**Lemma 5.2.** If $g \leq f(\mathcal{A}_2)$, on $f$ and $T_f$ be non-negative simple functions, where $T_f \in C_0^\infty(\mathbb{R})$. Suppose that $T_f$ is non-negative (i.e. $T_f \geq 0$).

**Theorem 5.2.** $(\gamma_2$-Int) is a consequence of exact $\mathcal{A}_2$-monotonicity.

Note that here we do not assert $\gamma_2$ to be the best constant. However, in Lemma 5.3 it actually is. This is seen in Example 5.3 below. On the contrary, in Lemma 5.2 and Theorem 5.1 it is not in general. This is indicated in Remark 5.1, referring to the limit case $p = \infty$.

In these instances $\gamma_2$ cannot, in general, be replaced by the constant 1 is shown by Examples 5.1 and 5.2. There the case $p = (1, p)$ is considered. In particular, we conclude that, generally, $\mathcal{A}_2$ is not exactly $K$-adequate.

**Remark 5.1.** By Lemma 5.1 the deviation between the $\mathcal{A}_2$ and the $K_2$ (for $K$ quasi-orders of $\gamma_2$). For $p$-values close to 1, $\gamma_2$ is close to 1 too. This is not surprising, since $\mathcal{A}_2$ is modelled after the $K$-functional for $(L_1, L_\infty)$ (where $\mathcal{A}_2 = K_2 = K$). Considering the optimal partitions of $f$ involved in calculating $\gamma_2$ and $\gamma_2$, it seems equally natural that $\gamma_2$ increases with $p_1, p_1$. One readily verifies that the extremal value 2 is approached in the limit $p_1$ and/or $p_1 = \infty$.

Trying to characterize the exact interpolation spaces in this limit case, a sufficient condition is as usual given by Theorem 1.1. What necessity concerns, $\mathcal{A}_2$, thus also Theorem 4.3, loose their sense. However, considering instead Theorem 5.2, a by continuity argument it can be extended to the present case. Thereby, as was remarked above, $\gamma_2 = 2$. But if particular $p_1 = 1$, we know from Theorem C that the best constant is 1. We thus conclude that the constant $\gamma_2$ in Lemma 5.2 and Theorem 5.1 is not, in general, the best one. For the couple $(L_1, L_\infty)$, $1 < \beta < \infty$, in [17] was proved analogous of Lemma 5.2 and Theorem 5.1 with constants estimated by $2^{3\beta}$, $1/p + 1/p' = 1$. Concerning this value, cf. also [2], [9].

**Remark 5.2.** If $\beta = (p, p)$, $\gamma_2 = 2^{3\beta}$. For this case, in Theorem 5.3 we thus obtained the same estimate as in [96].

Showing that the constant 1 does not do in Lemma 5.3 and Theorem 5.2, we need the following partial converse of Lemma 4.2.

**Lemma 5.3.** If $g \leq f(\mathcal{A}_2)$, on $f$ and $T_f$ be non-negative simple functions, where $T_f \in C_0^\infty(\mathbb{R})$. Suppose that $T_f$ is non-negative (i.e. $T_f \geq 0$).

**Theorem 5.3.** $(\gamma_2$-Int) implies $(\gamma_2; K_2)$-monotonicity and, equivalently, $(\gamma_2; K)$-monotonicity.

As immediate consequences we obtain

**Theorem 5.4.** $(\gamma_2; K)$-monotonicity and, equivalently, $(\gamma_2; K)$-monotonicity.

**Theorem 5.5.** $(\gamma_2$-Int) is a consequence of exact $\mathcal{A}_2$-monotonicity.

**Theorem 5.6.** $(\gamma_2$-Int) implies $(\gamma_2; K_2)$-monotonicity.

**Theorem 5.7.** $(\gamma_2$-Int) is a consequence of exact $\mathcal{A}_2$-monotonicity.
\[(z_0)_n^k.\] Hence \(T\) is determined by a matrix \((t_{ij})\) in the sense that

\[(5.12) \quad g_j = \sum_{i=1}^{n} t_{ij} f_i \quad (j = 1, \ldots, n).\]

By the positivity of \(T\), \(t_{ij} \geq 0\).

Put \(a(F) = a_i, \beta(i) = \beta_i\). By assumption we have

\[\sup \|T x \|_{p_\mu} = \|g\|_{p_\mu}, \quad \|x\|_{p_\mu} = \|f\|_{p_\mu} \quad (\mu = 0, 1),\]

or equivalently

\[\sup \sum_i \sum_j t_{ij} x_i \|p_\mu \| = \|g\|_{p_\mu} \quad \sum_i x_i \|p_\mu \| = \|f\|_{p_\mu} \quad (\mu = 0, 1).\]

Since \(\|T x \|_{p_\mu} = 1, \|f\|_{p_\mu} = \|g\|_{p_\mu} \quad (\mu = 0, 1)\), in both instances the supremum is attained for \(x = f\). Hence, using Lagrangian multipliers \(\lambda_\mu\), we get

\[(5.13) \quad \sum_i \sum_j t_{ij} x_i \|^\mu \lambda_\mu = \lambda_\mu \|f\|_{p_\mu} \quad (\mu = 0, 1).\]

