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Bounded complete Finsler structures I

by

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Abstract. It is proved that any C^∞ Banach manifold satisfying a stability condition admits a complete bounded Finsler structure; in particular, any C^∞ separable Hilbert manifold admits a bounded complete Riemannian structure. In finite dimensions, only compact manifolds admit complete bounded Finsler structures. A "global" definition of Finsler structures (in the sense of Palais) is also given.

The principal purpose of this note is to resolve a problem proposed some years ago by Elworthy, [2]. We have, however, appended some easy remarks on related topics. In § 1, we give a definition of the notion of "Finsler structure" in the sense of Palais [8], which, although obvious, does not seem to have appeared before, and is at least of some theoretical interest. In § 2, we discuss Elworthy's problem in finite dimensions, and in § 3 we answer the problem in infinite dimensions.

§ 1. Suppose that E is a topological vector space whose topology admits a norm. Let $\mathfrak{N}$ denote the set of norms on E which define the given topology.

(a) $\mathfrak{N}$ is a cone, in the space of real-valued functions on E with pointwise addition and multiplication. That is, if $\lambda\in\boldsymbol{R}$ and $\lambda>0$, and $\nu\in\mathfrak{N}$, then $\lambda\nu\in\mathfrak{N}$; if $\nu_1,\nu_2\in\mathfrak{N}$, then $\nu_1+\nu_2\in\mathfrak{N}$.

Choose $\nu\in\mathfrak{N}$, and define $\Delta_\nu\colon \mathfrak{N}\times\mathfrak{N}\to\boldsymbol{R}$ by the following technique. For $\nu'\in\mathfrak{N}$, let $B(\nu')=\{x\in E\colon \nu'(x)\leqslant 1\}$ be the closed unit ball with respect to ν'. Thus the correspondence $\nu'\leftrightarrow B(\nu')$ is a bijection between $\mathfrak{N}$ and the set of bounded, absolutely convex, closed neighbourhoods of 0 in E. Now let $\Delta_\nu(\nu_1,\nu_2)$ be the Hausdorff distance between $B(\nu_1)$ and $B(\nu_2)$ in the metric on E defined by ν. Formally,

$$(*)\qquad \Delta_\nu(\nu_1,\nu_2)=\max\big(\alpha_\nu(B(\nu_1),B(\nu_2)),\ \alpha_\nu(B(\nu_2),B(\nu_1))\big),$$

where, for any two bounded nonnull sets A, A' in E,

$$\alpha_\nu(A,A')=\sup_{x\in A}\inf_{y\in B}\nu(x-y).$$

(b) Δ_ν is a metric on $\mathfrak{N}$. This is immediate from the fact that $B(\mu)$, for $\mu \in \mathfrak{N}$, is closed with respect to ν, and the Hausdorff distance is a metric on the class of all nonnull bounded closed sets.

(c) If $\nu' \in \mathfrak{N}$, then $\exists k, K \in \boldsymbol{R}$ such that $0 < k \leqslant K$ and, for any $\nu_1, \nu_2 \in \mathfrak{N}$,

$$k\Delta_{\nu'}(\nu_1, \nu_2) \leqslant \Delta_\nu(\nu_1, \nu_2) \leqslant K\Delta_{\nu'}(\nu_1, \nu_2);$$

that is, $\Delta_{\nu'}$ and Δ_ν are "metrically equivalent" metrics on $\mathfrak{N}$. This follows from $(*)$, since there exist k, K such that, for all $x \in E$,

$$k\nu'(x) \leqslant \nu(x) \leqslant K\nu'(x).$$

If E is complete, the set of all bounded, closed, nonnull, absolutely convex sets in E is complete with respect to Δ_ν. (This is standard.) It is easy to see that $\mathfrak{N}$ is a dense open subset of this space. Thus, $\mathfrak{N}$ is incomplete and locally complete, and the ideal points of its completion correspond to the bounded closed nonnull absolutely convex non-absorbing sets in E, i.e. those whose linear spans are proper subspaces of E.

The above definition of the topology on $\mathfrak{N}$ instantly proves metrisability but is otherwise inconvenient. However:

(d) Let $\nu_0 \in \mathfrak{N}$. A base of nbds for ν_0 is given by sets of the form

$$\{\mu \in \mathfrak{N}\colon (\forall x \in E)\ (1-\varepsilon)\nu_0(x) \leqslant \mu(x) \leqslant (1+\varepsilon)\nu_0(x)\},$$

where $\varepsilon \in (0, 1)$.