On multiplying both sides by \(f_i\) and summing after \(i\), by means of (5.13) we obtain

\[\sum_i \|g\|_{p_\mu} = \lambda_\mu \sum_i \|f\|_{p_\mu} \quad (\mu = 0, 1).\]

Thus, since \(\|g\|_{p_\mu} = \|f\|_{p_\mu}, \lambda_\mu = 1 \quad (\mu = 0, 1)\). Put

\[\theta_\mu = \lambda_\mu \|f\|_{p_\mu}.\]

The equations (5.13) and (5.13) then are equivalent to

\[\sum_{i=1}^{n} \theta_\mu = 1 \quad (j = 1, \ldots, n),\]

\[\sum_{i=1}^{n} \theta_\mu \|g\|_{p_\mu} \|f\|_{p_\mu} \quad (i = 1, \ldots, n, \mu = 0, 1).\]

Since moreover \(\theta_\mu \geq 0\), application of Lemma A.2 yields the assertion \(g \leq f(x_{\mu})\).

**Example 5.1.** Let \(\bar{x} = (1, p), a_i, b_i = 1 \quad (\mu = 0, 1)\). We construct \(f\) and \(g\) with \(f \leq f([K_{\mu}])\) such that \(T f \neq g\) for all \(T \in \mathcal{A}([A, B])\). To this end we note that by the proof of Lemma 3.1 there exist simple functions

\[f = \sum_i f_i, \quad \theta = \sum_i \theta_i, \quad \text{having } X_{\mu}\text{-functions as in Figure 5.2, such that}\]

\[g \leq f([K_{\mu}]), \quad g \leq f(X_{\mu}).\]

From the behaviours at 0 and \(\infty\) it follows that \(\|f\|_{p_\mu} = \|g\|_{p_\mu} \quad (\mu = 0, 1)\). Without restrictions we may assume \(\|f\|_{1\mu} = \|g\|_{1\mu} = 1\). Let \(\mathcal{A}\) and \(\mathcal{D}\) be the

\[\text{spaces generated by } \{x_{\mu}\} \text{ and } \{y_{\mu}\}, \text{ respectively. Let } \mathcal{F} \text{ be the subspace of } \mathcal{A} \text{ defined by } \varphi \geq 0, \varphi|_{\mu} = 0.\]

Now suppose that \(T \in \mathcal{A}(A, B), T f = g\) and, without restrictions, \(T: \mathcal{A} \to \mathcal{D}\). The functional on \(L_1\)

\[\varphi \mapsto \int\! T f \, d\mu\]

then has a norm not exceeding 1. Since it attains the value 1 on \(\mathcal{F}\), at the interior point \(f\), it is identically 1 on \(\mathcal{F}\). But then \(T f \geq 0\) for \(\varphi \in \mathcal{F}\), since otherwise

\[\int\! T f \, d\mu > \int\! T f \, d\mu = 1,\]

contrary to the hypothesis. Hence \(T\) must be non-negative. But this is impossible by virtue of Lemma 5.4. We conclude that there does not exist an operator \(T \in \mathcal{A}(A, B)\) such that \(T f = g\).

**Example 5.2.** By means of the preceding example, we now construct an exact interpolation space with respect to \(\{l_{\mu}, L_{\mu}\}\) (or generally \(\{l_{\mu}, L_{\mu}\}\)), \(1 < p < \infty\), which is not exactly \(K\)-monotonic. In fact, with \(f, g\) as in Example 5.1, let \(A\) be defined by (cf. [1], p. 99)

\[\|g\|_{\mu} = \inf_{x \in \alpha} \|T x \|_{\alpha},\]

It is readily verified that \(A\) is an exact interpolation space with respect to \(A\). Obviously, \(\|f\|_{\mu} \leq 1\). But expressed in terms of \(A\), the content of the preceding example is that \(\|g\|_{\mu} > 1\). Hence \(\|g\|_{\mu} \geq \|f\|_{\mu}\), which proves that \(A\) is not exactly \(K\)-monotonic.

**Example 5.3.** We show that \(y_{\mu}\) in Lemma 5.3 cannot be replaced by any smaller constant. Referring to the proof of Lemma 5.1, let \(y(t) = \min(1, 6)\) and \(g(t) = \frac{t}{1 + \frac{t}{6}}\), where \(t, \xi, \eta\) are the optimal values in (5.8'). Let \(g = z_{AB}\) with \(\beta(0) = 1\) and let \(f^* = f_{AB}^* + f_{AB}^*\) be a 2-valued
simple function with

\[ \|F^n\|_{L^p} = \xi, \quad \|F^n\|_{L^q} = \eta, \quad \|F^n\|_{L^p} = \zeta, \quad \|F^n\|_{L^q} = \eta. \]

Then \( X^n(t, g) = \psi(t) \) and \( X^n(t, f') \nrightarrow \psi(t) \), \( n \to \infty \), cf. Figure 5.3.

\[ \text{Fig. 5.3} \]

Define \( T^g \) by

\[ T^g f^n = \tilde{g}, \quad T^g f^n = \eta g. \]

Then \( T^g f^n = g \) and, as is readily seen, \( T^g \in X^n_{1+n}(A; B) \) with \( g(n) \to 0 \), \( n \to \infty \). However, by the proof of Lemma 5.3, \( T^g f^n \leq \gamma g f^n[X^n] \), uniformly in \( n \), where the constant \( \gamma \) cannot be improved. Taking into account the homogeneity, this proves the assertion.