The proof is easy. This readily implies that:

(e) $\mathfrak{N}$ is a topological cone; that is, the maps

$$(0, \infty)\times\mathfrak{N} \to \mathfrak{N} \quad \text{and} \quad \mathfrak{N}\times\mathfrak{N} \to \mathfrak{N}$$

of (a) are both continuous. Also, $E\times\mathfrak{N} \to \boldsymbol{R}$: $(x, \nu) \mapsto \nu(x)$ is continuous.

Let $G = \mathrm{GL}(E)$ be the general linear group of E (of bounded linear transformations with bounded inverse), given the topology of convergence in operator-norm. There is an obvious right action of G on $\mathfrak{N}$:

$$\mathfrak{N}\times G \to \mathfrak{N}\colon (\nu, T) \mapsto \nu T,$$

where, for each $x \in E$, $\nu T(x) = \nu(Tx)$.

(f) The action $\mathfrak{N}\times G \to \mathfrak{N}$ is continuous. (From (d).)

Suppose that $p\colon \mathscr{E} \to X$ is a vector bundle of class C^0 with fibre E. Recall (see for instance [5]) that this, by definition, means that the transition functions between local trivialisations of the bundle are continuous as maps into G. Let $\tilde{p}\colon \mathscr{G} \to X$ be the associated principal right G-bundle. $\mathfrak{N}$ is a *left* G-module in an obvious way (define $T\cdot\nu = \nu\cdot T^{-1}$). So there is an $\mathfrak{N}$-bundle associated with $\mathscr{G}$, which we may call $\mathscr{F}$: its points are the equivalence classes of $\mathscr{G}\times\mathfrak{N}$ by the relation $(gT, \mu) \sim (g, T\mu)$ (for $g \in \mathscr{G}$, $\mu \in \mathfrak{N}$, $T \in G$), and the projection $\pi\colon \mathscr{F} \to X$ sends the equivalence class of (g, μ) to $\tilde{p}(g)$. Let $q\colon \mathscr{E}\times_X\mathscr{F} \to X$ be the fibre product of $\mathscr{E}$ and $\mathscr{F}$ over X (i.e. the Whitney sum). Then:

(g) There is a natural map $\mathscr{E}\times_X\mathscr{F} \to \boldsymbol{R}$. For $\mathscr{E}$ is the E-bundle associated with $\mathscr{G}$ by the natural action $G\times E \to E$. If $e \in \mathscr{E}$, $f \in \mathscr{F}$, and $p(e) = \pi(f)$ (so that $(e, f) \in \mathscr{E}\times_X\mathscr{F}$), then e is the equivalence class of $(g, \xi) \in \mathscr{G}\times E$ and f is the equivalence class of $(h, \mu) \in \mathscr{G}\times\mathfrak{N}$. Since $p(e) = \pi(f)$, $\tilde{p}(g) = \tilde{p}(h)$, and there is a unique $T_0 \in G$ with $h = gT_0$. Write

$$\alpha(e, f) = T_0\mu(\xi).$$

This is well-defined: if $T \in G$, $S \in G$, then $hT = (gS)S^{-1}T_0T$, and by the definitions $(S^{-1}T_0T)(T^{-1}\mu)(S^{-1}\xi) = T_0\mu(\xi)$. Furthermore, α is continuous on $\mathscr{E}\times_X\mathscr{F}$, as is easily verified by working in local trivialisations.

(h) If we fix $f \in \mathscr{F}$, the map $e \mapsto \alpha(e, f)$ is a norm (giving the right topology) on the fibre $p^{-1}(\pi(f))$ of $\mathscr{E}$. This follows from the above definition of α. Moreover, and again by definition, any such norm on the given fibre of $\mathscr{E}$ is representable as an element of the corresponding fibre of $\mathscr{F}$ via the map α. So it would be natural to call $\mathscr{F}$ the "bundle of fibre-norms" of $\mathscr{E}$. It should be noted that, in the language of differential geometry, $\mathscr{F}$ is covariantly related to $\mathscr{E}$—that is, representations of points of $\mathscr{F}$ and of points of $\mathscr{E}$ in local trivialisations which correspond to each other transform contragrediently.