6. A remark on the Lorentz case. We finally show how Lemma A.2 applies to yield a short proof of the results of Lorentz-Shimogaki [16] for interpolation of Lorentz spaces. (For Corollary 6.1 below, another simplified proof was given in [29].)

Let \( \varphi \) be a positive, decreasing function on \( \mathbb{R}_+ \) and let \( \Phi(t) = \frac{\varphi(tx)}{x} \) for \( x \geq 0 \).

Define the Lorentz space \( \Lambda(p) \) by means of the norm

\[ \|f\|_{\Lambda(p)} = \frac{\infty}{\infty} \int_0^\infty f(x) \varphi(x) dx = \frac{\infty}{\infty} \int_0^\infty f(x) d\varphi(x). \]

In particular, if \( f = \chi_{[0,1]} \) we have \( \|f\|_{\Lambda(p)} = \Phi(\varphi) \).

The following theorem and its corollary are in essence equivalent to Theorem 4 of [16].

**THEOREM 6.1.** Let \( A = (\Lambda(\psi), A(\varphi)), \quad B = (\Lambda(\psi), A(\varphi)). \) Then \( A = \Lambda(\varphi) \) and \( B = \Lambda(\varphi) \) obey (ExInt) if and only if they are exactly \( K \)-monotonic.

**Proof.** That \( K \)-monotonicity is sufficient for (ExInt) follows as usual from Theorem 2.1. What remains is the necessity.

It is well known that (cf. [29]))

\[ K(t, f; A) = \int f(t) d[\min\{(\Lambda(\psi)(\varphi), \Phi(\varphi))\}], \]

\[ K(t, g; B) = \int g(t) d[\min\{(\Lambda(\psi)(\varphi), \Phi(\varphi))\}]. \]

As in the proofs of Lemma 4.3 and Theorem 4.3, the problem can be reduced to the case of non-negative elementary functions \( f, g \), i.e.

\[ f = \sum_{i=1}^{\infty} f_i X_{0,i}, \quad g = \sum_{i=1}^{\infty} g_i X_{0,i}, \quad g_i < f_i. \]

Put

\[ f_i = f_i - f_{i-1}, \quad E_i = \bigcup_{k=0}^{i-1} E_k, \quad g_i = g_i - g_{i-1}, \quad G_i = \bigcup_{k=0}^{i-1} G_k. \]

Then

\[ f = \sum_{i=1}^{\infty} f_i X_{0,i}, \quad g = \sum_{i=1}^{\infty} g_i X_{0,i}. \]

Putting \( \text{meas}(E_i) = \alpha_i, \text{meas}(G_j) = \beta_j \), the condition \( g \leq f[K] \) means

\[ \sum_{i=1}^{\infty} \gamma_i \text{meas}(E_i) \text{meas}(G_j) \leq \sum_{i=1}^{\infty} \gamma_i \text{meas}(E_i) \text{meas}(G_j). \]

But then, by Lemma A.2, there exists a matrix \( \Theta = (\theta_{ij}) \) such that

\[ \sum_{i=1}^{\infty} \theta_{ij} = 1, \quad (j \in J), \quad \sum_{i=1}^{\infty} \theta_{ij} = 1, \quad (i \in I). \]

Now define operators \( T_i \) by

\[ T_i X_{0,i} = \frac{1}{f_i} \sum_{j=1}^{\infty} \theta_{ij} f_j X_{0,j}. \]

and, generally, for locally integrable functions \( h \) by

\[ T_i h = \frac{1}{\chi_i} \int h(x) T_i [x_{0,i}]. \]

Then, by (6.2),

\[ \|T_i X_{0,i}\|_{\Lambda(p)} = \frac{1}{f_i} \sum_{j=1}^{\infty} \theta_{ij} \|X_{0,j}\|_{\Lambda(p)}. \]

\[ \|X_{0,i}\|_{\Lambda(p)} = \frac{1}{f_i} \sum_{j=1}^{\infty} \theta_{ij} \|X_{0,j}\|_{\Lambda(p)}. \]

(\( \mu = 0, 1 \).)

\[ 4 \quad \text{Studia Mathematica XXII.3} \]
Making two integrations by parts and using the fact that \( \phi_\alpha(s) \) decreases with \( \alpha \), we get

\[ \|T_h L_{\alpha}(\phi)\|_{L_{\alpha}(\phi)} \leq \frac{1}{\alpha} \int \chi \frac{d\nu}{\chi} \| T_h L_{\alpha}(\phi)\|_{L_{\alpha}(\phi)} \leq \frac{\phi_\alpha(q)}{\alpha} \int \nu \frac{d\nu}{\nu} \leq \frac{\phi_\alpha(q)}{\alpha} \int \nu \frac{d\nu}{\nu} \]

\[ = \phi_\alpha(q) \left( a \psi_\alpha(q) + \int \frac{\phi_\alpha(q)}{\alpha} \nu \frac{d\nu}{\nu} \right) \]

\[ = \phi_\alpha(q) \left( a \psi_\alpha(q) + \int \frac{\phi_\alpha(q)}{\alpha} \nu \frac{d\nu}{\nu} \right) \]

\[ \leq \phi_\alpha(q) \left( a \psi_\alpha(q) + \int \phi_\alpha(q) \nu \frac{d\nu}{\nu} \right) \]

\[ = \int \nu \phi_\alpha(q) \frac{d\nu}{\nu} = \| T_h L_{\alpha}(\phi) \|_{L_{\alpha}(\phi)} \quad (\mu = 0, 1). \]

In other words, \( T_h L_{\alpha}(\phi) \) decreases with \( \alpha \); assuming that \( A(\phi), A(\psi) \) obey \((\text{ExInt})\), we get

\[ \|T_h L_{\alpha}(\phi)\|_{L_{\alpha}(\phi)} \leq \|T_h L_{\alpha}(\psi)\|_{L_{\alpha}(\psi)}, \quad \text{if} \quad h \in A(\phi). \]

Thus, for \( h = f_1 \mathcal{E}_1 \cdot \sum f_i g_i \psi_i(\beta_i) \leq f_i \phi_i(q) \quad (i \in I). \]

A summation after \( i \) now yields, by virtue of \((6.1)\),

\[ \| g \|_{L_{\alpha}(\phi)} = \sum g_i \psi_i(\beta_i) = \sum f_i \phi_i(q) \leq \sum f_i \psi_i(q) = \| f \|_{L_{\alpha}(\psi)}, \]

i.e. \( \| g \|_{L_{\alpha}(\phi)} \leq \| f \|_{L_{\alpha}(\psi)} \). This proves the proof. \( \blacksquare \)

In [16] also the following condition on \( A(\phi), A(\psi) \) was dealt with

\[ \frac{\psi(y)}{\phi(x)} \leq \max \left( \frac{\psi(y)}{\phi(x)}, \frac{\psi(y)}{\phi(x)} \right) \quad \text{for all} \quad x, y > 0. \]

We also introduce the slightly less restrictive

\[ \frac{\psi(y)}{\phi(x)} \leq \frac{\psi(y)}{\phi(x)} + \frac{\psi(y)}{\phi(x)} \quad \text{for all} \quad x, y > 0. \]

**Corollary 6.1.** Let \( \bar{A}, \bar{B} \) and \( A, B \) be the same as in Theorem 6.1. Then

(i) \((\text{ExInt})\) implies \((\text{L.S})\),

(ii) \((\text{L.S'})\) implies \((\text{L.S})\).

**Proof.** Note that, by the above theorem, \((\text{ExInt})\) and \((\text{L.S})\) are equivalent to exact \(K\) and \((2;K)\)-monotonicity, respectively. In the proof to follow we argue throughout in terms of \(K\)-monotonicity.

(i) Let

\[ g = \chi_{[0, \infty)}, \quad f = \max \left( \frac{\psi(y)}{\phi(x)}, \frac{\psi(y)}{\phi(x)} \right) \chi_{[0, \infty)}. \]

One readily verifies that \( g \leq f \). Assuming exact \(K\)-monotonicity, we obtain \( g \leq f \). But this is exactly the same as \((\text{L.S})\).

(ii) Suppose that \((\text{L.S'})\) is valid and that \( g = f \). Then we have relations \((6.1)\) and \((6.2)\). On dividing \((6.2)\) by \( \phi_\alpha(q) \) and adding the two inequalities \((\mu = 0, 1)\), by means of \((\text{L.S'})\) we get

\[ 2f' \geq \sum g_i \psi_i(\beta_i) \quad \text{for all} \quad i \in I. \]

Hence

\[ \sum g_i \psi_i(\beta_i) \leq 2f_i \psi_i(q). \]

In view of \((6.1)\), a summation after \( i \) now yields

\[ \sum g_i \psi_i(\beta_i) \leq 2 \sum f_i \psi_i(q), \]

i.e. \( \| g \|_{L_{\alpha}(\phi)} \leq 2 \| f \|_{L_{\alpha}(\psi)} \). This proves the assertion. \( \blacksquare \)

**Remark 6.1.** Concerning the family of Lorentz spaces \( L_{\alpha,\beta} \) defined in Remark 4.2, any couple \((L_{\alpha,\beta}, L_{\alpha,\beta})\) of such spaces is \(K\)-adequate, \( 1 \leq p, q, \alpha \leq \infty, \mu \geq 0, 1 \). This is a consequence of the result of [9], cited at the end of Section 1, and the fact that \( J_{\alpha,\beta} = (J_{\alpha,\beta})^{-1/\mu} \).

(That the couple \((L_{\alpha,\beta}, L_{\alpha,\beta})\) is \(K\)-adequate, in fact exactly, was first proved in [23].) In particular, the couple \((L_{1,\infty}, L_{1,\infty})\) is \(K\)-adequate. Hence every interpolation space with respect to this couple is \(K\)-monotonic, although not necessarily in the exact sense. Theorem 6.1 thus may be considered as a sharpening of this result for interpolation spaces which themselves are Lorentz spaces.