DEFINITION. A *Finsler structure* on $\mathscr{E}$ is a continuous section of $\mathscr{F}$. If M is a C^1 manifold, modelled on some normed space, a Finsler structure on M is, by definition, a Finsler structure on its tangent bundle.

If X is paracompact, $\mathscr{E}$ admits a Finsler structure by virtue of (e) and the existence of C^0 partitions of unity.

It is easy to check (by using (d)) that this is equivalent to the original definition of Palais in [8], and its only advantage over that formulation is its naturality—it does not require *a posteriori* verifications that it is well-defined. Notice, however, that if $\mathscr{E}$ were a C^r bundle $(r \geqslant 1)$ and $\mathfrak{N}_0$ were some subset of $\mathfrak{N}$ admitting a C^r structure such that, for $T \in G$, $\mathfrak{N}_0T \subseteq \mathfrak{N}_0$, and the map $\mathfrak{N}_0\times G \to \mathfrak{N}_0$ is C^r, then we could define analogously a bundle $\mathscr{F}_0$ of "fibre-norms of type $\mathfrak{N}_0$" (relative to the given C^r structure on $\mathfrak{N}_0$). $\mathscr{F}_0$ is a C^r bundle, and the C^r-sections are the "Finsler structures of type $\mathfrak{N}_0$". Riemannian structures are the obvious examples, but there are others which may sometimes be of interest.

§ 2. If M is a connected C^1 manifold, a Finsler structure on M induces a metric on M which defines the topology of M [9]. When this metric is, for instance, bounded or complete, one may say that the original Finsler structure is bounded or complete.

THEOREM. *Let M be a connected C^1 manifold of finite dimension, perhaps with boundary. M admits a bounded complete Finsler structure if and only if it is compact.*

It is clear that if M is compact, it admits Finsler structures (for it is paracompact), and all Finsler structures on M are bounded and complete. So it is only necessary to prove that the existence of a bounded complete Finsler structure implies compactness. The proof is in several foreseeable steps, but there are certain technical complications, and the arguments I present do not seem to appear in the literature. There are five steps, (a)–(e) below.

(a) M admits a compatible C^∞ structure (see, e.g., Munkres [6]). So we may assume it is C^∞, though the Finsler structure is only C^0.

(b) A connected C^∞ manifold without boundary, of finite dimension, admits a bounded complete C^∞ *Riemannian* structure only if it is compact. This is well known: see, e.g., [4], pp. 172–176.

(c) A connected finite-dimensional C^∞ manifold *with boundary* can have a bounded complete C^∞ Riemannian structure only if it is compact. This must also be well known, but I have never seen a proof. For completeness I give one here. The boundary of M is ∂M.

Using the given Riemannian structure g, construct a geodesic collaring of ∂M. This will consist of a nbd U of $\partial M\times 0$ in $\partial M\times I$,

$$U = \{(x,s)\colon\ x\in\partial M \text{ and } 0\leqslant s<\varphi(x)\}$$

(for some positive C^∞ function φ on ∂M), and a diffeomorphism

$$f\colon\ U,\partial M\times 0\to M,\partial M$$

of U with an open nbd of ∂M in M, which restricts itself to the identification $\partial M\times 0=\partial M$, such that, for each $x\in\partial M$, $f(x,s)$ is the geodesic through x in the direction of the inward normal to ∂M at x (for $0\leqslant s<\varphi(x)$).

Let g_1 be the C^∞ Riemannian structure on $f(U)$ induced via f from the restriction to U of the product structure on $\partial M\times I$ (where ∂M is given the structure induced from g). Then g_1 and g agree at points of ∂M. The structure g induces a Finsler structure ν on M, and g_1 induces a Finsler structure ν_1 on $f(U)$; ν_1 and ν agree on ∂M. Since ν_1 and ν are *continuous* sections of the bundle of fibre-norms, there is a positive C^∞ function ψ on ∂M such that, for each $x\in\partial M$, $\psi(x)\leqslant\varphi(x)$ and, whenever $0\leqslant s<\psi(x)$, for each tangent vector $\xi\in T_{f(x,s)}M$,

$$\nu_1\big(f(x,s)\big)\xi\leqslant 2\nu\big(f(x,s)\big)\xi\leqslant 4\nu_1\big(f(x,s)\big)\xi.$$