For a related result, showing that \( \bar{A} = (A(\phi), L_{\alpha}) \), \( \bar{B} = (B_{1, L_{\alpha}}) \) are exactly \(K\)-adequate, see [10].
Appendix. Some matrix lemmata. Let $R^+_n$ be the positive (vector-) quadrant, i.e. $\bar{x} = (x', x) \in R^+_n$ iff $x', x > 0$. On $R^+_n$ are defined order relations (inclination) by

$$\bar{x} \in \bar{y} \iff x'^1/y' < y'^1/y'^1,$$

$$\bar{x} \geq \bar{y} \iff x'^1/y'^1 \leq y'^1/y'^1.$$

We consider sequences (possibly non-finite) $X = (\bar{y}_i)_{i \in I} \subset R^+_n$, $I \subset N$, such that $\bar{y}_i \in \bar{y}_{i+1}$ and

$$\sum_{i \in I} y_i < \infty, \quad \sum_{i \in I} \bar{y}_i < \infty. \quad (A.1)$$

If $Y = (y_i)_{i \in I}$ is another such sequence and $\Theta = (\theta_{i,j})_{i \in I, j \in J}$ a matrix, we agree to write

$$\Theta X = Y \iff \sum_{j \in J} \theta_{i,j} \bar{y}_j = y_i \ (i \in I),$$

$$\Theta X \leq Y \iff \sum_{j \in J} \theta_{i,j} \bar{y}_j \leq y_i \ (i \in I).$$

Here $\bar{x} \leq \bar{y}$ stands for $x' \leq y'$, $x'^1 \leq y'^1$. Let $\mathcal{D}$ denote the set of matrices $\Theta = (\theta_{i,j})_{i \in I, j \in J}$ such that

$$\theta_{i,j} \geq 0, \quad \sum_{j \in J} \theta_{i,j} = 1 \ (i \in I, j \in J),$$

and such that the non-zero elements are distributed in accordance with the figure:

![Graph](image)

Thus, in particular, each row contains only finitely many non-zero elements, and for all but finitely many $i$ there exists a number $i = i(j)$ such that $\theta_{i,j} = 0$ for $i > i(j)$ or $i < i(j)$.

To start with we assume that $I$ and $J$ are finite. Then $\mathcal{D}$ is the set of all (finite) stochastic matrices. To every $X = (\bar{y}_i)_{i \in I}$ we associate the set (of vectors)

$$\omega_X = \left\{ \sum_{j \in J} \bar{y}_j \bar{z}_j \mid 0 \leq \bar{z}_j \leq 1, \ j \in J \right\}.$$

Using the point $P$ as origin, let $\omega_{X,P}$ be its affine representative. $\omega_{X,P}$ then is a convex parallelepiped, cf. Figure A.2. By $\gamma_{X,P}$ and $\bar{y}_{X,P}$ we denote the boundary polygons, and also the functions on $R$ having them as graphs.

![Graph](image)

**Lemma A.1.** Let $I, J$ be finite. Then the following conditions are equivalent

(i) $X = \Theta X$ with $\Theta \in \mathcal{D}$,

(ii) $\omega_Y \subset \omega_X$, $\sum_{j \in J} \bar{y}_j = \sum_{i \in I} \bar{y}_i$,

(iii) $\gamma_{X,P} \subset \gamma_{Y,P}$.

**Proof.** The equivalence between (ii) and (iii) is obvious. As a consequence of (i), $\sum_{i \in I} \bar{y}_i \in \omega_X$ for every $I \subset I$. Hence $\omega_Y \subset \omega_X$. Since obviously $\sum_{i \in I} \bar{y}_i = \sum_{j \in J} \bar{y}_j$, we have verified that (i) implies (ii).

We now prove that (iii) implies (i). Thus let $\gamma_{X,P} \subset \gamma_{Y,P}$. To construct $\theta$ we use induction over the number of $\bar{z}$-vectors $n$. For $n = 1$ and 2, the statement is obvious. Suppose it is true for $n - 1$. Let $I$ be chosen in accordance with Figure A.3 (along the "tangent" of $\gamma_{Y,P}$ through the corner $P + \bar{z}_i$ of $\gamma_{X,P}$).
Since

\[ \gamma_{\alpha_i} \leq \gamma_{\alpha_{i-1}}, \quad \forall \alpha_{i-1}, \alpha_i \in \mathcal{P}, \quad \forall \alpha_{i-1}, \alpha_i, \alpha_{i+1} \leq \gamma_{\alpha_{i+1}}, \ldots, \alpha_{n-1}, \alpha_n, \alpha_{n+1}, \]

the induction hypothesis yields

\[ \gamma_i = \theta_i \alpha_i + \lambda_i \beta_i, \]

\[ \gamma_{i+1} = \theta_{i+1} \alpha_{i+1} \]

with \( \sum_{i=1}^{h} \theta_i = 1, \) \( \sum_{i=1}^{k} \lambda_i = 1, \)

and

\[ \gamma_{i+1} = \theta_{i+1} \alpha_{i+1} \ldots + \theta_{i+1,n} \alpha_n, \]

\[ \gamma_m = \theta_{m+1} \alpha_{m+1} \ldots + \theta_{m+n} \alpha_n, \]

\[ \epsilon = \alpha_i \beta_i + \ldots + \alpha_m \epsilon_m \quad \text{with} \quad \mu_j + \sum_{i=1}^{n} \theta_i = 1 \quad (j = 2, \ldots, n). \]

Inserting the expression for \( \epsilon \) into the first \( k \) equations, we obtain a scheme of coefficients having the properties stated.