Suppose that $\omega\colon I\to I$ is a C^∞ function taking the value 0 on the interval $0\leqslant s\leqslant 0.2$ and the value 1 on the interval $0.8\leqslant s\leqslant 1$. Define

a C^∞ Riemannian structure g_2 on M as follows: g_2 agrees with g at all points outside $f(U)$ and inside $f(U)$:

$$g_2\big(f(x,s)\big)=\omega\big(s/\psi(x)\big)g\big(f(x,s)\big)+\big(1-\omega\big(s/\psi(x)\big)\big)g_1\big(f(x,s)\big).$$

The structure g_2 is clearly a well-defined C^∞ Riemannian structure on M. It gives rise to a Finsler structure ν_2. One instantly infers that, for every $y\in M$ and $\xi\in T_yM$,

$$\big(\nu_2(y)\xi\big)^2\leqslant 4\big(\nu(y)\xi\big)^2\leqslant 16\big(\nu_2(y)\xi\big)^2,$$

and consequently

$$\nu_2(y)\xi\leqslant 2\nu(y)\xi\leqslant 4\nu_2(y)\xi.$$

The same inequality therefore holds for path-lengths computed with respect to ν_2 and ν. It follows that ν_2 must also be complete and bounded.

Now "double up" M by means of f. Formally, let

$$N_1=(M\times\{-1,+1\})\cup\{(x,s)\colon\ x\in\partial M,\ |s|<\psi(x)/3\},$$

with a Riemannian structure defined as follows: on $M\times\{-1\}$ and $M\times\{+1\}$ it corresponds in an obvious way to g_2, and on

$$\{(x,s)\colon\ x\in\partial M,\ |s|<\psi(x)/3\}\subset M\times(-1,+1)$$

it is the product structure. Call this structure h_1. Define N to be the quotient of N_1 by the identifications

$$\big(f(x,s),+1\big)\sim(x,s),\qquad \big(f(x,s),-1\big)\sim(x,-s),$$

for $x\in M$, $0\leqslant s<\psi(x)/3$. N then inherits a C^∞ Riemannian structure h from N_1,h_1, because, by definition, the above identifications are structure-preserving. Write M_+, M_-, and ∂M for the images in N of $M\times\{+1\}$, $M\times\{-1\}$ and $\partial M\times\{+1\}$, respectively. The image ∂M is the image of three distinct copies of ∂M in N_1; no confusion will arise from this use of the notation. ∂M is clearly a closed C^∞ submanifold of N.

We shall prove that h is complete and bounded. Since N has empty boundary, it will therefore be compact, by (b); and since M is diffeomorphic with the closed submanifold M_+ of N, it is also compact.

Define the "folding" map $D\colon N\to M_+$ by identification from a similar map for N_1, $D_1\colon (x,s)\to(x,|s|)$ (where either $x\in M$ and $s=\pm 1$, or $x\in\partial M$ and $|s|<\psi(x)/3$). D is C^0 on N and C^∞ on $N\setminus\partial M$, and its restriction to either of the two components of $N\setminus\partial M$ is a Riemannian isomorphism with $M_+\setminus\partial M$. Hence, if p is a piecewise C^1 path in N which meets ∂M transversely (and so only finitely often), $D\cdot p$ is a piecewise C^1 path of the same h-length as p and lies entirely in M_+. Now, the distance between two points of N, in the metric induced from h, may be computed from paths meeting ∂M transversely (see below). If both points are in M_+, then any piecewise C^1 path q joining them in N and meeting ∂M transversely may be replaced by the path $D\cdot q$, which has the same h-length. Thus the metric on M_+ induced by h cannot exceed the restriction of the metric on N induced by h. It follows that they are the same; since the metric on M_+ is calculated from paths in M_+, and the metric on N from paths in N, the metric on N cannot exceed on M_+ the metric in M_+. But (M_+, h) is Riemannian-isomorphic with (M, g_2), and so is complete and bounded. It is thus complete and bounded as a subset of N with the metric given by h. So is M_-, by symmetry.

Hence N is the union of two bounded sets and is bounded. Also it is the union of two complete subsets, and so is complete (any Cauchy sequence has a subsequence lying entirely in one of the complete subsets, and so convergent).

It remains to prove the statement that a piecewise C^1 path in N may be replaced by another, transversal to ∂M, whose h-length is greater by no more than ε, and which has the same end-points. It is well known that there is a C^∞ path between the same end-points of h-length greater by no more than $\varepsilon/3$. If one or both end-points lie on ∂M, it is clear that a small further modification may, if necessary, be made to ensure that the tangents at these points do not lie in the tangent space to ∂M, without increasing the length by more than $\varepsilon/3$. Finally, we must show that a C^∞ path transversal to M at the end-points may be C^1-approximated arbitrarily closely by a C^∞ path with the same end-points everywhere transversal to ∂M. This, however, is a very simple case of the stronger version of the Thom transversality theorem: see Stong [11], Thom [12], Wall [13], etc. (The method transcribed by Stong seems preferable, and may be adapted to C^1-approximation with little effort.)

This concludes the proof of (c). I have given it in such detail chiefly for the sake of the following observation: the argument does not work, at least as given above, for general Finsler structures. The difficulty is that it is *a priori* impossible to approximate the given Finsler structure uniformly over M by a structure compatible with the procedure of "doubling-up". In finite dimensions, it can be done by means of (d) below, but this does not simplify the proof here.

(d) The following result is well known (John [3]).

THEOREM. *For any norm* $\|\ \|$ *on* $\boldsymbol{R}^n$, *there exists an inner product such that the induced norm* $|\ |$ *satisfies, for all* $x \in \boldsymbol{R}^n$, *the inequality*

$$n^{-1/2}\|x\| \leqslant |x| \leqslant n^{1/2}\|x\|.$$

Since this is a basic result in the theory of the so-called *Mazur distance* between norms on a given topological vector space, it is appropriate to remark here that there is no significant relation between the distance

we have defined in § 1(a) and the Mazur distance. The Mazur distance measures the difference in *shape* between the unit balls of the norms in question; my distance measures only the difference in size.

(e) If M is a C^∞ manifold, with or without boundary, of finite dimension n, and admits a bounded complete Finsler structure, then it admits a bounded complete C^∞ Riemannian structure. This ends the proof of the theorem.

Given $x \in M$, and a C^∞ chart about x, $\varphi\colon U \to \boldsymbol{R}^n_+$, the Finsler structure on M gives a norm $\|\ \|$ on T_xM, and hence on $\boldsymbol{R}^n$, which the chart φ identifies with T_xM. Now, by (d), there is an inner product on $\boldsymbol{R}^n$ inducing a norm $|\ |$ with, for all $\xi \in \boldsymbol{R}^n$,

$$n^{-1/2}\|\xi\| \leqslant |\xi| \leqslant n^{1/2}\|\xi\|.$$

Since φ further identifies $\boldsymbol{R}^n$ with T_yM for each $y \in U$, this inner product gives a C^∞ Riemannian structure on U. The set of points of U at which the norm given by this Riemannian structure lies (for each element of the tangent space) between $(2n)^{-1/2}\nu(y)$ and $(2n)^{1/2}\nu(y)$, where we write ν for the Finsler structure (thus $\|\ \| = \nu(x)$), is a nbd of x by the definition of Finsler structure. Thus M may be covered by open sets on which are defined C^∞ Riemannian structures satisfying these inequalities with respect to ν. Piecing them together by a C^∞ partition of unity, we obtain a C^∞ Riemannian structure on the whole of M, which, as is easily checked, satisfies these inequalities everywhere. From the consideration of path-lengths it follows that this Riemannian structure must be complete and bounded.

§ 3. The question raised in Elworthy's paper [2] is this: what infinite-dimensional C^1-connected Banach manifolds admit complete bounded Finsler structures? Behind the question perhaps lies a conjecture, namely that some "metrical" property which is necessarily true for compact manifolds may be possible only for some interesting class of infinite-dimensional manifolds. This seems at least plausible, and in such generality I do not know whether it is true or not. The answer to the original question, however, is rather surprisingly easy, although in some sense incomplete, since it leans heavily on the results of infinite-dimensional differential topology (for which, and for further references, see the same paper of Elworthy). It seems to me that this is inevitable, because we have fewer means of constructing Finsler structures with specific properties than of constructing diffeomorphisms.