As a corollary we get the theorem of Hardy, Littlewood and Pólya, cited in Section 2. In fact, considering the particular vectors \( x_i = (x_i, 1), \)

\( y_i = (y_i, 1) \) \( (i = 1, \ldots, n), \)

the equations for the second components in \( Y = X \) with \( \theta \in \mathcal{P} \) say that every row sum of \( \theta \) is equal to 1. Hence (i) of Lemma A.1 is equivalent to \( "y - \theta x" \) with \( \theta \in \mathcal{P}\), \( \theta \) stands for the doubly stochastic matrices. On the other hand, by virtue of Figure A.4, the condition (iii) is equivalent to \( "\sum_{i=1}^{n} \gamma_i \leq \sum_{i=1}^{n} \beta_i" \) for \( k = 1, \ldots, n - 1, \) \( \sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \beta_i. \)

Thus Lemma A.1 reduces to Theorem HLP in this case.

Now consider the general (i.e. not necessarily finite) case. To \( X = (\tilde{x}_i)_n, \)
we assign the piecewise linear function \( \gamma_X \) defined by having a graph with vertices at the points (cf. Figure A.5)

\[ P_m = \left( \sum_{i \in m} \gamma_i, \sum_{j \in m} \gamma_j \right). \]

If \( \sum_{i \in m} \gamma_i \) is finite, let \( \gamma_X(t) = 0 \) for \( t < -\sum_{i \in m} \gamma_i. \) (That the coordinates of \( P_m \)
are finite is ensured by the assumption (A.1)). \( \gamma_X \) thus is located in the second quadrant and has the coordinate axes as (possibly attained) asymptotes.

Our main result reads

**Lemma A.2.** The following statements are equivalent:

(i) \( \theta X \leq Y \) with \( \theta \in \mathcal{P}, \)

(ii) \( \gamma_X \leq \gamma_Y, \)

(iii) \( \sum_{i \in m} (\gamma_i, m) \leq \sum_{j \in m} (\gamma_j, m), \quad \sum_{i \in m} (\gamma_i, m) \leq \sum_{j \in m} (\gamma_j, m), \quad \sum_{i \in m} (\gamma_i, m) = \sum_{j \in m} (\gamma_j, m) \)

**(Proof.** (i) \( \Rightarrow \) (ii). If \( X, Y \) are finite, this is exactly what is expressed by Lemma A.1. In that case the following lengthy argument thus should be excluded.)
Dealing with the non-finite case, to fix the ideas let $I = J = Z$.
Beginning with the implication (ii) $\Rightarrow$ (i)', let $x = y$, $\sum y_j = \sum z_j$. The situation is illustrated in Figure A.5, where $\sum y_j$ is divergent, $\sum z_j$ convergent. Without restrictions we may assume that $y_j \in y_k$, $z_j \in z_k$ strictly. Drawing from the vertices of $y_k$ tangents of $y_k$, the vectors $\bar{u}_k$ are constructed, $k = \pm 1, \pm 2, \ldots$. In this way the region $D$ bordered by $y_k$ and $y_k$ is decomposed into subregions $D_k$, $k \in \mathbb{Z}$. Let $X = (\bar{u}_j)_{j \in \mathbb{Z}}$ and $Y = (\bar{y}_j)_{j \in \mathbb{Z}}$ be the subsets of $X$ and $Y$ associated with $D_k$.

![Figure A.5](image)

Now to every $D_k$, Lemma A.1 is applicable. It provides us with the $\mathcal{S}$-matrices $\Theta_k$ such that
\begin{align*}
(\bar{u}_{k+1}, Y_k) &= \Theta_k(\bar{u}_k, X_k), \quad k \geq 1, \\
(\bar{u}_{k-1}, Y_k) &= \Theta_k(\bar{u}_k, X_k), \quad k \leq -1.
\end{align*}
(A.2)

On eliminating successively the $\bar{u}_k$, $k = 1, 2, \ldots, k = -1, -2, \ldots$, we find that
\[
\text{if } i \in I_k \text{ then } \bar{y}_i = \sum_{j \in J_k} \theta_0 y_j \quad (k \in \mathbb{Z}).
\]

Hence $Y = \Theta X$ with $\Theta$ having the shape of Figure A.1.