(a) THEOREM. *Let M be a C^1-Banach manifold (perhaps with boundary) admitting a complete Finsler structure. Suppose N is a connected noncompact C^1 Banach manifold without boundary which admits a bounded complete Finsler structure. Then $M \times N$ admits a bounded complete Finsler structure.*

Proof. Since N is noncompact and paracompact, it admits a positive continuous real-valued function f whose infimum, never attained, is 0. Let ν_M be a complete Finsler structure on M and ν_N a complete bounded Finsler structure on N, and define a Finsler structure ν on $M\times N$ by the following formula (in which $x\in M$, $y\in N$; $\xi\in T_xM$, $\eta\in T_yM$; and $(\xi,\eta)\in T_{(x,y)}(M\times N)$ by canonical identification):

$$\nu(\xi,\eta)=f(y)\nu_M(x)\xi+\nu_N(y)\eta. \tag{1}$$

As usual, the verification that ν is a Finsler structure is a tedious triviality. Observe that the projection $\pi\colon M\times N\to N$ does not increase the length of paths, and so does not increase distances either.

We show first that ν is bounded. In fact, let k be the diameter of N with respect to the metric induced from ν_N, and let $\varepsilon>0$. Suppose, for $i=1,2$, that $(x_i,y_i)\in M\times N$. Choose a path q between x_1 and x_2 in M, of length l, say, with respect to ν_M, and piecewise C^1. Choose $y\in N$ so $3f(y)l<\varepsilon$; choose in N piecewise C^1 paths p_i from y_i to y of length less than $k+\varepsilon/3$. Now define $p\colon I\to M\times N$:

$$\begin{aligned}
&\text{for}\quad 0\leqslant t\leqslant 1/3, && p(t)=(x_1,p_1(3t));\\
&\text{for}\quad 1/3\leqslant t\leqslant 2/3, && p(t)=(q(3t-1),y);\\
&\text{for}\quad 2/3\leqslant t\leqslant 1, && p(t)=(x_2,p_2(3-3t)).
\end{aligned}$$

Hence p is piecewise C^1. Its length with respect to ν is the sum of the lengths of the three segments set out in the formulae. All vectors tangent to the first of these segments have zero component in the M-direction, so that, from (1), its ν-length is exactly the ν_N-length of p_1; likewise the last segment has the same length as p_2. The vectors tangent to the second segment have zero component in the N-direction, and so, from (1), its ν-length is exactly $f(y)l$. So the length of p does not, on the whole, exceed $2k+\varepsilon$. As (x_1,y_1), ε, and (x_2,y_2) were arbitrary, this shows that the diameter of $M\times N$ does not exceed $2k$.

We now demonstrate the completeness of ν. Write d_N for the metric on N induced by ν_N, d_M for the ν_M-metric on M, d for the ν-metric on $M\times N$. Suppose that $((x_n,y_n))$ is a d-Cauchy sequence in $M\times N$. Then, since π is distance-nonincreasing, (y_n) is d_N-Cauchy in N and has therefore a limit $y\in N$. $\exists\,\delta>0$ such that, when $z\in N$ and $d_N(y,z)<\delta$, $f(z)>f(y)/2$. But, on the other hand, $d((x_m,y_m),(x_n,y_n))<\delta/2$, $d_N(y,y_n)<\delta/2$, for all sufficiently large m and n. For such values of m and n, the d-distance between (x_m,y_m) and (x_n,y_n) may be computed from piecewise C^1 paths $p\colon I\to M\times N$ such that, for each $t\in I$, $d_N(\pi p(t),y)<\delta$; for, if p is a path of length less than $\delta/2$ (with respect to ν) between two such points, πp is a path of ν_N-length less than $\delta/2$, starting from a point within d_N-distance $\delta/2$ of y. Write σ for the projection of $M\times N$ on M. Now, for paths p of the type under consideration, $f(\pi p(t))>f(y)/2$ for all $t\in I$, and (1) shows that the ν_M-length of σp cannot exceed $2/f(y)$ times the ν-length of p. Consequently, for large enough m and n,

$$d_M(x_m,x_n)\leqslant(2/f(y))\,d((x_m,y_m),(x_n,y_n)),$$

so that (x_m) is d_M-Cauchy in M and so has a limit in M, x.

Therefore $(x_n,y_n)\to(x,y)$ in $M\times N$, as desired. ν is a bounded complete Finsler structure on $M\times N$.