What remains to prove is that all column sums equal 1. Let us talk about $\theta_k$ as "the $\bar{y}_k$-coefficient for $\bar{y}_i$". In the analogous way we describe the coefficients of the equations (A.2). Beginning with $j \in J_k$, $k \geq 1$, put
\[
a_0 = \text{the } \bar{y}_k\text{-coefficient for } \bar{u}_{k+1},
\]
\[a_0 = \text{the } \bar{u}_k\text{-coefficient for } \bar{u}_{k+1} \text{ if } \mu > k.
\]

By their construction the elements $\theta_{ik}$ of the $j$th column can be interpreted in the following way:
\[
\begin{align*}
\text{if } i &\in I_{k+1}, \mu < k: & \theta_{ik} = 0, \\
\text{if } i &\in I_k: & \theta_{ik} = \text{the } \bar{y}_k\text{-coefficient for } \bar{y}_i, \\
\text{if } i &\in I_{k+1}: & \theta_{ik} = a_0 \theta_{ik+1}, \text{(the } \bar{u}_{k+1}\text{-coefficient for } \bar{y}_i), \\
\text{if } i &\in I_k: & \theta_{ik} = a_0 a_{k+1} \ldots a_{\mu-1} \text{ (the } \bar{u}_k\text{-coefficient for } \bar{y}_i).
\end{align*}
\]

Put
\[
\beta_k = \text{the sum of the } \bar{y}_k\text{-coefficients for } \bar{y}_i, \quad i \in I_k,
\]
\[
\beta_\mu = \text{the sum of the } \bar{u}_k\text{-coefficients for } \bar{y}_i, \quad i \in I_\mu \text{ if } \mu > k.
\]

Since $a_0 + \beta_k$ is a column sum in $\Theta_k$, we have
\[
(A.3) \quad a_0 + \beta_\mu = 1, \quad \mu > k.
\]

The problem is to determine the sum of the series
\[
\beta_k + a_0 \beta_{k+1} + a_0 a_{k+1} \beta_{k+2} + \ldots
\]

In view of (A.3) its partial sums can be written
\[
s_n = \beta_k + \ldots + a_0 \ldots a_{n-1} \beta_n = 1 - a_0 \ldots a_n.
\]

But here the product $a_0 \ldots a_n$ has the geometrical significance of the $\bar{y}_k$-coefficient for $\bar{u}_{k+1}$ when in the iterative process above $\bar{u}_{k+1}$ is expressed as a linear combination of $\bar{y}_j$, $j \in \bigcup J_k$. This coefficient thus is majorized by the length of the projection of $\bar{u}_{k+1}$ onto $\bar{y}_j$ along the $\bar{y}_k$ direction, where $j_k$ is the greatest integer of $J_k$, cf. Figure A.6. But this length obviously tends to zero. Hence $s_n \to 1$, as $n \to \infty$, which proves the assertion if $j \in J_k$, $k \geq 1$. In the same way one treats the case $k \leq -1$. Thereafter, by a combination of these two cases one settles the case $k = 0$ (where the sum has to be taken over all integers). By this we have proved the implication (ii) $\Rightarrow$ (i)'.
In other words, \( Q_k = \gamma X, \) with \( X_k = (\theta_j x_{ij} < n_k), P_{n_k} = (-\sum_{i < n_k} x_i, \sum_{j < n_k} x_j). \)

Hence all vertices of \( \gamma X \) lie above \( \gamma Y \), which proves the assertion.

(i) \( \Rightarrow \) (ii). Let \( \gamma X \leq \gamma Y \) (but not necessarily \( \sum x_i = \sum y_i \)). Choose the points \( R_1, R_2 \) in Figure A.6 so that the segment \( R_1 R_2 \) intersects \( \gamma X \). From these points we draw the tangents of \( \gamma_Y \). Also connect them with \( \gamma_X \).

Let \( \Sigma = (\theta_j x_{ij}) \) and \( H = (\theta_i y_{ij}) \) be defined by the figure. Then \( \gamma X \leq \gamma Y \), \( \sum x_i = \sum y_i \), and thus, by the equivalence just proved, \( H = \theta \Sigma \) with \( \theta \in \mathcal{S} \).

Since \( \forall x \leq \forall y, \forall z \leq \forall y \) \( (i \in I, j \in J) \), it follows that \( \forall \leq \forall \) with \( \theta \in \mathcal{S} \).

Conversely, let \( X' = \gamma X \leq Y \) with \( \theta \in \mathcal{S} \). Then, by the equivalence between (i) and (ii), \( \gamma X \leq \gamma Y \). One easily verifies that \( \gamma Y \leq \gamma Y \).

Hence \( \gamma X \leq \gamma Y \), which concludes this part of the proof.

(ii) \( \Rightarrow \) (iii). Here we use the Legendre transform

\[ L_\psi(t) = \inf_{\phi \in \mathcal{S}} (\psi(\phi) - t \phi). \]

Since \( \gamma X \) and \( \gamma Y \) are convex, there holds the equivalence

\[ \gamma X \leq \gamma Y \Leftrightarrow \psi X \leq \psi Y. \]

Thus, proving that

\[ L_\gamma(t) = \sum_{i < j} \min (x_i, y_j), \]

the assertion will follow.
To this end, referring to Figure A.8, we have drawn the tangent of $\gamma_X$ in the $(t, 1)$-direction, $\mathcal{L}_X(t)$, which can be represented as the distance $|OM|$. Let the point of tangency be $P_1$: $(-\sum j^2, \sum j^2)$. By the figure we have

$\mathcal{L}_X(t) = |OM| = |ON| + |NM| = \sum j^2 + t \sum j^2$.