(b) Remarks. Every C^1 Banach manifold admits a complete Finsler structure (provided, of course, that it is paracompact). This fact, which is still not sufficiently known, was proved by Penot in his thesis [10], following the method of Nomizu–Ozeki [7]; thus, indeed, any Finsler structure is conformally equivalent to a complete one. I have, nevertheless, given the statement of the theorem without appealing to this result, because essentially the same proof may apply in cases where the existence of complete structures is not automatic. For instance, if we define a C^∞ Finsler structure on a C^∞ manifold as a Finsler structure which is C^∞ as a function on TM (see § 1(g), (h)), then the theorem is valid for such C^∞ Finsler structures, with the hypothesis that N is C^∞ and admits a positive unbounded analytic function. In this case the existence of a C^∞ Finsler structure, and a fortiori of a complete one, is not assured *a priori*.

(c) Definition. Let M and N be C^r Banach manifolds (perhaps with boundary), $r\geqslant 1$. M is said to be *C^r-stable by N* if there is a C^r-diffeomorphism of M with $M\times N$. (Notice that, in fact, N must be without boundary; but, for this definition, it may be compact.)

An important class of theorems in infinite-dimensional differential topology concerns situations in which a manifold M is C^r-stable by some other manifold N, most usually a Banach space of infinite dimension. For such theorems, see Elworthy [2]. One then obtains, from Theorem (a),

(d) Theorem. *Suppose that M is a C^1 Banach manifold (perhaps with boundary), admitting a complete Finsler structure and C^1-stable by a connected noncompact manifold N which admits a complete bounded Finsler structure. Then M itself admits a bounded complete Finsler structure.*

(Indeed, the structure on $M\times N$ may be transferred by the diffeomorphism to a bounded complete structure on M.)

(e) Theorem. *Any separable C^∞ Hilbert manifold (with or without boundary) admits a bounded complete C^∞ Riemannian structure.*

Proof. Let N be the unit sphere of a separable Hilbert space, with a Riemannian metric induced from the inner product. This makes N bounded and complete. It is C^∞-diffeomorphic to a separable Hilbert space, by Bessaga [1], and so $M\times N$ is C^∞-diffeomorphic to M by Theorem 22 of

[2], for instance. The proof of Theorem (a) can now be followed word for word, except that formula (1) requires alteration. If g_M, g_N are the Riemannian structures on M and N, respectively (both C^∞ and complete; g_N is bounded), and if f is a C^∞ function on N which is everywhere strictly positive and has infimum zero, define for $i = 1, 2$, $\xi_i \in T_x M$, $\eta_i \in T_y N$,

$$g(x, y)\big((\xi_1, \eta_1), (\xi_2, \eta_2)\big) = \big(f(y)\big)^2 g_M(x)(\xi_1, \xi_2) + g_N(\eta_1, \eta_2).$$

The previous proof now shows that g is a complete bounded C^∞ Riemannian structure on $M \times N$, which may be transferred by the given diffeomorphism to M.

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Interpolation of weighted L_p-spaces

by

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Abstract. We characterize the interpolation spaces with respect to couples of weighted L_p-spaces. This is done in terms of the K-functional of Peetre. The main tool is a generalization of the theorem of Hardy, Littlewood, and Polya on doubly stochastic matrices.

Contents

0. Introduction. We are concerned with the following problem: Let $\bar A = \{A_0, A_1\} = \{L_{p_0 a_0}, L_{p_1 a_1}\}$, $\bar B = \{B_0, B_1\} = \{L_{p_0 b_0}, L_{p_1 b_1}\}$, $0 < p_0, p_1 \leqslant \infty$, be two couples of weighted L_p-spaces assigned to some, not necessarily the same, measure spaces. Then characterize *all* interpolation spaces with respect to $\bar A, \bar B$, i.e. all spaces A, B obeying

$$T\colon A_\mu \to B_\mu \ (\mu = 0, 1) \quad \text{implies} \quad T\colon A \to B, \tag{0.1}$$
$$\|T\|_{\mathcal{L}(A;B)} \leqslant c \max_{\mu=0,1} \|T\|_{\mathcal{L}(A_\mu;B_\mu)}$$

(operator norms). A both necessary and sufficient condition is found to be

$$K(t, g; \bar B) \leqslant K(t, f; \bar A),\ t > 0, \quad \text{implies} \quad \|g\|_B \leqslant C\|f\|_A \tag{0.2}$$

("K-monotonicity"), where K is the Peetre functional

$$K(t, f; \bar A) = \inf_{f=f_0+f_1} (\|f_0\|_{A_0} + t\|f_1\|_{A_1}). \tag{0.3}$$