But here

- if $j < k$, then $\exists \in (1, 1)$, i.e. $x_j < x_k$, $x_k = \min(x_k, x_j)$,
- if $j > k$, then $(1, 1) \notin \mathbb{R}$, i.e. $x_j < x_k$, $x_k = \min(x_k, x_j)$.

This proves (A.4).

![Diagram of Figure A.8](image)

Then

(i) (A.6) is a consequence of $\gamma_X \leq \gamma_Y$ if and only if $\psi \in \mathcal{F}_w$, (ii) (A.6) is valid for every $\psi \in \mathcal{F}_w$ if and only if $\gamma_X \leq \gamma_Y$.

Proof. The “if” parts of (i) and (ii) coincide. To verify them, let $\mathfrak{X}, \mathfrak{Y}$ obey (iii) of Lemma A.2 and let $\phi \in \mathcal{F}_w$ be given by (A.5). Then

$\sum j^2 \leq \sum y_j^2$, $\sum y_j^2 \leq \sum y_i^2$,

and by an integration

$\sum j^2 \int \min(x_j, x_k), t \, dw_j(t) \leq \sum y_i^2 \int \min(y_j, y_k), t \, dw_j(t)$.

Combining these three inequalities, we get (A.6).

Verifying the “only if” part of (i), suppose that $\gamma_X \leq \gamma_Y$ implies (A.6). Let $\mathcal{E}_1 = (x_1, 1), \mathcal{E}_2 = (x_2, 1), y_1 = y_2 = ((x_1 + x_2)/2, 1)$. Then $\gamma_X = \mathcal{E}_X$ with $\theta \in \mathfrak{S}$. Hence, by (A.6),

$\psi(x_1) + \psi(x_2) \leq 2\psi((x_1 + x_2)/2)$,

which proves the concavity. To see that $\psi$ is positive, let $\mathfrak{X}$ and $\mathfrak{Y}$ consist of the single vectors $\mathfrak{X} = (a, 1)$ and $\mathfrak{Y} = (2\mathfrak{X}, 2)$, respectively. Then $\gamma_X \leq \gamma_Y$, so that $\psi(a) \leq 2\psi(a)$, i.e. $\psi \geq 0$. Thus $\psi \in \mathcal{F}_w$.

The necessity part of (iii), finally, follows immediately by taking in (A.6) the particular function $\psi(s) = \min(s, 0)$. We then obtain condition (iii) of Lemma A.2.

Remark A.1. These lemmata can be extended in various directions. Firstly, it is not necessary to restrict oneself to the quadrant $\mathbb{R}_+^2$. In fact, there exists an analogue of Lemma A.1 for arbitrary $\mathfrak{X} = (\mathfrak{X}, \mathfrak{Y})$, $\mathfrak{Y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. In particular, such a result applies to complex vectors $(x_1, \ldots, x_n) \in C^n$, $(y_1, \ldots, y_n) \in C^n$.

Above we have been confined to vectors $\mathfrak{X} (j \in J)$ which are, or can be, arranged in a non-decreasing order. However, by an argument similar to that of the proof of Lemma A.2 in the transition from the finite to the non-finite case, Lemma A.2 can be extended to $\mathfrak{X}$ and $\mathfrak{Y}$ being countable unions of monotonic sets.

In formulating Lemma A.1 we could also have used $\mathfrak{X}_{j, \mathfrak{Y}, \mathfrak{P}} \equiv \mathfrak{Y}_{j, \mathfrak{P}}$, which obviously is equivalent to $\mathfrak{X}_{j, \mathfrak{Y}, \mathfrak{P}} \leq \mathfrak{Y}_{j, \mathfrak{P}}$ (cf. Figure A.1). In passing from Lemma A.1 to Lemma A.2 we used the translates into the second quadrant $\mathfrak{X}_P$ and $\mathfrak{Y}_P$ of $\mathfrak{X}_{j, \mathfrak{P}}$ and $\mathfrak{Y}_{j, \mathfrak{P}}$, having the axes as asymptotes. Here the assumption (A.1) enabled us to consider the non-finite case. Translating in the same way $\mathfrak{X}_{j, \mathfrak{P}}$ and $\mathfrak{Y}_{j, \mathfrak{P}}$, we obtain what we define as $\mathfrak{X}_{j, \mathfrak{P}}$ and $\mathfrak{Y}_{j, \mathfrak{P}}$. Assuming $\sum x_{j, \mathfrak{P}}$ or $\sum x_{j, \mathfrak{P}}$ to be finite, also in this case the sequences are allowed to be non-finite. Arguing exactly as in the proof of Lemma A.2,
we find that the following conditions are equivalent:

(i) \( \Theta X \supseteq \Theta Y \) with \( \Theta \in \mathcal{G} \),

(ii) \( f_x \supseteq f_\gamma \),

(iii) \( \sum_{j \in \mathcal{J}} \max (t_j, t_{ij}) \supseteq \sum_{i \in \mathcal{I}} \max (y_i, t_{ij}) \), \( t > 0 \).

If \( \sum t_j = \sum t_i \), then (ii) and (iii) are equivalent to

(iv) \( \Theta X = \Theta Y \) with \( \Theta \in \mathcal{G} \).

References

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